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(Teoriya bifurkatsii dinamicheskikh sistem na ploskosti)

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PREFACE

The present monograph is a direct continuation of our *Qualitative Theory of Second-Order Dynamic Systems* (QT) published in 1966. It may also be considered as a second volume of the definitive treatise on dynamic systems and their applications planned by A. A. Andronov back in the 1940s. All this notwithstanding, however, *Theory of Bifurcations* can be treated as an independent volume and the reader is only expected to be acquainted with the basic concepts of the qualitative theory of differential equations on a plane.

In distinction from QT, the greater part of which is devoted to the classical theory (of Poincaré and Bendixson), *Theory of Bifurcations* presents relatively recent results which were obtained during the last three decades and published — in part or completely — in a number of letters and papers in scientific journals. These results are closely linked to the theory of oscillations and have by now found many important uses in physics and engineering.*

The present book, like QT, was begun by A. A. Andronov, E. A. Leontovich, and A. G. Maier and completed by E. A. Leontovich and I. I. Gordon. N. A. Gubar' and R. M. Mints also took part in the preparation of the monograph, the former being responsible for Chapter VIII and the latter for part of Chapter XIV. The final version was prepared by I. I. Gordon.

The main results presented in Chapters III through VII were derived by A. A. Andronov and L. S. Pontryagin, and those in Chapters IX through XII by A. A. Andronov and E. A. Leontovich. Chapter VIII is based on the work of N. A. Gubar' and the results of Chapter XIII are due to E. A. Leontovich, A. G. Maier, and L. S. Pontryagin. The general editing of the book was undertaken by Yu. M. Romanovskii.

The book naturally falls into two parts — the theory of structurally stable systems (Chapters I through VII) and the theory of bifurcations (Chapters VIII through XIV). The second part is largely independent of the first, and the reader will only require some basic information from Chapters I, II, IV, and V.

Although the book contains numerous references to QT, many of these refer to proofs of well known or relatively simple and obvious propositions contained in QT, and the reader may safely ignore these references.

Each chapter includes a brief introductory summary. These chapter introductions were written in such a way as to enable the reader to form a clear idea of the contents of each chapter and to decide what chapters deserve detailed study and what can be skipped.

* Some data on structurally stable dynamic systems and bifurcations (without exhaustive proof) will be found in the second edition of A. Andronov, A. Vitt, and S. Khaikin, *Theory of Oscillations* (Moscow, 1959).

The book contains numerous drawings and worked-out examples illustrating the various mathematical propositions. Unfortunately, space limitations prevented us from including many more remarkable examples which arise from applications (see, e.g., /2, 3/).

The sections, theorems, definitions, figures, and examples are numbered continuously through the book. The numbering of lemmas and equations is restricted to each section. In the Appendix at the end of the book, the equations and lemmas are numbered according to the subsections.

The reference §21.2, (5) is to equation (5) in subsection 2 of §21. The reference (7) is to equation (7) of the current section. The reference QT, §8.5, Lemma 4 is to Lemma 4 in subsection 5 of §8 in QT.

A list of bibliographical references directly related to the subject matter of the present volume will be found at the end of the book. References to the sources in this bibliography are indicated by numbers between slashes.

E. A. Leontovich
I. I. Gordon

Gor'kii, 1966

Table of Contents

INTRODUCTION

The subject-matter of the qualitative theory of dynamic systems is formulated in QT. The theory is concerned with the topological structure of the partition into paths of the domain of definition of the dynamic system. A number of topics relating to second-order systems (defined in a plane region or on a sphere) are treated in QT. In particular, the different kinds of paths of different systems are identified, the limit sets of these paths are described, and methods for investigating the configuration of paths in the neighborhood of an equilibrium state are given. A large part of QT is devoted to establishing the minimum information about the paths of a dynamic system needed in order to determine its topological structure in a region. This problem is completely solved in QT under certain limitations on the class of dynamic systems being considered: it is established that the topological structure of a dynamic system is determined by the character and the configuration of the so-called singular paths (equilibrium states, limit cycles, and separatrices). All the relevant information can be presented in the form of a certain finite scheme. The basic problem in elucidating the topological structure of a dynamic system is thus to find the particular scheme of the system. So far, however, no regular methods have been devised enabling us to establish the existence of limit cycles of a dynamic system, their configuration, and the configuration of the separatrices. Only individual particular techniques are available which permit solving — and sometimes quite successfully — a number of particular problems related to the existence and behavior of limit cycles and separatrices. The most useful of these techniques are described in QT, together with examples of their application.

In QT the approach to the qualitative structure of dynamic systems is static — it is assumed that the system does not change. On the other hand, the main problems treated in the present volume are concerned with the changes in the topological structure of a dynamic system when the system itself changes. As in QT, we are dealing with autonomous systems on a plane, i.e., systems of the form

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y).$$

Let a system of this kind be defined in some region G. What happens to the topological structure of the partition of this region into paths when the system — i.e., the functions P and Q on the right — changes?

This question is obviously of independent mathematical significance. It is also highly important for various applications. The point is that dynamic systems corresponding — under certain idealizations — to physical

or technical problems invariably contain a certain number of parameters, and we are generally interested in the changes in the topological structure of the system when the parameters are varied. In particular, it is important to find the partition of the parameter space into regions each corresponding to identical topological structure and to determine the change in the topological structure when the system moves across the boundary of two such regions in the parameter space.

The changes in topological structure evidently need be considered only for the case of *small changes* in the system. One of the most important classes of dynamic systems comprises those systems whose topological structure in a given region *does not change* under small modifications of the right-hand sides of the system. Such systems, generally known as *structurally stable*, were first introduced by A. Andronov and L. Pontryagin in /4/ under the name of *coarse systems* or *systèmes grossiers*.

Structural stability of a dynamic system is particularly important in applications, e.g., in various physical problems. The values of the parameters entering the right-hand sides of the system are generally linked with the particular physical problem being considered and are only known to some approximation. If small changes in these parameters — within the experimental margin of error — lead to a change in the topological structure of the dynamic system, i.e., if the system is *structurally unstable*, the topological structure of the system is clearly not a suitable criterion for analyzing the physical phenomenon. Conversely, if the system is structurally stable, its structure may be directly related to the properties of the physical phenomena. It is interesting to remark in this connection that Andronov and Pontryagin's term, *coarse systems*, was originally proposed in contradistinction from *fine systems* whose topological structure would break under the action of arbitrarily small external disturbances.

The first problem to be considered is that of the distinctive or identifying characteristics of structurally stable systems. For systems defined in a bounded plane region, this problem was essentially solved in A. Andronov and L. Pontryagin's original paper /4/, and subsequently elaborated in /5/ and /6/.

The necessary and sufficient conditions of structural stability for plane regions are relatively simple (see Introduction to Chapter VI), but a rigorous derivation of these conditions necessitates detailed scrutiny of a whole range of important concepts and scrupulous proofs. The first half of the present volume (Chapters I through VII) is entirely devoted to the theory of structurally stable systems and, in particular, to the derivation of these conditions of structural stability. The book deals only with structural stability of systems in a bounded plane region and on a sphere. Note, however, that the concept of structural stability has been extended and investigated, especially in the last decade, for a number of other objects also. Peixoto considered the conditions of structural stability of dynamic systems on an arbitrary closed surface (see /7/). Gudkov /8/ introduced the concept of structural stability of algebraic curves. Structural stability of many-dimensional dynamic systems is treated in /32, 33, 36/.

Structurally stable systems are the rule, so to say, in the metric space whose points are dynamic systems defined in some region. It is shown in Chapter VI that structurally stable systems form an open everywhere dense set in this space. The set of structurally stable systems is partitioned into components, each consisting of structurally stable systems of identical topological structure. The "partitions" between these components consist of structurally unstable dynamic systems. When a dynamic system is altered or modified, its topological structure will change only if the system passes through an intermediate stage of structural instability. The *theory of bifurcations*, which is concerned with changes in the topological structure of dynamic systems, therefore appropriately concentrates on structurally unstable systems. Structurally unstable systems are also of interest in applications: the so-called conservative systems (see Chapter XIII), often encountered in physics, are structurally unstable. We are thus naturally led to a detailed examination of structurally unstable systems.

The first step in this direction evidently involves a classification of structurally unstable systems. Structurally unstable systems can be divided into "less structurally unstable" and "more structurally unstable." This leads to a classification according to the *degrees of structural instability*, originally introduced in /9/. The least structurally unstable systems in this classification are the *systems of the first degree of structural instability*: under small changes, these systems either go to a structurally unstable system or retain their topological structure. The complete conditions for a system to be of the first degree of structural instability were derived for plane systems (see /9, 10/; these conditions are derived in Chapter XII). A dynamic system of the first degree of structural instability was found to have one and only one structurally unstable singular path, i.e., it has either a multiple equilibrium state, or a multiple limit cycle, or a saddle-to-saddle separatrix.* To establish the bifurcations of a system of the first degree of structural instability, it suffices to consider the changes in its topological structure in the neighborhood of its structurally unstable singular path. Bifurcations involving a change in the number of limit cycles — i.e., bifurcations in which limit cycles are *created* or *destroyed* — are of particular interest in theory and in applications. Dynamic systems of the first degree of structural instability may only exhibit the following instances of creation of limit cycles: from the multiple focus of the system, from the multiple cycle, from a loop of a saddle-point separatrix, from a loop of a saddle-node separatrix. The various cases are treated separately in Chapters IX through XII. Note that the material presented in this chapter applies to bifurcations of systems of both first and higher degrees of structural instability.

The concepts of structural instability, degrees of structural instability, and especially examination of the simplest bifurcations leads to a number of techniques for the investigation of particular differential equations. These techniques were successfully applied to a number of equations of special physical interest (see, e.g., /2, 3, 20, 25—28/).

* Certain additional conditions should also be satisfied; they are formulated in Chapter XII.

Chapter XIV is entirely devoted to examples of dynamic systems, analyzed by the tools of the theory of bifurcations.

Chapter XIII occupies a somewhat special position in the book: it treats the creation of limit cycles from closed paths of conservative systems.

Chapter I

MULTIPLICITY OF ROOTS OF FUNCTIONS AND MULTIPLICITY OF INTERSECTION POINTS OF TWO CURVES

INTRODUCTION

In this chapter we consider some fairly elementary concepts relating to the root of a function and the intersection point of two curves. In §1 we define the δ-closeness of two functions to rank r and the multiplicity of a root of a function. Roughly speaking, a root x_0 of a function $f(x)$ is said to be of multiplicity $r \geqslant 1$ if functions $\tilde{f}(x)$ "sufficiently" close to $f(x)$ cannot have more than r roots in a "sufficiently" small neighborhood of x_0, but there is any number of functions sufficiently close to $f(x)$ which have exactly r roots in any arbitrarily small neighborhood of x_0. The necessary and sufficient condition of r-multiplicity of a root is derived (Theorem 5), which states that $f(x_0) = f'(x_0) = \ldots f^{(r-1)}(x_0) = 0$, $f^{(r)}(x_0) \neq 0$. It follows from this condition that for analytical functions (and in particular, for polynomials), the root multiplicity defined in this chapter coincides with the usual concept of multiplicity.

The multiplicity of a common point of two curves, analogous to the concept of root multiplicity of a function, is introduced in §2, and the necessary and sufficient conditions are established for r-multiplicity of a common point (x_0, y_0) of two curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$, when $r = 1$ and $r = 2$. If $r = 1$, the common point is said to be simple or structurally stable. The necessary and sufficient condition of structural stability of a point (x_0, y_0) is simply

$$\Delta_0 = \begin{vmatrix} F'_{1x}(x_0, y_0) & F'_{1y}(x_0, y_0) \\ F'_{2x}(x_0, y_0) & F'_{2y}(x_0, y_0) \end{vmatrix} \neq 0 \text{ (Theorem 6).}$$

The conditions for a common point with $r = 2$ are more complicated (see Theorem 7).

§1. MULTIPLICITY OF A ROOT OF A FUNCTION

1. δ-closeness to rank r

We will consider functions defined at all points $M(x_1, x_2, \ldots, x_n)$ of some open (or closed) region G (or $\bar{G}_1$) in n-dimensional euclidean space E_n. In

applications we will be mainly concerned with the cases $n = 1$ or $n = 2$. Here it is assumed, however, that n is any natural number.

A function is said to be a function of class k in G (or $\bar{G}_1$), where k is a natural number, if it is continuous and continuously differentiable up to order k, inclusive, in its domain of definition; a function is said to be a function of the analytical class in some region if it is analytical in that region.*

Let $F_0(x_1, x_2, \ldots, x_n)$ be a function of class k or an analytical function in G (or $\bar{G}_1$), δ is some positive number, r a natural number such that $r \leqslant k$ if F_0 is a function of class k.

Definition 1. A function $F(x_1, x_2, \ldots, x_n)$ of class $k_1 \geqslant r$ or analytical in G ($\bar{G}_1$) is said to be δ-close to rank r to the function $F_0(x_1, x_2, \ldots, x_n)$ in the region G($\bar{G}_1$) if at any point of the region

$$|F - F_0| < \delta, \quad |F^{(l)}_{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}} - F^{(l)}_{0 x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}| < \delta,$$

where $l = 1, 2, \ldots, r$, all α_i are non-negative numbers and $\alpha_1 + \alpha_2 + \ldots \ldots + \alpha_n = l$.

Clearly if two functions are δ-close to rank r in some region G, they are δ-close to any rank $r_1 < r$ in that region; moreover, for any δ_1, $\delta_1 \geqslant \delta$, they are δ_1-close to rank r in any subregion of G.

If everywhere in the region we only have the one inequality

$$|F - F_0| < \delta,$$

i.e., only the functions as such are δ-close, but not their derivatives, the functions F and F_0 are said to be δ-close to rank 0. In what follows, with rare exceptions, we will always consider δ-closeness at least to rank 1. Therefore the expression "two δ-close functions" is to be understood as qualifying two functions which are δ-close to rank $r \geqslant 1$. The most interesting case is that of a function depending on one or several parameters which for any ("arbitrarily small") $\delta > 0$ can be made δ-close to any required rank to a given function $F_0(x)$ by an appropriate choice of the parameters.

Let us consider simple examples of functions which are δ-close to rank r to a given function $F_0(x)$ when $n = 1$ and $F_0(x) \equiv 0$. To be specific, we will only consider functions defined on the segment $[-1, +1]$.

Example 1. Let $f(x)$ be a function of class k on $[-1, +1]$, δ some positive number. Then the function $\mu f(x)$ for any sufficiently small (in magnitude) μ is δ-close to 0 to rank k. If $f(x)$ is an analytical function, then for any natural number r and sufficiently small μ, the function $\mu f(x)$ is δ-close to rank r to zero (i.e., to the function $F(x) \equiv 0$).

Example 2. Consider the function

$$F_\mu(x) = \mu \sin \frac{x}{\mu^2}$$

($\mu > 0$). For any given $\delta > 0$, an appropriate choice of a sufficiently small μ

* If $\bar{G}_1$ is closed and M_0 is a boundary point, partial derivatives (of any order and type) do not necessarily exist at M_0. In this case, a partial derivative at M_0 is defined as the limit value of the corresponding partial derivative at an inner point M when M tends to M_0 (see /11/, Vol. I, Sec. 258, p. 589). An analytical function in a closed region $\bar{G}_1$ is evidently defined in some larger open domain.

will clearly make this function δ-close to zero, but only to rank 0. Indeed,

$$F'_\mu(x) = \frac{1}{\mu}\cos\frac{x}{\mu^2}$$

and as μ decreases, the upper bound of $F'_\mu(x)$ increases to infinity. Therefore, if δ is sufficiently small, we can never choose a μ that will ensure δ-closeness of $F_\mu(x)$ to zero to rank higher than 0. It is also clear that, by an appropriate choice of a sufficiently small μ, the function

$$\Phi_\mu(x) = \mu^{2m+1}\sin\frac{x}{\mu^2}$$

can be made arbitrarily close to zero to rank m, but never to rank greater than m.

We will now give without proof two theorems which are repeatedly used in what follows. One is Weierstrass's classical theorem on polynomial approximation to functions (see /23/, Sec. 109, Theorem 1), reformulated in terms of our new concept of δ-closeness to rank r, and the other readily follows from the former.

Theorem 1. Let $F(x_1, x_2, \ldots, x_n)$ *be a function of class* k *defined in a closed bounded region* $\bar{G}$ *of the space* E_n. *For any* $\varepsilon > 0$ *and* $r \leqslant k$ *there exists a polynomial* $\Phi(x_1, x_2, \ldots, x_n)$ *which is* ε*-close to rank* r *to the function* $F(x_1, x_2, \ldots, x_n)$ *in* $\bar{G}$.

Theorem 2. Let $M_0(x_1^0, x_2^0, \ldots, x_n^0)$ *be a point in a closed bounded region* $\bar{G}$, *and* $F(x_1, x_2, \ldots, x_n) = F(M)$ *a function of class* k *defined in that region. For any* $\varepsilon > 0$ *and* $r \leqslant k$ *there exists a polynomial* $\Phi(x_1, x_2, \ldots, x_n) = \Phi(M)$ *which is* ε*-close to rank* r *to the function* $F(M)$ *in* $\bar{G}$ *such that*

$$\Phi(M_0) = F(M_0),$$
$$\Phi^{(l)}_{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}(M_0) = F^{(l)}_{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}(M_0),$$

where $l = 1, 2, \ldots, r$, *all* α_i *are non-negative whole numbers, and* $\alpha_1 + \alpha_2 + \ldots + \alpha_n = l$.

2. The theorem of a small increment of implicit functions

We now proceed to prove a theorem which can be called the theorem of a small increment of implicit functions. Let $F(x, y, z)$ be a function of class k defined in the parallelepiped Δ

$$x_1 \leqslant x \leqslant x_2, \quad y_1 \leqslant y \leqslant y_2, \quad z_1 \leqslant z \leqslant z_2 \qquad (\Delta)$$

of the three-dimensional space E_3, such that at any point of Δ

$$F'_z(x, y, z) \neq 0. \qquad (1)$$

Let further

$$z = \varphi(x, y)$$

be a function defined everywhere in the rectangle R

$$x_1 \leqslant x \leqslant x_2, \quad y_1 \leqslant y \leqslant y_2 \qquad (R)$$

in the plane (x, y), such that inside this rectangle

$$z_1 < \varphi(x, y) < z_2 \tag{2}$$

and

$$F(x, y, \varphi(x, y)) \equiv 0. \tag{3}$$

From (1), (2), and (3) it follows that $\varphi(x, y)$ is the unique solution of the equation

$$F(x, y, z) = 0 \tag{4}$$

in Δ, and the surface

$$z = \varphi(x, y)$$

has no common points either with the top or the bottom of Δ. Moreover, in virtue of the theorem of implicit functions, $\varphi(x, y)$ is a function of class k.

Theorem 3. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any function $\tilde{F}(x, y, z)$ defined in Δ which is δ-close to rank $r (1 \leqslant r \leqslant k)$ to $F(x, y, z)$ the equation*

$$\tilde{F}(x, y, z) = 0 \tag{5}$$

has a solution

$$z = \tilde{\varphi}(x, y),$$

defined in the rectangle R, such that (a) $\tilde{\varphi}(x, y)$ is the only solution of equation (5) in Δ; (b) $\tilde{\varphi}(x, y)$ is a function of class $\geqslant r$ and is ε-close to the function $\varphi(x, y)$ to rank r.

Proof. According to our assumptions, inequality (2) is satisfied at any point of the rectangle R:

$$z_1 < \varphi(x, y) < z_2.$$

Therefore, if $\varepsilon_1 > 0$ is sufficiently small, we have

$$z_1 < \varphi(x, y) \pm \varepsilon_1 < z_2. \tag{6}$$

We choose $\varepsilon_1 > 0$ which satisfies condition (6), such that $\varepsilon_1 < \varepsilon$. By (1) in Δ

$$F'_z(x, y, z) \neq 0.$$

To fix ideas, let in Δ

$$F'_z(x, y, z) > 0. \tag{7}$$

From (3), (6), and (7) it follows that in R

$$F(x, y, \varphi(x, y) - \varepsilon_1) < 0, \quad F(x, y, \varphi(x, y) + \varepsilon_1) > 0. \tag{8}$$

Now since F and φ are continuous, we conclude from (8) that at any point in R

$$F(x, y, \varphi(x, y) - \varepsilon_1) < -c, \quad F(x, y, \varphi(x, y) + \varepsilon_1) > c, \tag{9}$$

where c is a positive number. Moreover, from inequality (7) we conclude that in Δ

$$F'_z(x, y, z) > m, \tag{10}$$

where m is a positive number.

Let δ be such that

$$\delta > 0, \quad \delta \leqslant \frac{m}{2}, \quad \delta \leqslant \frac{c}{2}. \tag{11}$$

Consider any function $\tilde{F}(x, y, z)$ defined in Δ which is δ-close to rank r to the function $F(x, y, z)$ in Δ. From the definition of δ-closeness and from relations (9), (10), and (11) it follows that in Δ

$$\tilde{F}'_z(x, y, z) > \frac{m}{2} \tag{12}$$

and that

$$\tilde{F}(x, y, \varphi(x, y) - \varepsilon_1) < -\frac{c}{2} < 0, \quad \tilde{F}(x, y, \varphi(x, y) + \varepsilon_1) > \frac{c}{2} > 0. \tag{13}$$

Then clearly the equation

$$\tilde{F}(x, y, z) = 0 \tag{5}$$

has a unique solution $z = \tilde{\varphi}(x, y)$ in Δ, which is defined in R and which everywhere in R satisfies the inequality

$$|\tilde{\varphi}(x, y) - \varphi(x, y)| < \varepsilon_1 < \varepsilon. \tag{14}$$

By the theorem of implicit functions we now conclude that the solution $z = \tilde{\varphi}(x, y)$ of equation (5) is a function of class $\geqslant r$.

It remains to show that $\delta > 0$ can be chosen so that condition (b) of the theorem is satisfied. To this end we note that the partial derivatives of $\varphi(x, y)$ are obtained successively from the set of equations

$$\begin{gathered} F'_x + F'_z\varphi'_x = 0, \\ F'_y + F'_z\varphi'_y = 0, \\ F''_{xx} + 2F''_{xz}\varphi'_x + F''_{zz}(\varphi'_x)^2 + F'_z\varphi''_{xx} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{gathered} \tag{15}$$

and the partial derivatives of $\tilde{\varphi}(x, y)$ are calculated from analogous equations:

$$\begin{gathered} \tilde{F}'_x + \tilde{F}'_z\tilde{\varphi}'_x = 0, \\ \tilde{F}'_y + \tilde{F}'_z\tilde{\varphi}'_y = 0, \\ \tilde{F}''_{xx} + 2\tilde{F}''_{xz}\tilde{\varphi}'_x + \tilde{F}''_{zz}(\tilde{\varphi}'_x)^2 + \tilde{F}'_z\tilde{\varphi}''_{xx} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{gathered} \tag{16}$$

Since $F'_z(x, y, z) \neq 0$ nowhere in Δ, the partial derivatives of $\varphi(x, y)$ to order r inclusive are continuous functions of the arguments $F'_x, F'_y, F'_z, F''_{xx}, F''_{xy}, \ldots, F^{(r)}_{z^r}$, and the range of these arguments can be regarded as a closed region. Hence it clearly follows that if $\delta > 0$ is sufficiently small, e.g., $\delta < \delta_1$, and the function $\tilde{F}$ is δ-close to rank r to F, the function $\tilde{\varphi}(x, y)$ is ε-close to rank r to $\varphi(x, y)$. Thus any positive number δ smaller than $\frac{m}{2}$, $\frac{c}{2}$, and δ_1 satisfies the theorem. This completes the proof.

The previous theorem is naturally generalized to any number of functions and equations. We will formulate the corresponding theorem for two functions and two equations.

Let the functions

$$F_1(x, y, u, v) \text{ and } F_2(x, y, u, v)$$

of class $k \geqslant 1$ be defined in some parallelepiped Π

$$x_1 \leqslant x \leqslant x_2, \quad y_1 \leqslant y \leqslant y_2, \quad u_1 \leqslant u \leqslant u_2, \quad v_1 \leqslant v \leqslant v_2 \tag{Π}$$

in E_4. Moreover, suppose that

1) $\frac{\partial F_1}{\partial v}$ and $I = \frac{D(F_1, F_2)}{D(u, v)} = \begin{vmatrix} F'_{1u} & F'_{1v} \\ F'_{2u} & F'_{2v} \end{vmatrix}$ do not vanish in Π;

2) the equation

$$F_1(x, y, u, v) = 0$$

has in Π the solution

$$v = \theta(x, y, u),$$

defined at any point $x, y, u, x_1 \leqslant x \leqslant x_2, y_1 \leqslant y \leqslant y_2, u_1 \leqslant u \leqslant u_2$, such that $v_1 < \theta(x, y, u) < v_2$. Since $\frac{\partial F_1}{\partial v} \neq 0$, this solution is unique;

3) the equation

$$F_2(x, y, u, \theta(x, y, u)) = 0$$

has a solution $u = \varphi(x, y)$ in the rectangle R,

$$x_1 \leqslant x \leqslant x_2, \quad y_1 \leqslant y \leqslant y_2, \tag{R}$$

such that $u_1 < \varphi(x, y) < u_2$.

It follows from condition 1 that the derivative $\frac{\partial F_2(x, y, u, \theta(x, y, u))}{\partial u}$ does not vanish in the relevant region, and $\varphi(x, y)$ is the unique solution of the equation $F_2(x, y, u, \theta(x, y, u)) = 0$. Therefore the functions

$$u = \varphi(x, y) \text{ and } v = \theta(x, y, \varphi(x, y)) = \psi(x, y)$$

constitute the unique solution of the system

$$F_1(x, y, u, v) = 0, \quad F_2(x, y, u, v) = 0$$

in Π, which is defined for all $(x, y) \in R$. By the theorem of implicit functions, φ and ψ are both functions of class k.

Theorem 4. Under the above assumptions, for any $\varepsilon > 0$ *and* r, $1 \leqslant r \leqslant k$, *there exists* $\delta > 0$ *such that if the functions*

$$\tilde{F}_1(x, y, u, v) \text{ and } \tilde{F}_2(x, y, u, v)$$

are defined in Π *and are* δ*-close to rank* r *to the functions* F_1 *and* F_2, *respectively, the equations*

$$\tilde{F}_1(x, y, u, v) = 0, \quad \tilde{F}_2(x, y, u, v) = 0$$

have a unique solution in Π,

$$u=\tilde{\varphi}(x,\ y),\quad v=\tilde{\psi}(x,\ y),$$

which is defined for all $(x, y)\in R$, *and the functions* $\tilde{\varphi}$ *and* $\tilde{\psi}$ *are* δ*-close to rank* r *to the respective functions* φ *and* ψ.

Theorem 4 is proved by applying Theorem 3 first to the function $F_1(x, y, u, v)$* and then to the function $F_2(x, y, u, \theta(x, y, u))$, where $\theta(x, y, u)$ is the solution of the equation $F_1(x, y, u, v)=0$ for v.

Remark. Theorem 4 clearly remains valid if the functions F_1 and F_2 are independent of x and y, i.e., if we are concerned with equations of the form $F_1(u, v)=0$ and $F_2(u, v)=0$. The wording of the theorem can be changed without difficulty to conform to these new conditions. This also applies to Theorem 3.

3. Root multiplicity of a function of a single variable

The concept of root multiplicity is generally applied to the roots of analytical functions (in particular, polynomials) in connection with the calculation of derivatives and factorization of polynomials. In this subsection we will advance a general definition of root multiplicity for a function of a single variable in a form that will readily link up at a later stage with a number of other, more complex concepts in the theory of dynamic systems. For analytical functions, our definition will naturally coincide with the usual definition of multiplicity.

Let

$$y=F_0(x)$$

be a function defined on some segment $[x_1, x_2]$, where it is a function of class $k\geqslant 1$ or an analytical function, and let x_0 be a root of the equation

$$F_0(x)=0,$$

in $[x_1, x_2]$. For simplicity let $x_0 = 0$ (this can always be accomplished by changing over to a new variable $\bar{x}=x-x_0$) and suppose that $F_0(x)$ is defined for $|x|\leqslant a$, where a is some positive number. Let r be a natural number

Definition 2. The root 0 *of the equation*

$$F_0(x)=0$$

is called a root of multiplicity r *(or an* r*-tuple root) of this equation and also a root of multiplicity* r *of the function* $F_0(x)$, *if* $F_0(x)$ *is a function of class* $k\geqslant r$ *and the following conditions are satisfied:*

(a) there exist $\varepsilon_0>0$, $\delta_0>0$ *such that any equation* $F(x)=0$, *where* $F(x)$ *is a function of class* r *which is* δ_0*-close to rank* r *to the function* $F_0(x)$, *has at most* r *roots for* $|x|<\varepsilon_0$;

(b) for any positive $\varepsilon<\varepsilon_0$ *and* δ *there exists a function* $F(x)$, δ*-close to rank* r *to the function* $F_0(x)$ *such that the equation* $F(x)=0$ *has precisely* r *roots for* $|x|<\varepsilon$.

* Theorem 3 applies to functions of three variables. Nevertheless, it can be formulated and proved without modifications for a function of any number of variables.

A root of multiplicity 1 is called a simple or a structurally stable root of the equation.

A root of multiplicity r at the same time may be a root of a lower or a higher multiplicity.

Lemma 1. Let $F(x)$ be a function defined for $|x| < a$ such that

$$F(x) = x^n \Phi(x),$$

where n is a natural number, $\Phi(x)$ is a continuous function, and $\Phi(0) \neq 0$. Then for any $\varepsilon > 0$ and $\delta > 0$ there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that

$$|\alpha_i| < \delta \quad (i = 1, 2, \ldots, n-1),$$

and the function

$$\widetilde{F}(x) = \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{n-1} x^{n-1} + x^n \Phi(x)$$

has at least n different roots for $|x| < \varepsilon$.

Proof. By assumption, $\Phi(0) \neq 0$. Let $\Phi(0) > 0$, and choose $\varepsilon > 0$, $\delta > 0$. Since $\Phi(x)$ is continuous, there exists $\eta_1 > 0$, $\eta_1 < \varepsilon$ such that for all positive x, $x < \eta_1$,

$$F(x) = x^n \Phi(x) > 0.$$

We choose one of these x, say x_1, and keep it fixed. Then $0 < x_1 < \eta_1$ and $F(x_1) = x_1^n \Phi(x_1) > 0$. If α_{n-1} is of sufficiently small magnitude, the function $F_1(x) = \alpha_{n-1} x^{n-1} + x^n \Phi(x)$ at the point x_1 has the same sign as the function $F(x)$. The number α_{n-1} therefore can be chosen so that

$$|\alpha_{n-1}| < \delta, \ \alpha_{n-1} < 0 \quad \text{and} \quad F_1(x_1) = \alpha_{n-1} x_1^{n-1} + x_1^n \Phi(x_1) > 0.$$

Now consider the function

$$F_1(x) = x^{n-1} \ (\alpha_{n-1} + x\Phi(x)).$$

For all sufficiently small x, the sign of $F_1(x)$ coincides with the sign of α_{n-1}, i.e., it is negative. Thus, there exists a number $\eta_2 < x_1$ such that for all x, $0 < x < \eta_2$, $F_1(x) < 0$. We choose one of these x, say x_2, and keep it fixed. Then

$$0 < x_2 < \eta_2 < x_1 < \eta_1 < \varepsilon$$

and

$$F_1(x_2) = \alpha_{n-1} x_2^{n-1} + x_2^n \Phi(x_2) < 0.$$

As the next step, consider the function

$$F_2(x_2) = \alpha_{n-2} x^{n-2} + F_1(x) = x^{n-2} (\alpha_{n-2} + \alpha_{n-1} x + x^2 \Phi(x)).$$

The number α_{n-2} is chosen so that

$$\alpha_{n-2} > 0, \quad |\alpha_{n-2}| < \delta, \quad F_2(x_1) > 0, \ F_2(x_2) < 0.$$

For small positive x, $F_2(x) > 0$, and we can choose a number $\eta_3 < x_2$ and x_3, $0 < x_3 < \eta_3$ such that $F_2(x_3) > 0$. Continuing along the same lines, we obtain a function

$$\tilde{F}(x) = F_{n-1}(x) = \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{n-1} x^{n-1} + x^n \Phi(x)$$

and a set of numbers $x_1, x_2, \ldots, x_{n-1}, x_n$, such that

$$|\alpha_i| < \delta, \quad i = 1, 2, \ldots, n-1,$$
$$0 < x_n < x_{n-1} < \ldots < x_2 < x_1 < \varepsilon$$

and

$$\tilde{F}(x_1) > 0, \ \tilde{F}(x_2) < 0, \ \ldots, \ \tilde{F}(x_n) \left\{ \begin{array}{l} > 0 \text{ for odd } n, \\ < 0 \text{ for even } n. \end{array} \right.$$

Now, each of the intervals (x_{i+1}, x_i), $i = 1, 2, \ldots, n-1$, contains at least one root ξ_i of the function $\tilde{F}(x)$, and

$$0 < \xi_{n-1} < \xi_{n-2} < \ldots < \xi_3 < \xi_2 < \xi_1 < \varepsilon.$$

But $\tilde{F}(0) = 0$. Thus, for $|x| < \varepsilon$, $\tilde{F}(x)$ has at least n different roots. This completes the proof of the lemma.

Remark 1. Lemma 1 can be generalized as follows. Let $\Phi(x; \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ be a continuous function of all its arguments for $|x| \leqslant a$, $|\alpha_i| \leqslant \tau$ $(\tau > 0$, $i = 1, 2, \ldots, n-1)$, and let $\Phi(0; 0, 0, \ldots, 0) \neq 0$. The proposition of Lemma 1 is then also true, i.e., we can choose numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that their magnitudes are arbitrarily small and in any arbitrarily small neighborhood of the point $x = 0$ the function

$$\tilde{F}(x) = \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{n-1} x^{n-1} + x^n \Phi(x; \alpha_1, \ldots, \alpha_{n-1})$$

has at least n different roots.

An even more general proposition is the following: let c_{ik} be any real numbers, and the function $\Phi(x; \alpha_1, \ldots, \alpha_{n-1})$ as before. The above proposition is then also true for the function

$$\tilde{F}(x) = \alpha_1 x + (\alpha_2 + c_{21}\alpha_1) x^2 + (\alpha_3 + c_{32}\alpha_2 + c_{31}\alpha_1) x^3 + \ldots$$
$$\ldots + (\alpha_{n-1} + c_{n-1\, n-2}\alpha_{n-2} + \ldots + c_{n-1\, 1}\alpha_1) x^{n-1} + x^n \Phi(x; \alpha_1, \ldots, \alpha_{n-1}).$$

Both these general propositions are proved along the same lines as the lemma.

Remark 2. Sometimes an additional requirement is introduced into the conditions of Lemma 1, namely that some (fixed) coefficients α_i are zero. The function $\tilde{F}(x)$ in this case has the form

$$\tilde{F}(x) = \alpha_{k_1} x^{k_1} + \alpha_{k_2} x^{k_2} + \ldots + \alpha_{k_{s-1}} x^{k_{s-1}} + x^n \Phi(x),$$

where $1 \leqslant k_1 < k_2 < \ldots < k_{s-1} \leqslant n-1$, and $s < n$. In this case we can choose (sufficiently small) numbers α_{k_l} such that in any arbitrarily small neighborhood of the point $x = 0$ the function $\tilde{F}(x)$ has at least s different roots. This proposition is proved like the lemma. A similar remark applies to functions considered in Remark 1.

Theorem 5. The number $x = 0$ is a root of multiplicity r of the function $F_0(x)$ if and only if

$$F_0(0)=0, \quad F_0'(0)=0, \ \ldots, \ F_0^{(r-1)}(0)=0, \quad F_0^{(r)}(0)\neq 0 \tag{17}$$

($F_0(x)$ is naturally assumed to be a function of class $k \geqslant r$).

Proof. Let us first establish that conditions (17) are sufficient. Suppose that these conditions are satisfied. Let $|F_0^{(r)}(0)| = m$ $(m > 0)$. We choose $\varepsilon_0 > 0$ sufficiently small so that for all $|x| < \varepsilon_0$, we have

$$|F_0^{(r)}(x)| > \frac{m}{2}.$$

For δ_0 we take a positive number smaller than $\frac{m}{2}$ (e.g., $\delta_0 = \frac{m}{4}$). Then for any function $F(x)$ defined on the same segment as $F_0(x)$ and δ_0-close to rank r to $F_0(x)$, we clearly have for $|x| < \varepsilon_0$ the inequality

$$|F^{(r)}(x)| > \frac{m}{2} - \frac{m}{4} = \frac{m}{4} > 0. \tag{18}$$

Now suppose that some function $F(x)$, which is δ_0-close to rank r to $F_0(x)$, has at least $r+1$ roots in the interval $|x| < \varepsilon_0$. Then, by Rolle's theorem, the function $F'(x)$ has at least r roots in this interval, $F''(x)$ has at least $r-1$ roots, etc., and finally $F^{(r)}(x)$ has no roots altogether, at variance with inequality (18). Thus, if condition (17) is satisfied, any function $F(x)$, which is δ_0-close to rank r to $F_0(x)$, cannot have more than r roots for $|x| < \varepsilon_0$, i.e., condition (a) of Definition 2 holds true.

Let us now prove that condition (b) of Definition 2 is also satisfied. The function $F_0(x)$ can be written in the form

$$F_0(x) = x^r \Phi(x), \tag{19}$$

where $\Phi(x)$ is a continuous function and $\Phi(0) \neq 0$. Indeed, by Taylor's theorem and conditions (17), we have in the neighborhood of $x = 0$

$$F_0(x) = x^r \frac{F_0^{(r)}(\theta x)}{r!} \qquad (0 < \theta < 1).$$

Setting

$$\Phi(x) = \frac{F_0(x)}{x^r} \quad \text{for} \quad x \neq 0 \text{ and } \Phi(0) = \lim_{x \to 0} \Phi(x) = \frac{F_0^{(r)}(0)}{r!},$$

we conclude that representation (19) is applicable at any point of the segment $|x| \leqslant a$, $\Phi(x)$ is continuous in this segment, and $\Phi(0) \neq 0$.

From (19) and Lemma 1 it follows that for any $\varepsilon > 0$ and $\delta > 0$ there exists a function $\tilde{F}(x)$ of the form

$$\tilde{F}(x) = \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{r-1} x^{r-1} + x^r \Phi(x),$$

which is δ-close to rank r to $F(x)$ and has at least r different roots for $|x| < \varepsilon$. This means that condition (b) of Definition 2 also holds true, and the sufficiency of (17) has been fully established.

It now remains to prove that conditions (17) are necessary. Suppose that these conditions are not satisfied, i.e., either

1) there exists r_1, $1 \leqslant r_1 < r$, such that

$$F_0(0) = F_0'(0) = \ldots = F_0^{(r_1-1)}(0) = 0, \quad F_0^{(r_1)}(0) \neq 0,$$

or

2) $$F_0(0) = F_0'(0) = \ldots = F_0^{(r)}(0) = 0. \tag{20}$$

We will show that in either case $x = 0$ is not a root of multiplicity r. This will establish the necessity of conditions (17).

In case 1, the proposition is self-evident: because of sufficiency $x = 0$ is a root of multiplicity $r_1 < r$ and by definition it cannot be a root of multiplicity r.

Consider case 2. Let ε and δ be any positive numbers. By Theorem 2, there exists a polynomial $P(x)$ which is $\delta/2$-close to rank r to the function $F_0(x)$ and such that

$$P(0) = P'(0) = \ldots = P^{(r)}(0) = 0$$

(if the function $F_0(x)$ is a polynomial, $F_0(x)$ itself can be chosen as $P(x)$). $P(x)$ has the form $x^{r+1}\Phi(x)$, where $\Phi(x)$ is some polynomial. We may take $\Phi(0) \neq 0$ (otherwise, we may take for $P(x)$ the function $x^{r+1}(\Phi(x) + \gamma)$, where γ is a sufficiently small number, $\gamma \neq 0$). By Lemma 1 there exists a polynomial $\widetilde{P}(x)$ which is $\delta/2$-close to rank r to $P(x)$, and therefore δ-close to rank r to $F_0(x)$, and which has at least $r+1$ roots for $|x| < \varepsilon$. Hence, $x = 0$ is not a root of multiplicty r for $F_0(x)$. This completes the proof.

It follows from Theorem 5 that for analytical functions, and in particular for polynomials, root multiplicity in the sense of Definition 2 coincides with the normal concept of multiplicity.

Remark. Let $x = 0$ be a simple root of the function $F_0(x)$, i.e., $r = 1$, $F_0(0) = 0$, $F_0'(0) \neq 0$. Then there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any function $F(x)$ which is δ_0-close to $F_0(x)$ has precisely one root for $|x| < \varepsilon_0$, which is moreover a simple root. Furthermore, for any $\varepsilon < \varepsilon_0$, $\varepsilon > 0$, we can find $\delta > 0$ such that any function $F(x)$, δ-close to $F_0(x)$, has precisely one (simple) root for $|x| < \varepsilon$.

Indeed, for ε_0 we may take any number such that the derivative $F_0'(x)$ does not vanish on the segment $|x| \leqslant \varepsilon_0$. Then on this segment $|F_0'(x)| \geqslant m > 0$ and for δ_0 we may take, say, any positive number which satisfies the inequality

$$\delta_0 < \min\left\{\frac{m}{2},\ |F_0(-\varepsilon_0)|,\ |F_0(\varepsilon_0)|\right\}.$$

For $\varepsilon < \varepsilon_0$, δ is found by the same method.

We have shown that if $F_0(x)$ is of class r and condition (20) is satisfied, there exists a function $F(x)$ arbitrarily close to rank r to $F_0(x)$ which has at least $r+1$ roots in any small neighborhood of the point 0. This proposition can be strengthened by showing that under the same conditions there exists a function $F(x)$ arbitrarily close to rank r to $F_0(x)$ which has at least l roots

in any small neighborhood of 0, where l is a natural number. This proposition is formulated and proved in the following lemma.

Lemma 2. If

$$F_0(0)=F_0'(0)=\ldots=F_0^{(r)}(0)=0, \tag{20}$$

then for any positive ε ***and*** δ ***and any natural number*** l, ***there exists a function*** $F(x)$ ***which is*** δ***-close to rank*** r ***to the function*** $F_0(x)$ ***and has at least*** l ***roots for*** $|x|<\varepsilon$ (we recall that the entire treatment is confined to some segment $|x|<a$).

Proof. The function $F(x)$ whose existence is to be proved can be constructed by several different methods. We will describe here one of the possible constructions. Let $\varepsilon>0$, $\delta>0$, and l be given.

1) We choose $\varepsilon_1<\varepsilon$ so small that the function $F_0(x)$ is $\delta/2$-close to rank r to zero (i.e., to the zero function) for $0\leqslant x\leqslant\varepsilon_1$, and the polynomial

$$F_0(\varepsilon_1)+\frac{F_0'(\varepsilon_1)}{1}(x-\varepsilon_1)+\ldots+\frac{F_0^{(r)}(\varepsilon_1)}{r!}(x-\varepsilon_1)^r$$

is δ-close to zero to rank r for $0\leqslant x\leqslant a$. For ε_1 we may naturally take any sufficiently small positive number; this follows from conditions (20) and from the continuity of the function $F_0(x)$ and its derivatives.

2) We choose a large natural number N, so that each of the numbers $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{l}{N}$ is less than ε_1.

3) We choose $\mu>0$ small enough for the function

$$\varphi(x)=\mu x^{r+1}(x-\varepsilon_1)^{r+1}\sin\pi Nx$$

to be $\delta/2$-close to zero to rank r for $0\leqslant x\leqslant\varepsilon_1$. Note that the numbers $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{l}{N}$ are roots of the function $\varphi(x)$ which lie in the interval $0<x<\varepsilon_1$, and that the functions $\varphi(x)$, $\varphi'(x), \ldots, \varphi^{(r)}(x)$ vanish at the points $x=0$ and $x=\varepsilon_1$.

We now define the function $F(x)$ as follows:

$$F(x)=F_0(x) \text{ for } -a\leqslant x\leqslant 0;$$
$$F(x)=\varphi(x) \text{ for } 0\leqslant x\leqslant\varepsilon_1;$$
$$F(x)=F_0(x)-\left[F_0(\varepsilon_1)+\frac{F'(\varepsilon_1)}{1}(x-\varepsilon_1)+\ldots+\frac{F^{(r)}(\varepsilon_1)}{r!}(x-\varepsilon_1)^r\right] \text{ for } \varepsilon_1\leqslant x\leqslant a.$$

The function $F(x)$ constructed in this way is readily seen to be a function of class r which is δ-close to rank r to the function $F_0(x)$ and has at least l roots in the interval $(0, \varepsilon)$. This completes the proof of the lemma.

The function $F(x)$ constructed in the above proof is of class r, but in general its class is not higher than r. It is readily seen, however, that there exists a polynomial $P(x)$ which satisfies all the conditions of the lemma (i.e., a polynomial which is δ-close to rank r to $F_0(x)$ and has at least l roots in the ε-neighborhood of 0). Indeed, it follows from Theorem 5 that the roots $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{l}{N}$ are simple (structurally stable) roots of the function $F(x)$. But then, in virtue of the remark to Theorem 5, any polynomial $P(x)$ which is sufficiently close to $F(x)$ has at least l roots for $|x|<\varepsilon$, and thus satisfies the requirements of the lemma.

Note that Lemma 2 is also valid for $r = 0$. This means, in particular, that if the function $F_0(x)$ of class $k \geqslant 1$ has a simple root, $x = 0$ say, then for any $\varepsilon > 0$ and $\delta > 0$ there exists a function $F(x)$ (possibly a polynomial $P(x)$) which is δ-close to rank 0 to $F_0(x)$ and whose roots in the ε-neighborhood of the point 0 are more numerous than any chosen number. However, if closeness to rank 1 is required, $F(x)$ will have only a single root in the neighborhood of 0 for small δ (see remark to Theorem 5). This shows that the requirement of closeness to a certain rank is highly significant in the definition of multiple or simple roots.

Definition 3. A root $x = 0$ of the function $F_0(x)$ is said to be of infinite multiplicity or not of finite multiplicity if either

1) $F_0(x)$ is a function of class r but not of class $r+1$ $(r > 0)$ and

$$F_0(0) = F_0'(0) = \ldots = F_0^{(r)}(0),$$

or

2) $F_0(x)$ is differentiable to any order over the relevant segment, and all derivatives vanish at the point 0.

A root $x = 0$ of the function $F_0(x)$ is said to have a multiplicity higher than r (or to be a root of multiplicity $> r$, where r is a natural number) if it has a finite multiplicity $r' > r$ or if it has infinite multiplicity.

From Definitions 2 and 3 and Theorem 5 it follows that the root of each continuous function has a definite, finite or infinite, multiplicity. Each root of an analytical function which is not identically zero has a finite multiplicity.

4. Multiplicity of a root relative to a given class of functions

In our definition of a root of multiplicity r we assumed that all the functions ($F_0(x)$, $F(x)$, etc.) were functions of class r, without imposing any further restrictions. For certain purposes, however, it is worthwhile considering narrower classes of functions.

Let $\mathfrak{M}^{(r)}$ be the set of all functions of class r (where r is a non-negative integer) defined on the segment $|x| \leqslant a$. We will particularly concentrate on the following groups:

1) the set $\mathfrak{M}^{(k)}$ of all functions of class k;
2) the set $\mathfrak{M}_A$ of all analytical functions;
3) the set $\mathfrak{M}_P$ of all polynomials;
4) the set $\mathfrak{M}_n$ of all polynomials of degree $\leqslant n$.

Clearly, for any n and k,

$$\mathfrak{M}_n \subset \mathfrak{M}_P \subset \mathfrak{M}_A \subset \mathfrak{M}^{(k)}.$$

Moreover, if $n_1 < n_2$ and $k_1 < k_2$, we have

$$\mathfrak{M}_{n_1} \subset \mathfrak{M}_{n_2} \quad \text{and} \quad \mathfrak{M}^{(k_2)} \subset \mathfrak{M}^{(k_1)}.$$

We will now define root multiplicity relative to a given class $\mathfrak{M}$.

Definition 4. A root $x = 0$ of a function $F_0(x)$ of class $\mathfrak{M}$ is said to be a root of multiplicity r relative to the class $\mathfrak{M}$ if $\mathfrak{M} \subset \mathfrak{M}^{(r)}$ and the following conditions are satisfied:

(a) there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any function $F(x)$ of the class $\mathfrak{M}$ which is δ_0-close to rank r to $F_0(x)$ has at most r roots in the interval $|x| < \varepsilon_0$;

(b) for any positive $\varepsilon < \varepsilon_0$ and δ there exists a function $F(x)$ of the class $\mathfrak{M}$ which is δ-close to rank r to $F_0(x)$ and has exactly r roots in the interval $|x| < \varepsilon$.

If $\mathfrak{M}$ is one of the four classes above ($\mathfrak{M}^{(k)}$, $\mathfrak{M}_A$, $\mathfrak{M}_P$, and $\mathfrak{M}_n$), conditions (17) of Theorem 4,

$$F_0(0) = F_0'(0) = \ldots = F_0^{(r-1)}(0) = 0, \quad F_0^{(r)}(0) \neq 0,$$

are both necessary and sufficient for the root $x = 0$ of the function $F_0(x)$ to be a root of multiplicity r relative to the class $\mathfrak{M}$.

Indeed, it is readily seen that the proof of sufficiency in this case (i.e., for multiplicity relative to the class $\mathfrak{M}$) follows letter by letter the proof of Theorem 5.

The necessity of conditions (17) is also proved as in Theorem 5, except for the case $\mathfrak{M} = \mathfrak{M}_n$ and $r = n$. In this case, the equalities

$$F_0(0) = F_0'(0) = \ldots = F_0^{(r)} = 0$$

show that $F_0(x) \equiv 0$, which clearly contradicts condition (a) of Definition 4.

It follows from the above that if $F_0(x)$ is a function of class $\mathfrak{M}$, where $\mathfrak{M}$ is one of the four classes $\mathfrak{M}^{(k)}$, $\mathfrak{M}_A$, $\mathfrak{M}_P$, $\mathfrak{M}_n$, and $x = 0$ is a root of multiplicity r of this function in the sense of Definition 2 (i.e., relative to the class $\mathfrak{M}^{(r)}$), then $x = 0$ is a root of $F_0(x)$ of multiplicity r in the sense of Definition 4 (i.e., relative to the class $\mathfrak{M}$), and vice versa. Therefore, we do not have to consider multiplicity relative to any of the above particular classes of functions, and in all that follows root multiplicities are understood in the sense of Definition 2.

§2. THE MULTIPLICITY OF A COMMON POINT OF TWO CURVES

1. Definition of multiplicity

Let $F_1(x, y)$ and $F_2(x, y)$ be functions of class $k \geqslant 1$ defined in some closed region $\bar{G}$ of the plane (x, y).

Consider the set of equations

$$F_1(x, y) = 0, \quad F_2(x, y) = 0. \tag{1}$$

Let (C_1) and (C_2) be the curves described by these two equations, respectively. Let equations (1) have a simultaneous solution x_0, y_0 in $\bar{G}$, i.e.,

$$F_1(x_0, y_0) = 0, \quad F_2(x_0, y_0) = 0.$$

The point $M_0(x_0, y_0)$, which is a common point or a point of intersection of the curves (C_1) and (C_2), is invariably assumed to lie inside the region $\bar{G}$. Moreover, we may take $x_0 = y_0 = 0$. This assumption clearly does not detract from the generality of our argument.

Let r be a natural number not greater than the class of F_1 and F_2, i.e., $1 \leqslant r \leqslant k$ (if F_1 and F_2 are analytical functions, r is any natural number).

Definition 5. A common point $M_0(0, 0)$ of the curves (C_1) and (C_2) is said to be a common point of multiplicity r of these curves or a solution of multiplicity r of equations (1) *if the following conditions are satisfied:*

(a) There exist $\varepsilon_0 > 0$, $\delta_0 > 0$ such that any two curves $\bar{F}_1(x, y) = 0$ and $\bar{F}_2(x, y) = 0$, where $\bar{F}_1$ and $\bar{F}_2$ are functions of class r, δ_0-close to rank r to F_1 and F_2, respectively, have at most r common points in $U_{\varepsilon_0}(M_0)$.

(b) For any δ and $\varepsilon < \varepsilon_0$ ($\delta > 0$, $\varepsilon > 0$) there exist functions $\Phi_1(x, y)$ and $\Phi_2(x, y)$, δ-close to rank r to $F_1(x, y)$ and $F_2(x, y)$, such that the curves $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$ have exactly r common points in $U_\varepsilon(M_0)$.

An intersection point of multiplicity 1 is called a simple or structurally stable point of intersection of curves (C_1) and (C_2).

A common point M_0 of two curves (1) is said to be multiple or structurally unstable if it is not a simple intersection point. Note that a common point of two curves is not necessarily of finite multiplicity. Consider the following example.

Let $F_0(x)$ be a function of class r which is not a function of class $r+1$ and let $F_0(0) = F_0'(0) = \ldots = F_0^{(r)}(0)$.

$\bar{G}$ is chosen as some bounded closed region with $O(0, 0)$ as its interior point.

Consider two curves:

$$y = 0, \quad y - F_0(x) = 0. \tag{2}$$

$O(0, 0)$ is a common point of these curves. Let ε and δ be any two positive numbers, l some natural number, and $F(x)$ a function constructed as in Lemma 2 (§1. 3), i.e., a function which is δ-close to rank r to $F_0(x)$ and has at least l roots in the interval $|x| < \varepsilon$. The curves

$$y = 0, \quad y - F(x) = 0 \tag{3}$$

are then δ-close to rank r to the respective curves in (2) and have at least l common points in $U_\varepsilon(O)$. The intersection point $O(0, 0)$ of curves (2) thus does not have a finite multiplicity.

If a common point of two curves does not have a finite multiplicity, it is said to be an intersection point of infinite multiplicity. Finally, a common point $O(0, 0)$ of two curves is said to be of multiplicity higher than r either if it has finite multiplicity $r' > r$ or if it has infinite multiplicity (see Definition 3, §1).

2. Condition of simplicity for an intersection point of two curves

We will now consider the necessary and sufficient condition for an intersection point of two curves to be a simple (structurally stable) intersection point, i.e., a common point of multiplicity 1.

Theorem 6. The intersection point $O(0, 0)$ *of two curves*

$$F_1(x, y) = 0 \qquad (C_1)$$

$$F_2(x, y) = 0 \qquad (C_2)$$

is of multiplicity 1, i.e., simple (structurally stable), if and only if

$$\Delta_0 = \begin{vmatrix} F'_{1x}(0, 0) & F'_{1y}(0, 0) \\ F'_{2x}(0, 0) & F'_{2y}(0, 0) \end{vmatrix} \neq 0. \qquad (4)$$

Proof. We will first show that $\Delta_0 \neq 0$ constitutes a necessary condition. Suppose $\Delta_0 = 0$. Let ε and δ be some positive numbers. Let further $P_1(x, y)$ and $P_2(x, y)$ be polynomials which satisfy the following conditions:

1) $P_1(x, y)$ and $P_2(x, y)$ are $\delta/2$-close to rank 1 to the functions $F_1(x, y)$ and $F_2(x, y)$, respectively;

2) $P_i(0, 0) = 0$, $P'_{ix}(0, 0) = F'_{ix}(0, 0)$, $P'_{iy}(0, 0) = F'_{iy}(0, 0)$ $(i = 1, 2)$.

These polynomials exist by Theorem 2, §1. Two cases are possible: (a) at least one of the numbers $P'_{1x}(0, 0)$, $P'_{1y}(0, 0)$, $P'_{2x}(0, 0)$, $P'_{2y}(0, 0)$ does not vanish; (b) all these numbers are zero.

First let us consider case (a). The polynomials P_1 and P_2 in this case have the form

$$P_1(x, y) = A_1x + B_1y + C_1x^2 + \ldots.$$
$$P_2(x, y) = A_2x + B_2y + C_2x^2 + \ldots.$$

where $\Delta_0 = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$, but at least one of the coefficients A_1, B_1, A_2, B_2 does not vanish. For example, let $B_1 \neq 0$. Then either $B_2 \neq 0$ or $A_2 = B_2 = 0$. Consider the polynomials

$$\bar{P}_1(x, y) = A_1x + B_1y + \alpha_1x + C_1x^2 + \ldots = P_1(x, y) + \alpha_1x,$$
$$\bar{P}_2(x, y) = A_2x + B_2y + \alpha_2x + C_2x^2 + \ldots = P_2(x, y) + \alpha_2x$$

and the point $M_1\left(x_1, -\frac{A_1}{B_1}x_1\right)$, where $x_1 \neq 0$. We choose the coefficients α_1 and α_2 so that the two curves

$$\bar{P}_1(x, y) = 0 \quad \text{and} \quad \bar{P}_2(x, y) = 0$$

pass through the point M_1. Clearly α_1 and α_2 should satisfy the equations

$$\alpha_1x_1 + C_1x_1^2 + \ldots = 0,$$
$$\alpha_2x_1 + C_2x_1^2 + \ldots = 0.$$

Dividing through by x_1 $(x_1 \neq 0)$, we get

$$\alpha_1 = -C_1x_1 - \ldots, \qquad \alpha_2 = -C_2x_1 - \ldots$$

Thus α_1 and α_2 are uniquely determinable and can be as small as desired for sufficiently small x_1. We choose x_1 small enough for the point M_1 to lie inside $U_\varepsilon(O)$ and for the polynomials $\bar{P}_1(x, y)$ and $P_2(x, y)$ to be $\frac{\delta}{2}$-close to $P_1(x, y)$ and $P_2(x, y)$, respectively (and hence, δ-close to the original functions

$F_1(x, y)$ and $F_2(x, y)$, respectively). Then the curves $\bar{P}_1(x, y) = 0$ and $\bar{P}_2(x, y) = 0$ have at least two common points in $U_\varepsilon(O)$, namely the point O and the point M_1. This clearly signifies that the intersection point O of the curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$ in case (a) is not a simple (structurally stable) intersection point of these curves.

In case (b), when

$$P'_{1x}(0, 0) = P'_{1y}(0, 0) = P'_{2x}(0, 0) = P'_{2y}(0, 0) = 0,$$

we reason along the same lines as in case (a), but the previous point M_1 is now chosen as $M_1(x_1, 0)$. The number x_1 is chosen so that the polynomials

$$\bar{P}_1(x, y) = \alpha_1 x + P_1(x, y) \text{ and } \bar{P}_2(x, y) = \alpha_2 x + P_2(x, y)$$

are $\frac{\delta}{2}$-close to the polynomials P_1 and P_2, respectively, and the curves $\bar{P}_1(x, y) = 0$ and $\bar{P}_2(x, y) = 0$ pass through the point M. Thus again $O(0, 0)$ is not a simple point of curves (C_1) and (C_2). We have proved that the condition $\Delta = 0$ is necessary. The sufficiency of this condition follows directly from Theorem 4, if we assume that the functions F_1, F_2 and $\tilde{F}_1$, $\tilde{F}_2$ of that theorem are dependent on x and y only (see remark to Theorem 4). This completes the proof.

R e m a r k 1. If the original functions $F_1(x, y)$ and $F_2(x, y)$ are polynomials, we may take them as the polynomials P_1 and P_2. Then the polynomials $\bar{P}_1(x, y)$ and $\bar{P}_2(x, y)$ constructed in our proof of the necessity of condition (4) are of the same degree as the polynomials F_1 and F_2 respectively.

R e m a r k 2. It follows from condition (4) that if $O(0, 0)$ is a simple intersection point of curves (C_1) and (C_2), the angle between these curves at the point O is other than zero, i.e., the curves are not tangent to each other at this point.

R e m a r k 3. A simple (structurally stable) intersection point $O(0, 0)$ of the curves (C_1) and (C_2) has the following property: there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if the functions $\Phi_1(x, y)$ and $\Phi_2(x, y)$ are δ_0-close to the functions $F_1(x, y)$ and $F_2(x, y)$, respectively, the curves

$$\Phi_1(x, y) = 0 \text{ and } \Phi_2(x, y) = 0$$

have exactly one intersection point in $U_{\varepsilon_0}(O)$ and this intersection point is also simple. Moreover, regardless of how small $\varepsilon < \varepsilon_0$ is, δ_0 can be made sufficiently small so that this intersection point falls inside $U_\varepsilon(O)$.

The validity of this remark follows directly from Theorem 4 (§1.2) and Theorem 6.

3. Condition of duplicity for an intersection point of two curves

We have established the necessary and sufficient condition for a root of a function to be of multiplicity r (Theorem 5). The derivation of the necessary and sufficient condition for an r-tuple intersection point of two curves in the general case (any natural r) is far from being so elementary,

and it is therefore not considered here. For $r = 1$ the corresponding condition is given by Theorem 6. We will now derive the condition for a double intersection point of two curves, or an intersection point of multiplicity 2, which is often used in what follows.

As before, we consider two curves

$$F_1(x, y) = 0, \quad F_2(x, y) = 0$$

with a common point $O(0, 0)$. F_1 and F_2 are now assumed to be functions of class 2 in $\bar{G}$.

Theorem 7. A common point $O(0, 0)$ of the two curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$ is a double intersection point if and only if the following conditions are satisfied:

(a)

$$\Delta_0 = \begin{vmatrix} F'_{1x}(0, 0) & F'_{1y}(0, 0) \\ F'_{2x}(0, 0) & F'_{2y}(0, 0) \end{vmatrix} = 0;$$

(b) at least one of the elements in the determinant Δ_0 is other than zero;

(c) the number $x = 0$ is a double root of the function $F_2(x, \varphi(x))$, where $y = \varphi(x)$ is the solution of the equation $F_1(x, y) = 0$ for y in some sufficiently small rectangle $|x| \leqslant \alpha$, $|y| \leqslant \beta$ (this solution exists and is unique in virtue of condition (b) and the theorem of implicit functions; also $\varphi(0) = 0$).

If $F'_{1y}(0, 0) = 0$, but some other element of Δ_0 does not vanish, condition (c) should be appropriately reworded.

Proof. 1) Necessity. Condition (a), $\Delta_0 = 0$, is clearly necessary by Theorem 6. We will now show that condition (b) is necessary. Suppose that this condition is not satisfied, i.e.,

$$F'_{1x}(0, 0) = F'_{1y}(0, 0) = F'_{2x}(0, 0) = F'_{2y}(0, 0) = 0.$$

Let ε and δ be some positive numbers. By Theorem 2, there exist polynomials $P_1(x, y)$ and $P_2(x, y)$, $\frac{\delta}{2}$-close to rank 2 to the functions F_1 and F_2, respectively, such that

$$P_i(0, 0) = P'_{ix}(0, 0) = P'_{iy}(0, 0) = 0 \qquad (i = 1, 2)$$

(if F_1 and F_2 are polynomials, P_1 and P_2 are identified with F_1 and F_2). The polynomials P_1 and P_2 are written in the form

$$P_1(x, y) = A_1x^2 + 2B_1xy + C_1y^2 + \ldots,$$
$$P_2(x, y) = A_2x^2 + 2B_2xy + C_2y^2 + \ldots,$$

where the omitted terms are all of higher than second order.

The rest of the proof proceeds along the same lines as in Theorem 6. Consider the two polynomials

$$\tilde{P}_1(x, y) = \alpha_1x + \beta_1y + P_1(x, y),$$

$$\tilde{P}_2(x, y) = \alpha_2x + \beta_2y + P_2(x, y).$$

We choose α_1, β_1, α_2, β_2 so that the curves $\tilde{P}_1(x, y) = 0$ and $\tilde{P}_2(x, y) = 0$ pass through two fixed points $M_1(x_1, 0)$ and $N_1(0, y_1)$ ($x_1 \neq 0$, $y_1 \neq 0$). To this end, the following equalities should be satisfied:

$$\alpha_1 x_1 + P_1(x_1, 0) = 0, \quad \beta_1 y_1 + P_1(0, y_1) = 0$$

and

$$\alpha_2 x_1 + P_2(x_1, 0) = 0, \quad \beta_2 y_2 + P_2(0, y_1) = 0.$$

Dividing through by x_1 and y_1, respectively, we express α_i and β_i as polynomials in x_1 and y_1 without a free term. The numbers α_i and β_i therefore go to zero for $x_1 \to 0$ and $y_1 \to 0$, and we can choose x_1 and y_1 so small that the polynomials $\tilde{P}_1$ and $\tilde{P}_2$ are $\frac{\delta}{2}$-close to rank 2 to the polynomials P_1 and P_2 and therefore δ-close to rank 2 to the functions $F_1(x, y)$ and $F_2(x, y)$. If, moreover, $|x_1| < \varepsilon$, $|y_1| < \varepsilon$, the curves $\tilde{P}_1(x, y) = 0$ and $\tilde{P}_2(x, y) = 0$ have at least 3 common points in $U_\varepsilon(O)$: these are $O(0, 0)$, $M_1(x_1, 0)$, and $N_1(0, y_1)$. This clearly proves that the common point O of curves (C_1) and (C_2) is of multiplicity higher than 2. We have thus established that condition (b) is also necessary.

Let us now proceed with condition (c). Let $y = \varphi(x)$ be the unique solution of the equation

$$F_1(x, y) = 0$$

in some sufficiently small rectangle $|x| \leqslant \alpha$, $|y| \leqslant \beta$, such that $|\varphi(x)| < \beta$. Let

$$\theta(x) = F_2(x, \varphi(x)). \tag{5}$$

Since $\varphi(0) = 0$, we have $\theta(0) = 0$. Now, using the equality

$$\varphi'(0) = -\frac{F'_{1x}(0, 0)}{F'_{1y}(0, 0)},$$

we readily see that

$$\theta'(0) = -\frac{\Delta_0}{F'_{1y}(0, 0)},$$

i.e., $\theta'(0) = 0$ in virtue of condition (a).

Suppose that condition (c) is not satisfied, i.e., $x = 0$ is not a double root of the function $\theta(x)$. Since $\theta(0) = \theta'(0) = 0$, this assumption is equivalent to the equality $\theta''(0) = 0$. Note that the derivatives of the function $\varphi(x)$ at the point $x = 0$ are expressed in terms of the partial derivatives (up to corresponding orders, inclusive) of the function $F_1(x, y)$ at the point $O(0, 0)$ and the derivatives of the function $\theta(x)$ at the point O are expressed in terms of the partial derivatives of F_1 and F_2 at $O(0, 0)$.

Thus, let $\theta''(0, 0) = 0$. We choose some $\varepsilon > 0$ and $\delta > 0$. Let $\frac{\delta}{2}$ be a positive number smaller than δ_1 on which additional restrictions will now be imposed. Consider the polynomials $P_1(x, y)$ and $P_2(x, y)$, δ_1-close to rank 2 to the functions $F_1(x, y)$ and $F_2(x, y)$, respectively, such that their values and the values of their derivatives to second order inclusive at the

point $O(0, 0)$ coincide with the respective values of the functions $F_1(x, y)$ and $F_2(x, y)$ and their derivatives at that point (see Theorem 2). By Theorem 3 (the theorem of a small increment of implicit functions), if δ_1 is sufficiently small, there exists a function $y = \psi(x)$ which is the solution of the equation

$$P_1(x, y) = 0$$

for y in the rectangle $|x| \leqslant \alpha$, $|y| \leqslant \beta$. We choose δ_1 so small that the function $\psi(x)$ becomes sufficiently close to rank 2 to $\varphi(x)$ for $|x| \leqslant \alpha$, and the analog of $\theta(x)$, the function

$$\gamma(x) = P_2(x, \psi(x)),$$

becomes sufficiently close to rank 2 to the function

$$\theta(x) = F_2(x, \varphi(x)).$$

Clearly $\psi(0) = \gamma(0) = 0$. The derivatives $\psi'(0)$, $\psi''(0)$, $\gamma'(0)$ and $\gamma''(0)$ are expressed in terms of the partial derivatives of the polynomials P_1 and P_2 (to second order inclusive) at the point $O(0, 0)$. Seeing that the values of these derivatives coincide with the corresponding values of the derivatives of F_1 and F_2, we readily conclude that

$$\psi'(0) = \varphi'(0), \qquad \psi''(0) = \varphi''(0),$$
$$\gamma'(0) = \theta'(0) = 0, \qquad \gamma''(0) = \theta''(0) = 0.$$

$\gamma(x)$ is clearly an analytical function. If $\gamma(x) \equiv 0$, all the points $(x, \psi(x))$ for $|x| \leqslant \alpha$ are common points of the curves $P_1(x, y) = 0, P_2(x, y) = 0$, i.e., in any neighborhood of $O(0, 0)$ these curves have an infinite number of common points.

Now suppose that $\gamma(x)$ is not identically zero. Then it can be written in the form (since $\gamma(0) = \gamma'(0) = \gamma''(0) = 0$)

$$\gamma(x) = x^k \Phi(x),$$

where $k \geqslant 3$, and $\Phi(0) \neq 0$ (see §1, (19)).

Consider the polynomials

$$\widetilde{P}_1(x, y) = P_1(x, y) \quad \text{and} \quad \widetilde{P}_2(x, y) = \alpha_1 x + \alpha_2 x^2 + P_2(x, y).$$

Let

$$\widetilde{\gamma}(x) = \widetilde{P}_2(x, \psi(x)) = \alpha_1 x + \alpha_2 x^2 + x^k \Phi(x).$$

$\widetilde{\gamma}(x)$ is also an analytical function. By Lemma 1, §1, and Remark 2 to the lemma we know that for any $\varepsilon_1 > 0$ and $\delta_1 > 0$ we can choose two numbers $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ such that the polynomial $\widetilde{P}_2(x, y)$ is δ-close to rank 2 to the polynomial $P_2(x, y)$ and the equation

$$\widetilde{\gamma}(x) = 0,$$

apart from the root $x = 0$, has at least two more roots x_1 and x_2, $|x_1| < \varepsilon_1$, $|x_2| < \varepsilon_1$. This choice of α_1, α_2 ensures that the polynomials $\widetilde{P}_1$ and $\widetilde{P}_2$ are

δ-close to rank 2 to the functions F_1 and F_2, respectively, and the curves

$$\widetilde{P}_1(x, y) = 0, \quad \widetilde{P}_2(x, y) = 0$$

have at least three common points,

$$O(0, 0), \quad M_1(x_1, \psi(x_1)), \quad M_2(x_2, \psi(x_2)),$$

such that if $\varepsilon_1 > 0$ is sufficiently small, M_1 and M_2 lie in $U_\varepsilon(O)$. This means, however, that $O(0, 0)$ is not a double intersection point of the curves (C_1) and (C_2). The necessity of condition (c) is thus established.

Note that if F_1 and F_2 are polynomials, they can be used as P_1 and P_2. The polynomial $\widetilde{P}_2$ in this case is of the same degree as the polynomial F_2.

2) Sufficiency. Let conditions (a), (b), and (c) of Theorem 7 be satisfied. Since $F'_{1y}(0, 0) \neq 0$, by Theorem 3 for any $\delta_1 > 0$ we can find $\delta_0 > 0$ such that if the functions $\widetilde{F}_1(x, y)$ and $\widetilde{F}_2(x, y)$ are δ_0-close to rank 2 to the function F_1 and F_2 in $\overline{G}$, the following holds true;

1) the equation $\widetilde{F}_1(x, y) = 0$ has a unique solution $y = \widetilde{\varphi}(x)$ in the rectangle $|x| \leqslant \alpha$, $|y| \leqslant \beta$, and $\widetilde{\varphi}(x)$ is δ_1-close to rank 2 to $\varphi(x)$ for $|x| \leqslant \alpha$;

2) the function $\widetilde{\theta}(x) = \widetilde{F}_2(x, \widetilde{\varphi}(x))$ is δ-close to rank 2 to the function $\theta(x) = F_2(x, \varphi(x))$ for $|x| \leqslant \alpha$.

In virtue of condition (c), $x = 0$ is a double root of the function $\theta(x)$. Therefore, there exists $\varepsilon_1 > 0$ such that for sufficiently small δ_1 the equation

$$\widetilde{\theta}(x) = 0, \text{ i.e., } \widetilde{F}_2(x, \widetilde{\varphi}(x)) = 0$$

has at most two roots which are smaller than ε_1 in magnitude. Let ε_0 be a positive number smaller than ε_1 such that $U_{\varepsilon_0}(O)$ is entirely contained in the rectangle $|x| \leqslant \alpha$, $|y| \leqslant \beta$. Then, if δ_0 is sufficiently small and the functions $\widetilde{F}_1$ and $\widetilde{F}_2$ are δ_0-close to rank 2 to F_1 and F_2, respectively, the curves

$$\widetilde{F}_1(x, y) = 0 \quad \text{and} \quad \widetilde{F}_2(x, y) = 0$$

cannot have more than two common points in $U_{\varepsilon_0}(O)$, i.e., condition (a) of Definition 5 of the duplicity of point O is satisfied.

It now remains to show that condition (b) of Definition 5 is also satisfied, i.e., for any positive $\varepsilon < \varepsilon_0$ and δ there exist functions $\widetilde{F}_1$ and $\widetilde{F}_2$, δ-close to rank 2 to the functions F_1 and F_2, such that the curves

$$\widetilde{F}_1 = 0 \quad \text{and} \quad \widetilde{F}_2 = 0$$

have two common points in $U_\varepsilon(O)$.

Consider the functions

$$\widetilde{F}_1(x, y) \equiv F_1(x, y)$$

and

$$\widetilde{F}_2(x, y) = \alpha_1 x + F_2(x, y).$$

For these functions clearly $\widetilde{\varphi} \equiv \varphi(x)$ and

$$\widetilde{\theta}(x) = \widetilde{F}_2(x, \widetilde{\varphi}(x)) = \alpha_1 x + F_2(x, \varphi(x)) = \alpha_1 x + \theta(x).$$

Since $x=0$ is a double root of $\theta(x)$, this function can be written in the form

$$\theta(x)=x^2\Phi(x),$$

where $\Phi(x)$ is a continuous function and $\Phi(0)\neq 0$ (see § 1, (19)). Therefore

$$\tilde{\theta}(x)=\alpha_1 x+x^2\Phi(x).$$

By Lemma 1, §1, α_1 can be chosen so that both $x=0$ and x_1, $|x_1|<\varepsilon$, are roots of the function $\tilde{\theta}(x)$ and the point $M_1(x_1, \varphi(x_1))$ is inside $U_\varepsilon(O)$. This α_1, moreover, can be made so small that the function $\tilde{F}_2(x, y)$ will be δ-close to rank 2 to the function $F_2(x, y)$. Condition (b) of Definition 5 is thus satisfied, which completes the proof.

Remark. It is readily seen that condition (a) of Theorem 7 follows from conditions (b) and (c), and may therefore be omitted from the full statement of the theorem. We nevertheless preferred to include it in explicit form. Theorem 7 is a particular case of a more general proposition proved in Chapter VIII (§ 23.1, Theorem 33).

We conclude this section with a remark analogous to that in §4.1, where we discussed the multiplicity of a root relative to a given class of functions. Let $\mathfrak{M}$ be one of the following classes of the functions $\Phi(x, y)$ of two variables: the set $\mathfrak{M}^{(k)}$ of all functions of class $k>0$, the set $\mathfrak{M}_A$ of all analytical functions, the set $\mathfrak{M}_P$ of all polynomials, and the set $\mathfrak{M}_n$ of all polynomials of degree $\leqslant n$. For functions $F_1(x, y)$ and $F_2(x, y)$ from the class $\mathfrak{M}$, we define the multiplicity of a common point $O(0, 0)$ of the curves $F_1(x, y)=0$, $F_2(x, y)=0$ relative to the class $\mathfrak{M}$ (see §1.4, Definition 3). It is readily seen, however, that the various arguments leading to the final proof of Theorems 6 and 7 remain fully in force if functions F_1 and F_2 belong to the class $\mathfrak{M}$, and the multiplicity of their common point is considered relative to this class $\mathfrak{M}$, and not in the sense of Definition 5. Thus, the condition $\Delta_0\neq 0$ is the necessary and sufficient condition for simplicity of the common point of two curves relative to the class $\mathfrak{M}$, and conditions (a), (b), (c) of Theorem 7 are the necessary and sufficient conditions for duplicity of the common point relative to the class $\mathfrak{M}$ ($\mathfrak{M}$ is one of the classes $\mathfrak{M}^{(k)}$, $\mathfrak{M}_A$, $\mathfrak{M}_P$, $\mathfrak{M}_n$). There is thus no need (for $r=1$ or $r=2$) to consider the multiplicity of the common point of two curves relative to the class $\mathfrak{M}$, and in what follows simplicity and duplicity of intersection points are always understood in the sense of Definition 5.

Chapter II

DYNAMIC SYSTEMS CLOSE TO A GIVEN SYSTEM AND PROPERTIES OF THEIR PHASE PORTRAITS

INTRODUCTION

All the results of this chapter directly follow from the theorems of continuous dependence on the initial conditions and the right-hand sides. Although of no intrinsic significance and almost trivial, these results are absolutely essential for a rigorous treatment of the main material. This background chapter consists of two sections.

In §3, δ-close systems are defined and the relevant theorems of continuous dependence are stated (Theorems 8 and 9). Some properties of regular mappings are considered. The next section, §4, considers intersections of paths of close systems with arcs and cycles without contact; it is established that the behavior of the paths of system $(\tilde{A})$ which is sufficiently close to system (A) relative to arcs (or cycles) without contact is on the whole similar to the behavior of the paths of the original system (A). The reader who is interested in following the main line of argument can skim through this chapter, omitting the proof of the various lemmas and concentrating only on the relevant statements. The reader must acquaint himself, however, with the concepts of ε-close regions (Definition 7), ε-translation (Definition 8), and ε-identical partitions of two regions into paths (Definition 9), which are introduced in this chapter. The last of these definitions — ε-identical partition into paths — is the most significant: it is used in the definition of structural stability, which is the main subject of the book.

§3. CLOSENESS OF SOLUTIONS. REGULAR TRANSFORMATION OF CLOSE SYSTEMS

1. Theorems of closeness of solutions

We will consider systems of differential equations (dynamic systems) of the form

$$\frac{dx}{dt}=P(x, y), \qquad \frac{dy}{dt}=Q(x, y). \tag{A}$$

These systems are defined in a bounded region G of the plane (x, y); they are often considered, however, only in some closed subregion $\bar{G}^*$ of G.

System (A) is said to be a system of class k or of analytical class if P and Q are functions of class k or analytical in G.

Consider two systems defined in G,

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

and

$$\frac{dx}{dt} = \tilde{P}(x, y), \quad \frac{dy}{dt} = \tilde{Q}(x, y), \tag{$\tilde{A}$}$$

which both belong to some class k or are both analytical.

Definition 6. System ($\tilde{A}$) ***is said to be δ-close to rank r to system*** (A) ***in*** G ***(or*** $\bar{G}$*) ***if the functions*** $\tilde{P}$ ***and*** $\tilde{Q}$ ***in*** G ***(or*** $\bar{G}$*) ***are δ-close to rank r to*** P ***and*** Q, ***respectively*** (see Definition 1, §1).

Let

$$\tilde{P}(x, y) - P(x, y) = p(x, y), \quad \tilde{Q}(x, y) - Q(x, y) = q(x, y).$$

The functions $p(x, y)$ and $q(x, y)$ are the *increments* of the right-hand sides of system (A). If these functions are δ-close to rank r to zero, we will call them *δ-increments of rank r*. System ($\tilde{A}$) considered in conjunction with (A) is called *modified* (relative to system (A)), and is sometimes written in the form

$$\frac{dx}{dt} = P(x, y) + p(x, y), \quad \frac{dy}{dt} = Q(x, y) + q(x, y). \tag{$\tilde{A}$}$$

In what follows, if *system* ($\tilde{A}$) *is δ-close to rank 1 to system* (A), *we shall simply say that system* ($\tilde{A}$) *is δ-close to system* (A), omitting the qualification "to rank 1."

Consider two vector fields defined by system (A) and a δ-close system ($\tilde{A}$). At each point $M(x, y)$ of G two vectors are defined, $\mathbf{v}(P, Q)$ and $\tilde{\mathbf{v}}(\tilde{P}, \tilde{Q})$. Let $\mathbf{v} \neq 0$, $\tilde{\mathbf{v}} \neq 0$, and let θ be the angle between $\mathbf{v}$ and $\tilde{\mathbf{v}}$ (see QT, Appendix, §5.1). It is readily seen that this angle is infinitesimal for sufficiently small δ. Indeed,

$$\sin\theta = \frac{|Q\tilde{P} - P\tilde{Q}|}{\sqrt{P^2+Q^2}\sqrt{\tilde{P}^2+\tilde{Q}^2}}$$

and for $\tilde{P}$ and $\tilde{Q}$ which are close to P and Q $\sin\theta$ is close to zero while $\cos\theta$ is positive (close to $+1$).

Let us now formulate for these systems the theorem of continuous dependence of solutions on the increments of the right-hand sides and on the initial values. Let the systems (A) and ($\tilde{A}$) be defined in G. Let

$$x = \varphi(t - t_0, x_0, y_0), \quad y = \psi(t - t_0, x_0, y_0) \tag{1}$$

be the solution of system (A) corresponding to initial conditions t_0, x_0, y_0.

Solution (1) is defined for all t in some interval $\tau < t < T$. Let τ_1 and τ_2 be two numbers, such that $\tau < \tau_1 < t_0 < \tau_2 < T$. Let L be the path corresponding to solution (1). Let the segment of L corresponding to the values

of t, $\tau_1 \leqslant t \leqslant \tau_2$ be completely contained in the closed region $\bar{G}^*$ ($\bar{G}^* \subset G$), and let

$$x = \tilde{\varphi}(t-t_0, \tilde{x}_0, \tilde{y}_0), \quad y = \tilde{\psi}(t-t_0, \tilde{x}_0, \tilde{y}_0) \tag{2}$$

be the solution of system $(\tilde{\mathrm{A}})$ corresponding to the initial conditions $t_0, \tilde{x}_0, \tilde{y}_0$.

Theorem 8. For any $\varepsilon > 0$, there exist $\eta > 0$ and $\delta > 0$, such that if 1) $|\tilde{x}_0 - x_0| < \eta$, $|\tilde{y}_0 - y_0| < \eta$ and 2) system $(\tilde{\mathrm{A}})$ is δ-close to system (A) in G^*, solution (2) of system $(\tilde{\mathrm{A}})$ is defined for all t, $\tau_1 \leqslant t \leqslant \tau_2$, and in this time interval

$$|\tilde{\varphi}(t-t_0, \tilde{x}_0, \tilde{y}_0) - \varphi(t-t_0, x_0, y_0)| < \varepsilon,$$
$$|\tilde{\psi}(t-t_0, \tilde{x}_0, \tilde{y}_0) - \psi(t-t_0, x_0, y_0)| < \varepsilon.$$

Theorem 8 is a particular case of Theorem 2 from subsection 1 in the Appendix.

Remark. Since the functions $\varphi(t-t_0, x_0, y_0)$ and $\psi(t-t_0, x_0, y_0)$ are continuous, and are therefore uniformly continuous in t over the segment $\tau_1 \leqslant t \leqslant \tau_2$, Theorem 8 clearly can be strengthened as follows: for any $\varepsilon > 0$, we can choose $\eta > 0$ and $\delta > 0$ such that if in addition to conditions 1 and 2 above we also have condition 3) $|t' - t''| < \eta$ (t' and t'' are any two values from $[\tau_1, \tau_2]$), then

$$|\tilde{\varphi}(t''-t_0, \tilde{x}_0, \tilde{y}_0) - \varphi(t'-t_0, x_0, y_0)| < \varepsilon,$$
$$|\tilde{\psi}(t''-t_0, \tilde{x}_0, \tilde{y}_0) - \psi(t'-t_0, x_0, y_0)| < \varepsilon.$$

Theorem 8 can also be formulated in geometrical terms. This formulation is particularly convenient for what follows.

Let L be a path of system (A), M_0 and M_1 are the points of this path corresponding to t_0 and t_1; the arc M_0M_1 of L is completely contained in a closed region $\bar{G}^*$ ($\bar{G}^* \subset G$). $M(t)$ represents the point of L which corresponds to time t. Similarly, $\tilde{M}(t)$ is the point of the path $\tilde{L}$ of system $(\tilde{\mathrm{A}})$ corresponding to time t.

Geometrical form of Theorem 8. For any $\varepsilon > 0$, there exist $\eta > 0$ and $\delta > 0$, such that if system $(\tilde{\mathrm{A}})$ is δ-close to system (A) in $\bar{G}^*$ and the path $\tilde{L}$ at $t = t_0$ passes through the point $\tilde{M}_0 \in U_\eta(M_0)$, the corresponding motion along $\tilde{L}$ is uniquely determinable for any t, $t_0 \leqslant t \leqslant t_1$, and for these t, $\tilde{M}(t) \in U_\varepsilon(M(t))$. In particular $\tilde{M}_1 = \tilde{M}(t_1) \in U_\varepsilon(M_1)$ (Figure 1).

The next theorem is a generalization of Theorem 8. Let $\bar{F}$ be a closed bounded region, $\bar{F} \subset G$. Consider the solutions

$$x = \varphi(t-t_0, x_0, y_0), \quad y = \psi(t-t_0, x_0, y_0)$$

of system (A) corresponding to various points $M_0(x_0, y_0)$ of region $\bar{F}$. Suppose that each solution (1) is certainly defined in a closed interval

$$\tau_1(x_0, y_0) \leqslant t \leqslant \tau_2(x_0, y_0),$$

where $\tau_1(x_0, y_0)$ and $\tau_2(x_0, y_0)$ are continuous functions of x_0, y_0, such that

$$\tau_1(x_0, y_0) < t_0, \quad \tau_2(x_0, y_0) > t_0.$$

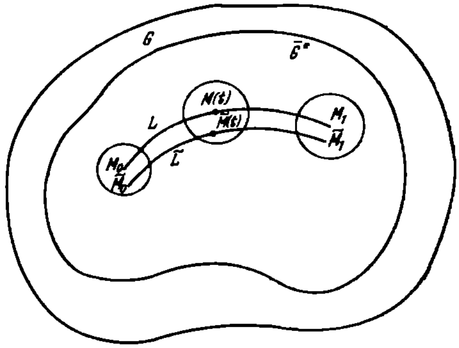

FIGURE 1

By QT, §1.9, Lemma 9, we may take $|\tau_1(x_0, y_0)-t_0| > h_0$, $|\tau_2(x_0, y_0)-t_0| > h_0$, where $h_0 > 0$ is some constant number (which is independent of the point $M_0(x_0, y_0) \in \bar{F}$).

Let $\bar{G}^*$ be a closed region which completely contains all the arcs of the path defined by solutions (1) as the point $M_0(x_0, y_0)$ goes over the entire region $\bar{F}$ and t varies from $\tau_1(x_0, y_0)$ to $\tau_2(x_0, y_0)$. $\bar{G}^*$ contains, in particular, the closed region $\bar{F}$.

Theorem 9. For any $\varepsilon > 0$, there exist $\delta > 0$ and $\eta > 0$ independent of the point $M_0(x_0, y_0) \in \bar{F}$, such that if 1) $|\tilde{x}_0 - x_0| < \eta$, $|\tilde{y}_0 - y_0| < \eta$ and 2) system $(\tilde{A})$ is δ-close to system (A) in $\bar{G}^*$, the solutions

$$\tilde{x} = \tilde{\varphi}(t-t_0, \tilde{x}_0, \tilde{y}_0), \qquad \tilde{y} = \tilde{\psi}(t-t_0, \tilde{x}_0, \tilde{y}_0) \tag{2}$$

of system $(\tilde{A})$ are defined for all t, $\tau_1(x_0, y_0) \leqslant t \leqslant \tau_2(x_0, y_0)$, and in this time interval

$$|\tilde{\varphi}(t-t_0, \tilde{x}_0, \tilde{y}_0) - \varphi(t-t_0, x_0, y_0)| < \varepsilon,$$
$$|\tilde{\psi}(t-t_0, \tilde{x}_0, \tilde{y}_0) - \psi(t-t_0, x_0, y_0)| < \varepsilon.$$

The proof of this theorem follows in the usual way from compactness considerations of region $\bar{F}$ and from Theorem 8. Note that instead of a closed bounded region, $\bar{F}$ can be any compact set.

Remark. The numbers $\eta > 0$ and $\delta > 0$ can be so chosen that if in addition to conditions 1 and 2 above we also have condition 3) $|t' - t''| < \eta$, where $\tau_1(x_0, y_0) \leqslant t' \leqslant \tau_2(x_0, y_0)$, $\tau_1(x_0, y_0) \leqslant t'' \leqslant \tau_2(x_0, y_0)$, then

$$|\tilde{\varphi}(t''-t_0, \tilde{x}_0, \tilde{y}_0) - \varphi(t'-t_0, x_0, y_0)| < \varepsilon,$$
$$|\tilde{\psi}(t''-t_0, \tilde{x}_0, \tilde{y}_0) - \psi(t'-t_0, x_0, y_0)| < \varepsilon$$

(see remark following Theorem 8).

In what follows we will repeatedly use the theorem of closeness of the functions φ and $\tilde{\varphi}$, and also of ψ and $\tilde{\psi}$ (see Appendix, subsection 1, Theorem 3).

2. ε-Closeness of regions. Lemmas of regular transformation

In this subsection we formulate two propositions concerning the substitution of variables in systems of differential equations. The proof is elementary and is therefore omitted.

First we have to define the concept of ε-closeness of regions.

Definition 7. The closed regions $\bar{G}_1$ and $\bar{G}_2$ are said to be ε-close if

1) each point in $\bar{G}_1$ is distant less than ε from $\bar{G}_2$, and conversely each point in $\bar{G}_2$ is distant less than ε from $\bar{G}_1$;

2) each boundary point of $\bar{G}_1$ is distant less than ε from the boundary of $\bar{G}_2$, and conversely each boundary point of $\bar{G}_2$ is distant less than ε from the boundary of $\bar{G}_1$. *

Clearly if $\bar{G}_2$ is the image of $\bar{G}_1$ under a topological mapping f and for any $M \in G_1$, $\rho(M, f(M)) < \varepsilon$, regions $\bar{G}_1$ and $\bar{G}_2$ are ε-close.

Consider a regular transformation of variables of class $k+1$

$$u = \varphi(x, y), \qquad v = \psi(x, y), \tag{3}$$

which is defined in some region of the plane (x, y) (see QT, Appendix, §6.1). We say that this transformation is δ-close to rank $r (r \leqslant k+1)$ to the identity transformation

$$u = x, \qquad v = y, \tag{4}$$

if the functions $\varphi(x, y)$ and $\psi(x, y)$ in this region are δ-close to rank r to the functions x and y, respectively.

Suppose that transformation (3) is considered in an open domain G, and $\bar{G}_1$ is a closed bounded region, $\bar{G}_1 \subset G$. Let

$$x = f(u, v), \qquad y = g(u, v) \tag{5}$$

be the inverse of transformation (3). $\bar{G}_1^*$ is the (u, v) image of $\bar{G}_1$ under transformation (3).

Theorem 4 (the theorem of a small increment of an implicit function) and the compactness of $\bar{G}_1$ clearly show that if transformation (3) is sufficiently close to rank r to the identity transformation (4), transformation (5) in $\bar{G}_1^*$ is arbitrarily close to rank r to the identity transformation

$$x = u, \qquad y = v.$$

If the plane (u, v) coincides with the plane (x, y) and the axes x and y coincide with the axes u and v, respectively, i.e., if the point $M^*(u, v)$ is in fact a point in the plane (x, y) with the coordinates u, v (relative to the system of coordinates defined in that plane), then, transformation (3) being δ-close to the identity transformation, we immediately conclude that $\bar{G}_1^*$ is δ_1-close to $\bar{G}_1$, where $\delta_1 = \sqrt{2}\,\delta$.

Consider a dynamic system of class k

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y) \tag{A}$$

* Simple examples will show that conditions 1 and 2 are independent.

defined in domain G of plane (x, y). We apply to this system a regular transformation (3) of class $k+1$:

$$u=\varphi(x, y), \quad v=\psi(x, y).$$

This gives a system

$$\frac{du}{dt}=P^*(u, v), \quad \frac{dv}{dt}=Q^*(u, v), \tag{A*}$$

where

$$P^*(u, v)=\varphi'_x(f(u, v), g(u, v))P(f(u, v), g(u, v))+ \\ +\varphi'_y(f(u, v), g(u, v))Q(f(u, v), g(u, v)),$$
$$Q^*(u, v)=\psi'_x(f(u, v), g(u, v))P(f(u, v), g(u, v))+ \\ +\psi'_y(f(u, v), g(u, v))Q(f(u, v)\ g(u, v)).$$

(A*), like (A), is clearly a system of class k.

Treating u, v as coordinates in the plane (x, y) and reverting to previous notation, we write system (A*) in the form

$$\frac{dx}{dt}=P^*(x, y), \quad \frac{dy}{dt}=Q^*(x, y). \tag{A*}$$

Let $\bar{G}_1 \subset G$ be a closed bounded region, and $\bar{G}_1^*$ its image under (3).

Lemma 1. For any $\varepsilon>0$, there exists $\delta>0$ such that if transformation (3) is δ-close to rank $k+1$ to the identity transformation in G, then $\bar{G}_1^* \subset G$ and system (A) in $\bar{G}_1^*$ is δ-close to rank k to system (A).*

Now consider two systems of class k defined in G (at this stage, it does not matter whether this region is closed or open):

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y), \tag{A}$$

$$\frac{dx}{dt}=\tilde{P}(x, y), \quad \frac{dy}{dt}=\tilde{Q}(x, y), \tag{$\tilde{A}$}$$

and let

$$u=\varphi(x, y), \quad v=\psi(x, y) \tag{3}$$

be a regular transformation of class $k+1$. Applying this transformation to (A) and ($\tilde{A}$), we get

$$\frac{du}{dt}=P^*(u, v), \quad \frac{dv}{dt}=Q^*(u, v) \tag{A*}$$

and

$$\frac{du}{dt}=\tilde{P}^*(u, v), \quad \frac{dv}{dt}=\tilde{Q}^*(u, v) \tag{$\tilde{A}$*}$$

which are both systems of class k defined in G^* (G^* is the image of G under (3)).

Lemma 2. For any $\varepsilon>0$ there exists $\delta>0$ such that if system ($\tilde{A}$) is δ-close to rank k to system (A) in G, then the system ($\tilde{A}$) is ε-close to rank k to system (A*) in G^*.*

In conclusion of this section, we consider some simple particular cases of increments to right-hand sides of a given system (A), which are often encountered in what follows.

Consider two systems defined in G:

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y), \tag{A}$$

$$\frac{dx}{dt}=\tilde{P}(x, y)=P(x, y)+p(x, y), \quad \frac{dy}{dt}=\tilde{Q}(x, y)=Q(x, y)+q(x, y). \tag{$\tilde{A}$}$$

The sine of the angle θ* between the direction of the field described by system (A) and the direction of the field of system ($\tilde{A}$) at any point in G is expressed by

$$\sin\theta=\frac{P\tilde{Q}-Q\tilde{P}}{\sqrt{P^2+Q^2}\sqrt{\tilde{P}^2+\tilde{Q}^2}}.$$

The angle θ is positive at any point where $P\tilde{Q}-Q\tilde{P}>0$, it is negative if $P\tilde{Q}-Q\tilde{P}<0$; wherever $P\tilde{Q}-Q\tilde{P}=0$ the fields of systems (A) and ($\tilde{A}$) are either parallel ($\theta=0$) or antiparallel ($\theta=\pi$), so that at the relevant points the paths of systems (A) and ($\tilde{A}$) are tangent to one another.

Consider increments of the form

$$p=-\mu fQ, \quad q=+\mu fP, \tag{6}$$

where $f=f(x, y)$ is some function of the same class as the functions $Q(x, y)$, $P(x, y)$, and μ is a parameter. If the relevant region is closed and bounded, then for sufficiently small μ, p and q are evidently arbitrarily small (to some rank r) increments. System ($\tilde{A}$) then has the form

$$\frac{dx}{dt}=P-\mu fQ=\tilde{P}, \quad \frac{dy}{dt}=Q+\mu fP=\tilde{Q}. \tag{7}$$

In particular, consider the case $f=1$. Then the modified system is

$$\frac{dx}{dt}=P-\mu Q, \quad \frac{dy}{dt}=Q+\mu P. \tag{8}$$

The sine of the angle θ between the directions of the field of system (A) and the field of system ($\tilde{A}$) is expressed by

$$\sin\theta=\frac{\tilde{Q}P-\tilde{P}Q}{\sqrt{P^2+Q^2}\sqrt{\tilde{Q}^2+\tilde{P}^2}}=\frac{\mu f}{\sqrt{1+\mu^2 f^2}}. \tag{9}$$

In our particular case ($f\equiv 1$), the angle between the fields of (A) and ($\tilde{A}$) is constant everywhere in the region,

$$\sin\theta=\frac{\mu}{\sqrt{1+\mu^2}}. \tag{10}$$

We say in this case that *the field of system* ($\tilde{A}$) *is rotated through a constant angle relative to the field of*

* The angle between two ordered vectors is defined as the angle not exceeding 180° through which the first vector should be rotated so as to coincide with the second vector.

system (A), or alternatively, that increments of the form (6) for $f \equiv 1$ produce a rotation of the field through a constant angle.*

Lemma 3. The states of equilibrium of system (7) *are those of system* (A) *and vice versa.*

Proof. System (7) is in equilibrium at any point where

$$P-\mu f Q=0, \quad \mu f P+Q=0. \tag{11}$$

Since (11) is a system of linear homogeneous equations for P and Q and the determinant of this system

$$\begin{vmatrix} 1 & -\mu f \\ \mu f & 1 \end{vmatrix} = 1+\mu^2 f^2$$

does not vanish for any x, y, equations (11) can be satisfied if and only if

$$P(x, y)=Q(x, y)=0,$$

i.e., if and only if the point x, y is a state of equilibrium of system (A). This completes the proof.

§4. INTERSECTION OF PATHS OF CLOSE SYSTEMS WITH ARCS AND CYCLES WITHOUT CONTACT

1. Intersection with one arc without contact

This section presents a number of elementary, almost self-evident propositions which are analogous to those discussed in QT, §3. The main difference is that together with system (A) we will also consider modified systems ($\tilde{\mathrm{A}}$). Since these propositions are repeatedly used in what follows, they will be given detailed proof.

Consider a system of class k

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y), \tag{A}$$

defined in G, and let $\bar{G}^*$ be a closed region, $\bar{G}^* \subset G$. The modified system

$$\frac{dx}{dt}=\tilde{P}(x, y), \quad \frac{dy}{dt}=\tilde{Q}(x, y) \tag{$\tilde{\mathrm{A}}$}$$

is also defined in G.

Consider an arc l or a cycle C, which are without contact for a path of system (A) completely contained in $\bar{G}^*$. The following self-evident proposition is given without proof.

Lemma 1. There exists $\delta_0>0$ such that if system ($\tilde{\mathrm{A}}$) *is δ_0-close to system* (A) *in $\bar{G}^*$, the arc l (the cycle C) is without contact for paths of system* ($\tilde{\mathrm{A}}$), *and these paths make with arc l (cycle C) an angle of the same sign* (see QT, Appendix, §5.5) *as the paths of system* (A).

* With increments of this form, the field vectors of system (A) are not only turned through a constant angle but also stretched in a ratio of $\sqrt{1+\mu^2}$. We are concerned only with directions, however.

To simplify further formulations, we will now introduce the concept of ε-translation.

Definition 8. The mapping f of a set E in a metric space R into the same space is called ε-translation if f is a topological mapping and if for each point $M \in E$ the distance between the original and the image is less than ε, $\rho(M, f(M)) < \varepsilon$.*

Now consider systems (A) and ($\tilde{\mathrm{A}}$) in regions H and $\tilde{H}$ (closed or open), respectively, $H \subset G$, $\tilde{H} \subset G$ (in particular, (A) and ($\tilde{\mathrm{A}}$) may coincide, and then the same system is considered in two different subregions of G; alternatively, H and $\tilde{H}$ may coincide).

Definition 9. A partition of H by the paths of system (A) and a partition of $\tilde{H}$ by the paths of system ($\tilde{\mathrm{A}}$) are said to be ε-identical, or in symbols

$$(H, A) \stackrel{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}),$$

*if there exists a mapping of H onto $\tilde{H}$ which is an ε-translation and which transforms the paths of system (A) into the paths of system ($\tilde{\mathrm{A}}$).***

Thus, the partitions of H and $\tilde{H}$ by the paths of (A) and ($\tilde{\mathrm{A}}$) are ε-identical if these partitions have the same topological structure and are "distorted" or "translated" one relative to the other by an amount less than ε. To ensure ε-identity

$$(H, A) \stackrel{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}).$$

H and $\tilde{H}$ should be homeomorphic and ε-close to each other. Moreover, it is necessary that the partitions of these regions by the paths of the corresponding systems have the same topological structure (see QT, §5, Definition V). These necessary conditions, however, are not sufficient in general, since even if they are satisfied, there may prove to be no topological mapping of H onto $\tilde{H}$ which conserves paths and is at the same time an ε-translation.

Now suppose that the arc l, which is an arc without contact for the paths of system (A), is defined by the parametric equations

$$x = f(s), \qquad y = g(s),$$

where $a \leqslant s \leqslant b$. According to the definition of an arc without contact, $f(s)$ and $g(s)$ are continuously differentiable continuous functions. Let $\delta_0 > 0$ be the number introduced in Lemma 1, i.e., such that for all modified systems ($\tilde{\mathrm{A}}$) δ_0-close to system (A) in $\bar{G}^*$ ($\bar{G}^* \subset G$), the arc l is an arc without contact. We will only consider modified systems ($\tilde{\mathrm{A}}$) which are δ_0-close to system (A).

* The term ε-translation is generally understood in a wider sense. Indeed, ε-translation is defined as a continuous mapping (not necessarily topological) which translates each point of the set by less than ε. However, we will have opportunity to use this concept only in relation to topological mappings, and therefore in what follows ε-translation is understood in the restricted sense of Definition 8.

** A mapping which transforms paths into paths is given in QT (§5, Definition V). This is a topological mapping, such that any two points of a path of system (A) are mapped into points of one path of system ($\tilde{\mathrm{A}}$), and any two points of a path of system (A) are mapped by the reverse mapping into two points of one path of system (A). In QT this mapping is called an identifying mapping. We will refer to it as a mapping which maps paths into paths or more briefly a path-conserving mapping.

Let

$$x=\varphi(t-t_0,\ x_0,\ y_0),\quad y=\psi(t-t_0,\ x_0,\ y_0) \tag{1}$$

be a general solution of system (A) and

$$x=\tilde{\varphi}(t-t_0,\ x_0,\ y_0),\quad y=\tilde{\psi}(t-t_0,\ x_0,\ y_0) \tag{2}$$

a general solution of system $(\tilde{A})$. Then the equations

$$x=\varphi(t-t_0,\ f(s),\ g(s))=\Phi(t,\ s),\quad y=\psi(t-t_0,\ f(s),\ g(s))=\Psi(t,\ s) \tag{3}$$

and respectively

$$x=\tilde{\varphi}((t-t_0),\ f(s),\ g(s))=\tilde{\Phi}(t,\ s),\quad y=\tilde{\psi}((t-t_0),\ f(s),\ g(s))=\tilde{\Psi}(t,\ s) \tag{4}$$

for any fixed s, $a\leqslant s\leqslant b$, are equations of that path of system (A) (and respectively $(\tilde{A})$) which for $t=t_0$ crosses the arc l at the point $(f(s),\ g(s))$ (i.e., at the point of the arc l which corresponds to the given value of the parameter s). According to the definition of the functions $\varphi,\ \psi,\ \tilde{\varphi},\ \tilde{\psi}$ we see that

$$\Phi(t_0,\ s)=\tilde{\Phi}(t_0,\ s)=f(s),\quad \Psi(t_0,\ s)=\tilde{\Psi}(t_0,\ s)=g(s). \tag{5}$$

Now suppose that at any point $M_0(x_0,\ y_0)$ of the arc l $(x_0=f(s),\ y_0=g(s),\ a\leqslant s\leqslant b)$ solution (1) is defined for all t, $t_0\leqslant t\leqslant\tau(s)$, * where $\tau(s)$ is a continuous function (in particular, it may be a constant), and that the corresponding arc of the path is completely contained in G^* and has no common points with the arc l, except M_0. Evidently, under these conditions, the functions $\Phi(t,\ s)$, $\Psi(t,\ s)$ are a priori defined everywhere in the closed region

$$a\leqslant s\leqslant b,\quad t_0\leqslant t\leqslant\tau(s). \tag{6}$$

By QT, §1.3, Lemma 5, the functions Φ and Ψ have continuous first-order partial derivatives in region (6), which are expressed by

$$\left.\begin{aligned}
&\Phi'_t(t,\ s)=\varphi'_t(t-t_0,\ f(s),\ g(s)),\\
&\Phi'_s(t,\ s)=\varphi'_{x_0}(t-t_0,\ f(s),\ g(s))\,f'(s)+\varphi'_{y_0}(t-t_0,\ f(s),\ g(s))\,g'(s),\\
&\Psi'_t(t,\ s)=\psi'_t(t-t_0,\ f(s),\ g(s)),\\
&\Psi'_s(t,\ s)=\psi'_{x_0}(t-t_0,\ f(s),\ g(s))\,f'(s)+\psi'_{y_0}(t-t_0,\ f(s),\ g(s))\,g'(s).
\end{aligned}\right\} \tag{7}$$

Further, by QT, §3.5, Lemma 8, equations (3)

$$x=\Phi(t,\ s),\quad y=\Psi(t,\ s)$$

define a regular mapping of region (6) onto some closed region K in the plane $(x,\ y)$ with the arc l as part of its boundary (see Figure 2; region (6) is shown in Figure 3). In the light of our assumptions, region K is contained in $\bar{G}^*$.

* Or for all t, $t_0>t>\tau(s)$. This case is analyzed precisely in the same way.

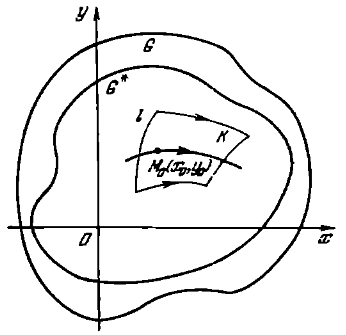

FIGURE 2.

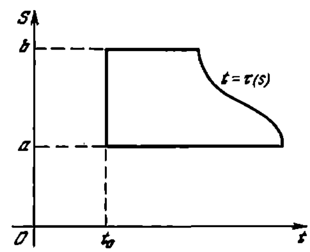

FIGURE 3

L e m m a 2. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if the system* $(\tilde{\mathrm{A}})$ *is* δ-*close to* (A) *in* $\bar{G}^*$, *the functions* $\tilde{\Phi}(t, s)$ *and* $\tilde{\Psi}(t, s)$ *corresponding to system* $(\tilde{\mathrm{A}})$ *are defined in region* (6), *have continuous first-order partial derivatives in this region, and are* ε-*close there to the functions* $\Phi(t, s)$ *and* $\Psi(t, s)$.

The proof of this lemma follows directly from Theorem 9 (§3) and from Appendix, subsection 1, Theorem 3.

L e m m a 3. *(a) There exist* $\delta > 0$ *and* $h > 0$ *such that if the system* $(\tilde{\mathrm{A}})$ *is* δ-*close in* $\bar{G}^*$ *to system* (A), *the mapping*

$$x = \tilde{\Phi}(t, s), \quad y = \tilde{\Psi}(t, s) \tag{4}$$

is a regular mapping of the rectangle

$$a \leqslant s \leqslant b, \quad |t - t_0| \leqslant h \tag{8}$$

in the plane (t, s) *onto the closed region* $\tilde{K}$ *in the plane* (x, y), *and* $\tilde{K}$ *is entirely contained in* $\bar{G}^*$.

(b) Let K *be a closed region which is the image of rectangle* (8) *under the transformation*

$$x = \Phi(t, s), \quad y = \Psi(t, s), \tag{3}$$

corresponding to the "initial" system (A). *For any* $\varepsilon > 0$, *there exists* $\delta^* > 0$ *such that if system* $(\tilde{\mathrm{A}})$ *is* δ^*-*close to* (A) *in* $\bar{G}^*$, $\tilde{K}$ *is* ε-*close to* K.

P r o o f. Let us first prove (a). According to the definition of a regular mapping (see QT, Appendix, §6.1), we have to prove that an appropriate choice of $\delta > 0$ and $h > 0$ will make (4) a one-to-one mapping in rectangle (8) and that everywhere in this rectangle

$$\tilde{\Delta}(t, s) = \begin{vmatrix} \tilde{\Phi}'_t(t, s) & \tilde{\Phi}'_s(t, s) \\ \tilde{\Psi}'_t(t, s) & \tilde{\Psi}'_s(t, s) \end{vmatrix} \neq 0. \tag{9}$$

By QT, §3.4, Lemma 3, there exists $h_0 > 0$ such that the mapping (3) is regular in the rectangle

$$a \leqslant s \leqslant b, \quad |t - t_0| \leqslant h_0. \tag{10}$$

Hence, at any point (t, s) of the rectangle (10), the determinant

$$\Delta(t, s) = \begin{vmatrix} \Phi_t'(t, s) & \Phi_s'(t, s) \\ \Psi_t'(t, s) & \Psi_s'(t, s) \end{vmatrix}$$

has the same sign. Suppose that this sign is positive. Then at any point of the rectangle (10) $\Delta(t, s) > c$, where c is some positive number. Hence, remembering that the rectangle is compact and the elements of the determinant Δ are continuous, we conclude that there exist $\eta > 0$ and $\sigma > 0$ with the following property:

if

$$|t_i - t_j| < \eta, \quad |s_i - s_j| < \sigma, \quad |t_i - t_0| \leqslant h_0, \quad a \leqslant s_i \leqslant b \qquad (i, j = 1, 2, 3, 4), \tag{11}$$

then

$$\begin{vmatrix} \Phi_t'(t_1, s_1) & \Phi_s'(t_2, s_2) \\ \Psi_t'(t_3, s_3) & \Psi_s'(t_4, s_4) \end{vmatrix} > \frac{c}{2}.$$

From the last relation and Lemmas 1 and 2 it follows that there exists $\delta_0 > 0$ such that if system $(\tilde{A})$ is δ_0-close to system (A) and the numbers t_i and s_i ($i = 1, 2, 3, 4$) satisfy (11), l is an arc without contact for the paths of system $(\tilde{A})$ and

$$\begin{vmatrix} \tilde{\Phi}_t'(t_1, s_1) & \tilde{\Phi}_s'(t_2, s_2) \\ \tilde{\Psi}_t'(t_3, s_3) & \tilde{\Psi}_s'(t_4, s_4) \end{vmatrix} > 0. \tag{12}$$

Now suppose that proposition (a) of the lemma is not true. Then for any $\delta > 0$ and $h > 0$, there is always a system $(\tilde{A})$ δ-close to system (A) for which (4) is not a regular mapping of the rectangle (8) into the plane (x, y). As δ and h we choose any two numbers satisfying the inequalities

$$\delta < \delta_0, \quad h < h_0, \quad h < \frac{\eta}{2}. \tag{13}$$

This choice clearly does not detract from the generality of our argument. $\tilde{\Phi}$ and $\tilde{\Psi}$ are single-valued functions. Now, by (12),

$$\tilde{\Delta} = \begin{vmatrix} \tilde{\Phi}_t'(t, s) & \tilde{\Phi}_s'(t, s) \\ \tilde{\Psi}_t'(t, s) & \tilde{\Psi}_s'(t, s) \end{vmatrix} > 0$$

in rectangle (10) and hence also in rectangle (8). Therefore mapping (4) is not regular only if it is not one-to-one, i.e., if in rectangle (8) there is at least one pair of different points (t', s') and (t'', s'') which are mapped under (4) into the same point of the plane (x, y), i.e., such that

$$\tilde{\Phi}(t', s') = \tilde{\Phi}(t'', s''), \quad \tilde{\Psi}(t', s') = \tilde{\Psi}(t'', s''). \tag{14}$$

Applying Taylor's expansion to these equalities, we see that

$$\begin{aligned}\tilde{\Phi}'_t(t_1, s_1)(t'-t'') + \tilde{\Phi}'_s(t_1, s_1)(s'-s'') &= 0,\\ \tilde{\Psi}'_t(t_2, s_2)(t'-t'') + \tilde{\Psi}'_s(t_2, s_2)(s'-s'') &= 0,\end{aligned} \tag{15}$$

where t_1 and t_2 are numbers lying between t' and t'', and s_1 and s_2 are numbers lying between s' and s''. We will now show that $|s' - s''| \geqslant \sigma$. Indeed, if $|s' - s''| < \sigma$, we have $|s_1 - s_2| < \sigma$. Furthermore, $|t_1 - t_2| < |t' - t''| < \eta$ and $|t_1 - t_0| < h_0$, $|t_2 - t_0| < h_0$ by (13). Hence, inequalities (11) are satisfied, and by (12)

$$\begin{vmatrix}\tilde{\Phi}'_t(t_1, s_1) & \tilde{\Phi}'_s(t_1, s_1)\\ \tilde{\Psi}'_t(t_2, s_2) & \tilde{\Psi}'_s(t_2, s_2)\end{vmatrix} > 0.$$

The last equality shows that system (15) considered with respect to $t' - t''$, $s' - s''$ has a nonvanishing determinant, i.e., $t' = t''$, $s' = s''$. This is at variance with the starting assumption that (t', s') and (t'', s'') are two different points in the rectangle (8). Hence $|s' - s''| \geqslant \sigma$.

Let us consider two sequences of positive numbers δ_i and h_i, $i = 1, 2, \ldots$, satisfying conditions (13), such that $\lim\limits_{i\to\infty}\delta_i = \lim\limits_{i\to\infty}h_i = 0$. We have seen that if proposition (a) of the lemma is not true, then for each pair δ_i and h_i there exists a system $(\tilde{A}_i)$ which is δ_i-close to system (A) and a pair of points (t'_i, s'_i), (t''_i, s''_i) such that $|s'_i - s''_i| \geqslant \sigma$ and

$$\tilde{\Phi}_i(t'_i, s'_i) = \tilde{\Phi}_i(t''_i, s''_i), \quad \tilde{\Psi}_i(t'_i, s'_i) = \tilde{\Psi}_i(t''_i, s''_i), \tag{16}$$

where $\tilde{\Phi}_i$ and $\tilde{\Psi}_i$ are the functions corresponding to system $(\tilde{A}_i)$. Since $|t'_i - t_0| < h_i \to 0$ and $|t''_i - t_0| < h_i \to 0$, we have $\lim\limits_{i\to\infty}t'_i = \lim\limits_{i\to\infty}t''_i = t_0$. If necessary we can pick out suitable subsequences, so that s'_i and s''_i can always be made to converge. Let $\lim\limits_{i\to\infty}s'_i = s'_0$, $\lim\limits_{i\to\infty}s''_i = s''_0$. Clearly $a \leqslant s'_0 \leqslant b$, $a \leqslant s''_0 \leqslant b$, and $|s'_0 - s''_0| \geqslant \sigma$. From the definition of $\tilde{\Phi}_i$ and $\tilde{\Psi}_i$ and by Theorem 8 (§3) together with the remark to that theorem we obtain the equalities $\lim\limits_{i\to\infty}\tilde{\Phi}_i(t'_i, s'_i) = \Phi(t_0, s'_0)$ and $\lim\limits_{i\to\infty}\tilde{\Phi}_i(t''_i, s''_i) = \Phi(t_0, s''_0)$ and similarly for Ψ_i. Therefore, taking the limit in (16), we get

$$\Phi(t_0, s'_0) = \Phi(t_0, s''_0), \quad \Psi(t_0, s'_0) = \Psi(t_0, s''_0)$$

or, from (5),

$$f(s'_0) = f(s''_0), \qquad g(s'_0) = g(s''_0).$$

These relations are at variance with the inequality $|s'_0 - s''_0| \geqslant \sigma$. Proposition (a) of the lemma is thus proved.

The validity of proposition (b) follows directly from Theorem 9 (§3), which completes the proof of the lemma.

R e m a r k. It follows directly from Lemma 3 that if $\delta > 0$ is sufficiently small and the system $(\tilde{A})$ is δ-close in $\bar{G}^*$ to the system (A), the paths of system $(\tilde{A})$ which meet at $t = t_0$ the arc l (arc without contact) have no common points with the arc l for all other values of t, $t \neq t_0$, $|t - t_0| \leqslant h$.

Let now $M_0(x_0, y_0)$ be an interior point of the arc l, corresponding to the value s_0 of the parameter s $(a < s_0 < b)$.

L e m m a 4. *For any $\varepsilon > 0$, $h_0 > 0$, there exist $\eta > 0$, $\delta > 0$ which satisfy the following condition: if system $(\tilde{\mathrm{A}})$ is δ-close to system (A) in $\bar{G}^*$, M is some point in $U_\eta(M_0)$, and $\tilde{L}$ is the path of system $(\tilde{\mathrm{A}})$ which at $t = t_0$ passes through the point M, then for some t^*, $|t^* - t_0| < h_0$, the path $\tilde{L}$ intersects the arc l at the point M^*, such that the arc MM^* of the path $\tilde{L}$ is entirely contained in $U_\varepsilon(M_0)$ (Figure 4).*

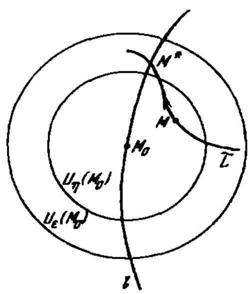

FIGURE 4

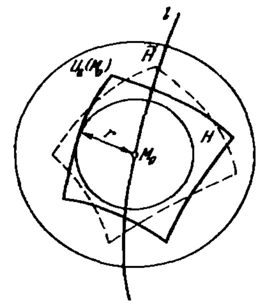

FIGURE 5

P r o o f . Fix $\varepsilon > 0$ and $h_0 > 0$. Without loss of generality we may take $U_\varepsilon(M_0) \subset \bar{G}^*$. Choose $\sigma > 0$ such that the part of the arc l corresponding to the values of the parameter s, $s_0 - \sigma \leqslant s \leqslant s_0 + \sigma$, is completely contained in $U_\varepsilon(M_0)$. By Lemma 3 (a), using the continuity of all the functions, we see that there exist $\delta > 0$ and h, $0 < h < h_0$, such that the mapping

$$x = \Phi(t, s), \quad y = \Psi(t, s), \tag{3}$$

corresponding to system (A), and also the mapping

$$x = \tilde{\Phi}(t, s), \quad y = \tilde{\Psi}(t, s), \tag{4}$$

corresponding to any system $(\tilde{\mathrm{A}})$ which is δ-close to system (A) in $\bar{G}^*$, are regular in the rectangle

$$|t - t_0| \leqslant h, \quad |s - s_0| \leqslant \sigma$$

in the plane (t, s) and map this rectangle onto a closed region H (or $\tilde{H}$) in the plane (x, y), $H \subset U_\varepsilon(M_0)$, $\tilde{H} \subset U_\varepsilon(M_0)$. M_0 is evidently an interior point of H and $\tilde{H}$. Let r be the distance of M_0 from the boundary of H (Figure 5). By Lemma 3(b), δ can be cosen so small that $\tilde{H}$ is $\frac{r}{3}$-close to H. Then the boundary of $\tilde{H}$ is $\frac{r}{3}$-close to the boundary of H. For η we can take any positive number smaller than $\frac{r}{3}$. Indeed, if $\eta < \frac{r}{3}$, it is readily seen that $U_\eta(M_0)$ is completely contained in each region $\tilde{H}$. But then from the

definition of $\bar{H}$ and the inequality $h < h_0$ we conclude that the particular η and δ that we have chosen satisfy the condition of the lemma. Q. E. D.

Remark. Lemma 4 can be generalized as follows. Let λ be a part of the arc without contact l which is contained entirely in l (i.e., the end-points of the arc λ do not coincide with the end points of l). Then for any $\varepsilon > 0$ and $h > 0$ there exist $\eta > 0$ and $\delta > 0$ which satisfy the following condition: if system $(\tilde{A})$ is δ-close to system (A) in $\bar{G}^*$, M is any point whose distance from the arc λ is less than η (i.e., $M \in U_\eta(\lambda)$), and $\tilde{L}$ is the path of the system $(\tilde{A})$ which at $t = t_0$ passes through the point M, then for some t^*, $|t^* - t_0| < h$, the path $\tilde{L}$ intersects the arc l at the point M^*, and the arc MM^* of the path $\tilde{L}$ is entirely contained in $U_\varepsilon(\lambda)$.

The validity of this proposition can be established in the usual way by reductio ad absurdum, using Lemma 4 and the compactness of the arc λ.

As before, let the arc without contact l lie inside the region $\bar{G}^*$. Consider some interior point $M_0(x_0, y_0)$ of this region which does not belong to l. Let the path L of system (A) pass through the point M_0 at $t = t_0$, and at $t = \tau \neq t_0$ it crosses the arc l at a point M, which is not an end-point of the arc l. Moreover, let the arc M_0M of L be entirely contained in G^* (Figure 6).

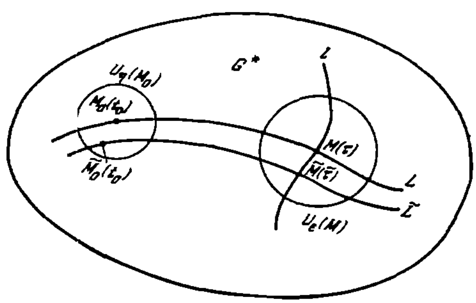

FIGURE 6

Lemma 5. For any $\varepsilon > 0$, $h > 0$, there exist $\eta > 0$ and $\delta > 0$ such that if system $(\tilde{A})$ is δ-close to system (A) in $\bar{G}^*$, $\tilde{M}_0 \in U_\eta(M_0)$, and $\tilde{L}$ is the path of system $(\tilde{A})$ which at $t = t_0$ passes through the point $\tilde{M}_0$, then at some $t = \tilde{\tau}$ the path $\tilde{L}$ crosses the arc l at the point $\tilde{M}$ so that (a) $|\tilde{\tau} - \tau| < h$; (b) $\tilde{M} \in U_\varepsilon(M)$; (c) the arc $\tilde{M}_0\tilde{M}$ of the path $\tilde{L}$ is contained entirely in G^*; (d) if $|\tilde{\tau} - t_0| \leqslant |\tau - t_0|$, then $\tilde{M}(t) \in U_\varepsilon(M(t))$ for all $t \in [\tilde{\tau}, t_0]$; if, on the other hand, $|\tilde{\tau} - t_0| > |\tau - t_0|$, then $\tilde{M}(t) \in U_\varepsilon(M(t))$ for all $t \in [\tau, t_0]$ and the arc of the path $\tilde{L}$ corresponding to $t \in [\tilde{\tau}, \tau_0]$ is entirely contained in $U_\varepsilon(M(\tau))$ (Figure 6).*

Lemma 5 is an obvious corollary from Theorem 8 (§3) and Lemma 4.

Remark. Lemma 5 can be generalized like the preceding lemma (see remark to Lemma 4). Indeed, consider a compact set F in G^* such that each path which at $t = t_0$ passes through the point $M_0(x_0, y_0)$ of this set intersects at some $t = \tau(x_0, y_0)$ the arc without contact l at the point $M(x_0, y_0)$, always remaining inside G until the intersection point is reached; then for given

* Here, as in the following, $M(t)$ is the point of the path of the relevant system which corresponds to the time t for the particular motion chosen along the path.

$\varepsilon > 0$ and $h > 0$, the numbers $\eta > 0$ and $\delta > 0$ entering Lemma 5 can be chosen independently of the point $M_0 \in F$, and will thus satisfy the lemma irrespective of the particular point $M_0 \in F$ that is taken. In what follows, the compact set F is generally identified with some arc without contact l' which has no common points with the arc l.

In Lemma 3 the functions $\Phi(t, s)$, $\Psi(t, s)$, and also $\tilde{\Phi}(t, s)$ and $\tilde{\Psi}(t, s)$, were considered only for t such that $|t - t_0| \leqslant h_0$, where $h_0 > 0$ is some sufficiently small number. Let us now again consider the case when the functions $\Phi(t, s)$ and $\Psi(t, s)$ are determined for all t and s in the closed region

$$a \leqslant s \leqslant b, \quad t_0 \leqslant t \leqslant \tau(s),^{*} \tag{6}$$

where $\tau(s)$ is a continuous function. We assume, as before, that for all t and s from this region, $(\Phi(t, s), \Psi(t, s)) \in G^{*}$ and that if $t_0 < t \leqslant \tau(s)$ and s, s' are any two numbers from $[a, b]$, at least one of the following two inequalities is satisfied:

$$\Phi(t, s) \neq \Phi(t_0, s'), \quad \Psi(t, s) \neq \Psi(t_0, s'),$$

i.e., any path of system (A) defined by the equations

$$x = \varphi(t - t_0, f(s), g(s)) = \Phi(t, s), \quad y = \psi(t - t_0, f(s), g(s)) = \Psi(t, s), \tag{3}$$

has no common points with the arc l for t going from t_0 to $\tau(s)$, except the one point at $t = t_0$.

We have noted above that under these conditions equations (3) define a regular mapping of the region (6) in the plane (t, s) onto some closed region K in the plane (x, y) (Figure 2).

Consider the following lemma.

Lemma 6. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if system $(\tilde{A})$ is δ-close in $\bar{G}^{*}$ to system (A), the functions $\tilde{\Phi}(t, s)$ and $\tilde{\Psi}(t, s)$ are defined in the region (6) and the equations

$$x = \tilde{\Phi}(t, s), \quad y = \tilde{\Psi}(t, s) \tag{4}$$

describe a regular mapping of (6) onto some closed region $\tilde{K}$, $\tilde{K} \subset G^{*}$, such that the regions K and $\tilde{K}$ are ε-close.

Proof. It suffices to show that for small δ the mapping (4) of the region (6) is regular, since the other propositions of the lemma are contained in Lemma 2. To establish the regularity of the mapping (6), it suffices to show that the paths of any system $(\tilde{A})$ which is δ-close to the system (A) have no common points with the arc l for t from the interval $t_0 < t \leqslant \tau(s)$.

By Lemma 3, there exist $\delta_1 > 0$ and $h > 0$ such that the paths of any system $(\tilde{A})$ δ_1-close to system (A) which intersect the arc l at $t = t_0$ have no common points with this arc for t from the interval $t_0 < t \leqslant t_0 + h$. Here h can be taken arbitrarily small. Let $t_0 + h < \tau(s)$ for all s, $a \leqslant s \leqslant b$. $\delta_1 > 0$ is chosen so small that all the conditions of the lemma, possibly except the regularity of the mapping, are satisfied for systems $(\tilde{A})$ which are δ_1-close to (A).

* Or $t_0 > t > \tau(s)$. Condition (6) is used without loss of generality.

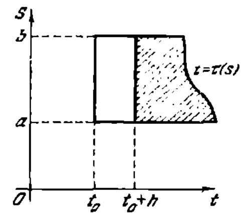

FIGURE 7

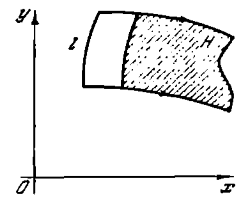

FIGURE 8

Consider the region

$$a \leqslant s \leqslant b, \quad t_0 + h \leqslant t \leqslant \tau(s) \tag{17}$$

in the plane (t, s) (the cross-hatched part of Figure 7). Because of the particular choice of δ_1 and h, mapping (3) maps this region onto some closed region H in the plane (x, y) at a positive distance from the arc l (Figure 8). If $\delta > 0, \delta < \delta_1$ is sufficiently small, and the system $(\tilde{\mathrm{A}})$ is δ-close to the system (A), mapping (4) maps the region (17) onto region $\tilde{H}$ which is sufficiently close to H and therefore has no common points with the arc l. Since $\delta < \delta_1$, δ clearly satisfies all the conditions of the lemma. Q. E. D.

2. Paths of close systems between two arcs without contact

Let us now consider the intersection of the paths of system (A) and of system $(\tilde{\mathrm{A}})$ close to (A) with two arcs without contact. We will prove two lemmas.

Let l_1 and l_2 be two arcs without contact for the paths of system (A) which lie in G^* and have no common points. Let

$$x = f_1(s), \quad y = g_1(s)$$

be the parametric equations of the arc l_1. Suppose that the path of system (A) which at $t = t_0$ passes through the point $(f_1(s), g_1(s))$ of the arc l_1 intersects the arc l_2 at some $t = \tau(s)$; the part of this path corresponding to t from the interval $t_0 \leqslant t \leqslant \tau(s)$ is entirely contained in G^* and has no common points with l_1 or l_2, except its two end-points (to fix ideas, we take $\tau(s) > t_0$).

In QT, §3.6, Lemma 9 it has been shown that $\tau(s)$ is a continuous and therefore bounded function of s. Let M_1 and M_2 be any two points of the arc l_1, other than its end-points, corresponding to the values s_1 and s_2 of the parameter s, $a < s_1 < s_2 < b$. Let further the paths L_1 and L_2 that pass through the points M_1 and M_2 at $t = t_0$ meet the arc l_2 at the points N_1 and N_2 for $\tau_1 = \tau(s_1)$ and $\tau_2 = \tau(s_2)$, respectively; N_1 and N_2 do not coincide with the end-points of l_2, either. Let Γ designate an elementary quadrangle (see QT, §3.6, Remark 2 to Lemma 10) limited by the segments M_1M_2 and N_1N_2 of the arcs l_1 and l_2 and by the segments M_1N_1 and M_2N_2 of the paths. Because of our assumptions, $\Gamma \subset G^*$ (Figure 9).

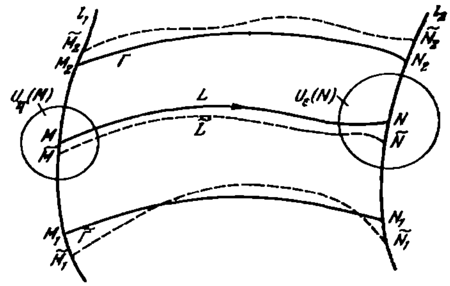

FIGURE 9

Lemma 7. For any $\varepsilon>0$ there exist $\eta>0$ and $\delta>0$ such that if system $(\tilde{A})$ is δ-close to system (A) then l_1 and l_2 are arcs without contact for the paths of system $(\tilde{A})$, and moreover

(a) if $\tilde{M}_1$ and $\tilde{M}_2$ are two points of the arc l_1 which lie in $U_\eta(M_1)$ and $U_\eta(M_2)$, respectively, and $\tilde{L}_1$ and $\tilde{L}_2$ are the paths of system $(\tilde{A})$ which pass through these points at $t=t_0$, the paths $\tilde{L}_1$ and $\tilde{L}_2$ at $t>t_0$ intersect the arc l_2 at points $\tilde{N}_1$ and $\tilde{N}_2$ which lie in $U_\varepsilon(N_1)$ and $U_\varepsilon(N_2)$, respectively, and the sections $\tilde{M}_1\tilde{N}_1$ and $\tilde{M}_2\tilde{N}_2$ of these paths, together with the sections $\tilde{M}_1\tilde{M}_2$ and $\tilde{N}_1\tilde{N}_2$ of the arcs l_1 and l_2, delineate a region $\tilde{\Gamma}$ which constitutes an elementary quadrangle for system $(\tilde{A})$;

(b) the elementary quadrangle $\tilde{\Gamma}$ is entirely contained in G^* and is ε-close to the elementary quadrangle Γ.

Proof. Consider the transformation corresponding to system (A):

$$x=\Phi(t, s), \quad y=\Psi(t, s). \tag{3}$$

By assumption, Φ and Ψ are defined for all t and s satisfying the inequalities

$$t_0\leqslant t\leqslant \tau(s), \qquad a\leqslant s\leqslant b, \tag{18}$$

and for all these t and s the points $(\Phi(t, s), \Psi(t, s))$ belong to G^*. By QT, §3.4, Lemma 3, it is readily seen, however, that the t interval in (18) always can be somewhat increased; therefore, there exists a certain $h_0>0$ such that the functions Φ and Ψ are also defined for

$$a\leqslant s\leqslant b, \qquad t_0\leqslant t\leqslant \tau(s)+h_0 \tag{19}$$

and for all these values of the parameters the points $(\Phi(t,s), \Psi(t, s))$ lie in G^*. The points $(\Phi(t, s), \Psi(t, s))$ for which $s_1\leqslant s\leqslant s_2$, $\tau(s)<t\leqslant \tau(s)+h_0$ evidently lie outside the elementary quadrangle Γ, i.e., these points and the interior points of l_2 close to Γ lie on the two sides of the arc l_2.

By Lemmas 1, 2, 5 and remark to Lemma 5, we see that for any $\varepsilon>0$ and $h>0$, $h<h_0$, there exist $\delta>0$ and $\eta>0$ such that l_1 and l_2 are arcs without contact for any system $(\tilde{A})$ which is δ-close to system (A) in $\bar{G}^*$, and moreover:

1) the functions $\tilde{\Phi}(t, s)$, $\tilde{\Psi}(t, s)$ are defined for all t and s ($a\leqslant s\leqslant b$, $t_0\leqslant t<\tau(s)+h_0$) and for these values of the parameters they are ε-close to the functions Φ and Ψ, respectively, the points $(\tilde{\Phi}(t, s), \tilde{\Psi}(t, s))$ lying in G^*;

2) if $M(f_1(s),\ g_1(s))$, $s_1 \leqslant s \leqslant s_2$ is a point of the section M_1M_2 of the arc l, L is the path of system (A) passing through this point at $t = t_0$, N is the point of the path L which belongs to the section N_1N_2 of the arc l_2 and corresponds to the time $\tau(s)$, $\tilde{M}(\tilde{s})$ is a point of the arc l_1, $\tilde{M} \in U_\eta(M)$, $\tilde{L}$ is the path of system $(\tilde{A})$ which at $t = t_0$ passes through the point $\tilde{M}$ (Figure 9), then the path $\tilde{L}$ crosses the arc l_2 at $t = \tilde{\tau}(\tilde{s})$, $|\tilde{\tau}(\tilde{s}) - \tau(s)| < h$ at a point $\tilde{N}$ which lies in $U_\varepsilon(N)$ (Figure 9).

Let $\tilde{s}_1(\tilde{s}_2)$ be the value of the parameter s corresponding to the point $\tilde{M}_1(\tilde{M}_2)$.

It follows from (1), in particular, that the paths $\tilde{L}_1$ and $\tilde{L}_2$ of system $(\tilde{A})$ passing through the points $\tilde{M}_1 \in U_\eta(M_1)$ and $\tilde{M}_2 \in U_\eta(M_2)$, respectively, meet the arc l_2 at the points $\tilde{N}_1 \in U_\varepsilon(N_1)$ and $\tilde{N}_2 \in U_\varepsilon(N_2)$. Moreover, arguing as before in Lemma 6, we can show that if $\delta > 0$ is sufficiently small, each path $\tilde{L}$ of system $(\tilde{A})$, which, at $t = t_0$ intersects the arc l_1 at the point $\tilde{M}(f(\tilde{s}),\ g(\tilde{s}))$, $\tilde{s}_1 \leqslant \tilde{s} \leqslant \tilde{s}_2$, and meets the arc l_2 at $t = \tilde{\tau}(\tilde{s})$, has no common points with l_1 and l_2 for intermediate values of t, $t_0 < t < \tilde{\tau}(s)$. But then the region $\tilde{\Gamma}$ limited by the sections $\tilde{M}_1\tilde{M}_2$, $\tilde{N}_1\tilde{N}_2$ of the arcs l_1 and l_2 and by the sections $\tilde{M}_1\tilde{N}_1$, $\tilde{M}_2\tilde{N}_2$ of the paths $\tilde{L}_1$ and $\tilde{L}_2$ is an elementary quadrangle of system $(\tilde{A})$. Part (a) of the lemma is thus proved. That for the given choice of $\delta > 0$, $\tilde{\Gamma}$ is ε-close to the elementary quadrangle Γ and $\tilde{\Gamma} \subset G^*$ follows directly from Theorem 9 (§3.1) and from proposition 2 above. This completes the proof of the lemma.

Remark. Let M_1' and M_2' be any two points of the arc l_1 which lie between M_1 and M_2. By Lemma 7 it is readily seen that if $\delta > 0$ is sufficiently small and system $(\tilde{A})$ is δ-close to system (A), the arcs of the paths of system $(\tilde{A})$ lying between the part $M_1'M_2'$ of the arc l_1 and the corresponding section of the arc l_2 belong to the elementary quadrangle Γ of the original system (A) (Figure 10).

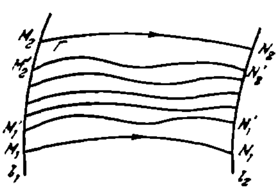

FIGURE 10

We will prove another lemma which pertains to elementary quadrangles Γ and $\tilde{\Gamma}$ made up of paths of two sufficiently close dynamic systems (A) and $(\tilde{A})$. This lemma is repeatedly used in what follows. We retain the same notation as before. Let s_1 and s_2, $a < s_1 < s_2 < b$, be the values of the parameter s along the arc l_1, corresponding to the points M_1 and M_2, $\tilde{s}_1$ and $\tilde{s}_2$ the values of s corresponding to the points $\tilde{M}_1$ and $\tilde{M}_2$ ($\tilde{M}_1 \in U_\eta(M_1)$, $\tilde{M}_2 \in U_\eta(M_2)$, $\eta > 0$ is sufficiently small). Then the points of the section M_iN_i of the path L_i $(i = 1, 2)$ which is part of the boundary of the quadrangle Γ correspond to t values from the interval $t_0 \leqslant t \leqslant \tau(s_i)$, and the points of the section $\tilde{M}_i\tilde{N}_i$ of the path $\tilde{L}_i$ $(i = 1, 2)$ which is part of the boundary of the quadrangle $\tilde{\Gamma}$ correspond to t values from the interval $t_0 \leqslant t \leqslant \tau(s_i)$ (Figure 9).

Consider a topological mapping φ of the section M_iN_i of the arc l_1 onto the section $\tilde{M}_1\tilde{M}_2$ such that $\varphi(M_1) = \tilde{M}_1$, $\varphi(M_2) = \tilde{M}_2$. A suitable mapping φ is the linear mapping defined by the equation

$$\tilde{s} = \tilde{s}_2 \frac{s - s_1}{s_2 - s_1} + \tilde{s}_1 \frac{s - s_2}{s_1 - s_2}, \tag{20}$$

where s is the value of the parameter in the equations of the arc l_1 corresponding to some point M from M_1M_2, and $\tilde{s}$ is the value of the same parameter corresponding to the point $\tilde{M}$ which is the image of the point M.

The mappings φ of the arc M_1M_2 onto the arc $\widetilde{M}_1\widetilde{M}_2$ are not restricted to linear mappings of the form (20), and in what follows other cases will also have to be considered. The general mapping can obviously be expressed in the form

$$\tilde{s} = \omega(s),$$

with equation (20) considered as a particular case. If the explicit expression of the mapping φ is not given, we will always define it by equation (20), however.

Lemma 8. *For any $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that if system $(\widetilde{A})$ is δ-close in G^* to system (A), and the distance between any point M of the section M_1M_2 of the arc l_1 and its image $\varphi(M)$ is less than η, i.e., $\rho(M, \varphi(M)) < \eta$, then there exists a path-conserving topological mapping T of the elementary quadrangle Γ onto $\widetilde{\Gamma}$ (also conserving the direction of motion along the paths) which coincides along the section M_1M_2 of the arc l_1 with the mapping φ and is in fact its ε-translation.*

Proof. Every point $P(x, y)$ of the elementary quadrangle Γ lies on the path L of system (A) which at $t = t_0$ passes through the point $M(f_1(s), g_1(s))$ of the arc l_1. Let the point $P(x, y)$ correspond to some time t. The numbers t, s can be regarded as curvilinear coordinates of the point $P \in \Gamma$. Here s varies from s_1 to s_2. If s is fixed, t varies from t_0 to $\tau(s)$, and the point P traces the section MN of the path L (Figure 11). The cartesian coordinates of the point P are x, y, and

$$x = \Phi(t, s) = \varphi(t - t_0, f(s), g(s)),$$
$$y = \Psi(t, s) = \psi(t - t_0, f(s), g(s)).$$

Similarly the point $\widetilde{P}(\tilde{x}, \tilde{y})$ of the elementary quadrangle $\widetilde{\Gamma}$ can be made to correspond to the curvilinear coordinates $\tilde{t}$, $\tilde{s}$, $\tilde{s}_1 \leqslant \tilde{s} \leqslant \tilde{s}_2$, $t_0 \leqslant \tilde{t} \leqslant \tilde{\tau}(\tilde{s})$, and

$$\tilde{x} = \widetilde{\Phi}(\tilde{t}, \tilde{s}), \quad y = \widetilde{\Psi}(\tilde{t}, \tilde{s}).$$

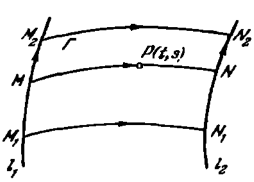

FIGURE 11

We will now define new curvilinear coordinates λ, s in the elementary quadrangle Γ:

$$s = s, \quad t = t_0 + \lambda(\tau(s) - t_0).$$

If s is fixed $(s_1 \leqslant s \leqslant s_2)$ and λ varies from 0 to 1, the point P travels along the entire section MN of L. In the elementary quadrangle $\widetilde{\Gamma}$ we introduce new

coordinates $\tilde{\lambda}$, $\tilde{s}$ by the analogous relations

$$\tilde{s} = \tilde{s}, \quad \tilde{t} = t_0 + \tilde{\lambda}(\tilde{\tau}(\tilde{s}) - t_0).$$

As $\tilde{t}$ goes from t_0 to $\tilde{\tau}(\tilde{s})$, $\tilde{\lambda}$ also varies from 0 to 1.

Let us now find the mapping T of the quadrangle Γ, assuming that each point $P(\lambda, s)$ of the quadrangle is mapped onto a new point $T(P) = \tilde{P}(\tilde{\lambda}, \tilde{s})$, where

$$\tilde{s} = \omega(s), \quad \tilde{\lambda} = \lambda.$$

In other words, a point P with the cartesian coordinates

$$x = \Phi(t_0 + \lambda(\tau(s) - t_0), s), \quad y = \Psi(t_0 + \lambda(\tau(s) - t_0), s)$$

is mapped under T into a point $\tilde{P}$ with the cartesian coordinates

$$\tilde{x} = \tilde{\Phi}(t_0 + \lambda(\tilde{\tau}(\omega(s)) - t_0), \omega(s)), \quad \tilde{y} = \tilde{\Psi}(t_0 + \lambda(\tilde{\tau}(\omega(s)) - t_0), \omega(s)).$$

The mapping T defined in this way is clearly a one-to-one mapping of the quadrangle Γ onto $\tilde{\Gamma}$; it maps the paths of system (A) into paths of system ($\tilde{\mathrm{A}}$) conserving the direction of motion, and coincides with the mapping φ along the section M_1M_2 of the arc l_1. It is readily seen that T is also a continuous mapping; this follows from the continuity of the functions Φ, Ψ, $\tilde{\Phi}$, $\tilde{\Psi}$, ω, τ, and $\tilde{\tau}$ (the last two functions are continuous by QT, §3.6, Lemma 9). T is therefore a topological mapping.

Finally from Lemma 2 and also from Lemma 5 and the corresponding remark it follows that if $\eta > 0$ and $\delta > 0$ are sufficiently small, then $\rho(P, T(P)) < \varepsilon$ for any $P \in \Gamma$. This completes the proof of the lemma.

Remark. It follows from Lemma 8 that if $\eta > 0$ and $\delta > 0$ are sufficiently small, the partitions of the elementary quadrangles Γ and $\tilde{\Gamma}$ by the paths of the corresponding systems are ε-identical (see Definition 9).

The following lemma is a stronger version of Lemma 8. We retain the same notation and further assume that, apart from the mapping φ of the section M_1M_2 of the arc l_1 onto the section $\tilde{M}_1\tilde{M}_2$ of the same arc, there is also given a mapping α of the section M_1N_1 of the path L_1 onto the section $\tilde{M}_1\tilde{N}_1$ of the path $\tilde{L}_1$, such that

$$\alpha(M_1) = \tilde{M}_1, \quad \alpha(N_1) = \tilde{N}_1,$$

and a mapping β of the section M_2N_2 of the path L_2 onto the section $\tilde{M}_2\tilde{N}_2$ of the path $\tilde{L}_2$, such that

$$\beta(M_2) = \tilde{M}_2, \quad \beta(N_2) = \tilde{N}_2$$

(Figure 9).

Lemma 9. *For any $\varepsilon > 0$, there exist $\delta > 0$ and $\eta > 0$ such that if system ($\tilde{\mathrm{A}}$) is δ-close to system (A) and the mappings φ, α, and β are η-translations (Definition 8, §4.1), there exists a path-conserving topological mapping T of the elementary quadrangle Γ onto $\tilde{\Gamma}$ (which also conserves the direction of motion) that coincides on the sections M_1M_2, M_1N_1, and M_2N_2 of the boundary of the elementary quadrangle Γ with the mappings φ, α, and β, respectively, and is an ε-translation.*

Proof. The argument constitutes a minor modification of the line of reasoning which established the proof of Lemma 8. As in the previous lemma, we introduce a coordinate λ varying from 0 to 1 along each section of the path of system (A) in Γ and along each section of the path of system $(\widetilde{A})$ in $\widetilde{\Gamma}$. Along the sections M_1M_2 and $\widetilde{M}_1\widetilde{M}_2$ of the arc l_1 we moreover introduce the coordinates μ and $\widetilde{\mu}$, respectively, which vary from 0 to 1 and are linearly related to the parameter s: $s=s_1(1-\mu)+s_2\mu$ along the arc M_1M_2 and $s=\widetilde{s}_1(1-\widetilde{\mu})+\widetilde{s}_2\widetilde{\mu}$ along the arc $\widetilde{M}_1\widetilde{M}_2$. The mappings φ, α, and β can be defined by the appropriate relations $\widetilde{\mu}=\varphi(\mu)$, $\widetilde{\lambda}_1=\alpha(\lambda)$, $\widetilde{\lambda}_2=\beta(\lambda)$. (We use here the same notation φ, α, β. This is not quite rigorous, but need not cause any confusion.) Here $\widetilde{\lambda}_i$ is the value of the parameter λ on the path $\widetilde{L}_i$ at the point corresponding to the point λ_i on the path L_i $(i=1,2)$. Clearly $\alpha(0)=\beta(0)=0$, $\alpha(1)=\beta(1)=1$, $\varphi(0)=0$, $\varphi(1)=1$.

For T we take a mapping which maps the point $P(x, y)\in\Gamma$ defined by the parameters μ, λ $(0\leqslant\mu\leqslant1,\ 0\leqslant\lambda\leqslant1)$ onto the point $\widetilde{P}(\widetilde{x}, \widetilde{y})$ with the parameters $\widetilde{\mu}$, $\widetilde{\lambda}$ related to μ and λ by the equalities

$$\widetilde{\mu}=\varphi(\mu),\qquad \widetilde{\lambda}=\alpha(\lambda)(1-\mu)+\beta(\lambda)\mu.$$

In other words, the point $P(x, y)\in\Gamma$ with the coordinates

$$x=\Phi(t_0+\lambda\tau(s_1(1-\mu)+s_2\mu),\ s_1(1-\mu)+s_2\mu),$$
$$y=\Psi(t_0+\lambda\tau(s_1(1-\mu)+s_2\mu),\ s_1(1-\mu)+s_2\mu)$$

is mapped onto a point $\widetilde{P}(\widetilde{x}, \widetilde{y})$ with the coordinates

$$\widetilde{x}=\widetilde{\Phi}(t_0+[\alpha(\lambda)(1-\mu)+\beta(\lambda)\mu]\widetilde{\tau}(\widetilde{s}_1(1-\varphi(\mu))+\widetilde{s}_2\varphi(\mu)),\ \ \widetilde{s}_1(1-\varphi(\mu))+\widetilde{s}_2\varphi(\mu)),$$
$$\widetilde{y}=\widetilde{\Psi}(t_0+[\alpha(\lambda)(1-\mu)+\beta(\lambda)\mu]\widetilde{\tau}(\widetilde{s}_1(1-\varphi(\mu))+\widetilde{s}_2\varphi(\mu)),\ \ \widetilde{s}_1(1-\varphi(\mu))+\widetilde{s}_2\varphi(\mu)).$$

The mapping T defined in this way is readily seen to be a path-conserving topological mapping of the elementary quadrangle Γ onto $\widetilde{\Gamma}$ which coincides with the mappings φ, α, β on the corresponding sections of the boundary. If now δ and η are sufficiently small, $\widetilde{\Phi}$ and $\widetilde{\Psi}$ can be made as close as is needed to Φ and Ψ, $\alpha(\lambda)$ and $\beta(\lambda)$ to λ, $\widetilde{\mu}$ and $\widetilde{\lambda}$ to μ and λ, and $\widetilde{s}_1$ and $\widetilde{s}_2$ to s_1 and s_2, respectively. But then for sufficiently small δ and η the mapping T is an ε-translation. Q. E. D.

Retaining the previous notation, let us consider the elementary quadrangle Γ formed by the arcs of the paths of system (A). Here, with system (A), we will only consider modified systems $(\widetilde{A})$ of one particular form, namely systems

$$\frac{dx}{dt}=\widetilde{P}(x, y),\qquad \frac{dy}{dt}=\widetilde{Q}(x, y), \tag{$\widetilde{A}$}$$

such that at any point of $\overline{G}^*$ which is not a state of equilibrium of system (A) we have

$$P\widetilde{Q}-Q\widetilde{P}\neq0. \tag{21}$$

Thus, at any nonequilibrium point in $\overline{G}^*$ we have either $P\widetilde{Q}-Q\widetilde{P}>0$ or $P\widetilde{Q}-Q\widetilde{P}<0$.

An example of such systems is provided by §3.2, where we described systems of the form

$$\frac{dx}{dt} = P - \mu f Q, \quad \frac{dy}{dt} = Q + \mu f P,$$

$f(x, y)$ being a function which does not vanish in $\bar{G}^*$.

Condition (21) clearly indicates that at those points in $\bar{G}^*$ which are not equilibrium states of system (A), the angle between the direction of the field of system (A) and that of system $(\tilde{A})$ retains a constant sign.

Together with the parametric equations of the arc l_1

$$x = f_1(s), \quad y = g_1(s)$$

we will consider parametric equations of the arc l_2

$$x = f_2(\bar{s}), \quad y = g_2(\bar{s}), \quad \bar{a} \leqslant \bar{s} \leqslant \bar{b}.$$

The parameter s along the arc l_2 is so chosen that the paths of system (A) intersecting the arcs l_1 and l_2 make with these arcs angles of the same sign. This clearly implies that the determinants

$$D_1 = \begin{vmatrix} P(f_1, g_1) & Q(f_1, g_1) \\ f_1' & g_1' \end{vmatrix} \quad \text{and} \quad D_2 = \begin{vmatrix} P(f_2, g_2) & Q(f_2, g_2) \\ f_2' & g_2' \end{vmatrix} \qquad (22)$$

both have the same sign. Suppose that $D_1 > 0$ and $D_2 > 0$ (this corresponds to the case schematically shown in Figure 12).

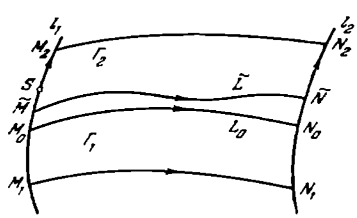

FIGURE 12

Let $\bar{s}_1$ and $\bar{s}_2$ be the values of the parameter $\bar{s}$ corresponding to the points N_1 and N_2 of the arc l_2. Also let M_0 be some point of the arc l_1 which lies between M_1 and M_2, s_0 the corresponding value of the parameter s $(s_1 < s_0 < s_2)$, L_0 the path of system (A) which passes through the point M_0 for $t = t_0$, N_0 the point at which the path L_0 meets the arc l_2 for $t = \tau(s_0)$, $\bar{s}_0$ the value of the parameter s corresponding to the point N_0 $(\bar{s}_1 < s_0 < \bar{s}_2)$. The section M_0N_0 of the path L_0 clearly partitions the elementary quadrangle Γ into two elementary quadrangles Γ_1 and Γ_2 and is part of the boundary of the two quadrangles (Figure 12). The points of the quadrangle Γ_1 (or Γ_2) are described by the coordinates

$$x = \Phi(t, s), \quad y = \Psi(t, s),$$

where

$$s_1 \leqslant s \leqslant s_0, \quad t_0 \leqslant t \leqslant \tau(s)$$

(or $s_0 \leqslant s \leqslant s_2$, $t_0 \leqslant t \leqslant \tau(s)$, respectively). The boundary of the quadrangle Γ_1 (Γ_2) clearly includes the section N_1N_0(N_0N_2) of the arc l_2 corresponding to the values $\bar{s}_1 \leqslant \bar{s} \leqslant \bar{s}_0$ ($\bar{s}_0 \leqslant \bar{s} \leqslant s_2$) of the parameter $\bar{s}$. The sections M_0M_2 and N_0N_2 of the arcs l_1 and l_2, respectively, lie on the positive side of the path L_0 and the sections M_0M_1 and N_0N_1 fall on its negative side.

Let S be some point of the section M_0M_2 of the arc l_1, other than M_2, and let s^* be the corresponding value of the parameter s. Clearly, $s_0 < s^* < s_2$.

Lemma 10. There exists $\delta > 0$ such that if system $(\tilde{\mathrm{A}})$ is δ-close to system (A) in $\bar{G}^*$ and at any point in $\bar{G}^*$

$$P\tilde{Q} - Q\tilde{P} > 0, \tag{23}$$

then any path $\tilde{L}$ of this system which at $t = t_0$ meets the section M_0S of the arc l_1 at the point $\tilde{M}$ will cross the section N_0N_2 of the arc l_2 at the point $\tilde{N}$ at $t = \tilde{\tau}$, so that the section $\tilde{M}\tilde{N}$ of $\tilde{L}$ has no common points with the arcs l_1 and l_2, except its two end points, and is completely contained in the elementary quadrangle Γ_2. A similar proposition holds true if the point s lies on the section M_0M_1 of the arc l_1 and $P\tilde{Q} - Q\tilde{P} < 0$. In this case, the section $\tilde{M}\tilde{N}$ of $\tilde{L}$ is entirely contained in the quadrangle Γ_1.

Proof. By the remark to Lemma 7, there exists $\delta > 0$ such that if system $(\tilde{\mathrm{A}})$ is δ-close to system (A) and its path $\tilde{L}$ passes through some point $\tilde{M}$ of the arc M_0S at $t = t_0$, then at $t = \tilde{\tau}$ the path $\tilde{L}$ meets the arc l_2 at some point $\tilde{N}$ so that the section $\tilde{M}\tilde{N}$ of $\tilde{L}$ has no common points with the arcs l_1 and l_2, except its end points, and is entirely contained in the starting quadrangle Γ. We will now show that if inequality (23) is additionally satisfied, the section $\tilde{M}\tilde{N}$ of $\tilde{L}$ is entirely contained in Γ_2.

Let T denote the parameter (time) along the path $\tilde{L}$ (so as to distinguish it from the parameter t of the paths of system (A)). The corresponding motion along the path $\tilde{L}$ is described by the equations

$$x = \tilde{\varphi}(T), \quad y = \tilde{\psi}(T). \tag{24}$$

The section $\tilde{M}\tilde{N}$ of $\tilde{L}$ is generated as T goes from t_0 to $\tilde{\tau}(\tilde{s})$, where $\tilde{s}$ is the value of the parameter on the arc l_1 corresponding to the point $\tilde{M}$. Because of the particular choice of δ, this section of $\tilde{L}$ is completely contained in the starting quadrangle Γ. Indeed, for any T, $t_0 \leqslant T \leqslant \tilde{\tau}(\tilde{s})$,

$$x = \Phi(t, s), \quad y = \Psi(t, s),$$

where s and t are some numbers satisfying the inequalities

$$s_1 \leqslant s \leqslant s_2, \quad t_0 \leqslant t \leqslant \tau(s). \tag{25}$$

This evidently means that the equations

$$\Phi(t, s) = \tilde{\varphi}(T), \quad \Psi(t, s) = \tilde{\psi}(T) \tag{26}$$

for any T from the interval $[t_0, \tilde{\tau}(\tilde{s})]$ have a unique solution for t and s in the region (25). We write this solution in the form

$$t = t(T), \quad s = s(T).$$

Let us calculate $\frac{ds}{dT}$. Differentiation of (26) with respect to T gives

$$\begin{aligned} \Phi_s'(t, s)\frac{ds}{dT} + \Phi_t'(t, s)\frac{dt}{dT} &= \tilde{\varphi}'(T), \\ \Psi_s'(t, s)\frac{ds}{dT} + \Psi_t'(t, s)\frac{dt}{dT} &= \tilde{\psi}'(T). \end{aligned} \tag{27}$$

The determinant of this system is

$$D = \Phi_s'(t, s)\Psi_t'(t, s) - \Phi_t'(t, s)\Psi_s'(t, s).$$

By QT, §3.5, Lemma 6 and Lemma 7,

$$D = -\begin{vmatrix} P(f_1, g_1) & Q(f_1, g_1) \\ f_1'(s) & g_1'(s) \end{vmatrix} I = -D_1 I,$$

where

$$I = e^{\int_{t_0}^{t}[P_x'(\varphi, \psi) + Q_y'(\varphi, \psi)]dt} > 0.$$

Since by assumption $D_1 > 0$, we have $D < 0$ and (27) is a Cramer system. Therefore,

$$\frac{ds}{dT} = \frac{\tilde{\varphi}'(T)\Psi_t'(t, s) - \tilde{\psi}'(T)\Phi_t'(t, s)}{-D_1 I}.$$

But

$$\begin{aligned} \Phi_t'(t, s) &= \varphi_t'(t - t_0, f_1(s), g_1(s)) = P(x, y), \\ \Psi_t'(t, s) &= \psi_t'(t - t_0, f_1(s), g_1(s)) = Q(x, y), \\ \tilde{\varphi}'(T) &= \tilde{P}(x, y), \quad \tilde{\psi}'(T) = \tilde{Q}(x, y). \end{aligned}$$

Therefore

$$\frac{ds}{dT} = \frac{\tilde{P}(x, y)Q(x, y) - P(x, y)\tilde{Q}(x, y)}{-P_1 I}.$$

Using (23) and the inequality $D = -D_1 I < 0$ we conclude that at any point of the section $\tilde{M}\tilde{N}$ of the path $\tilde{L}$, i.e., for all T, $t_0 \leqslant T \leqslant \tilde{\tau}(\tilde{s})$, $\frac{ds}{dT} > 0$, i.e., $s(T)$ is an increasing function of T. Since $T = t_0$, $s = \tilde{s} > s_0$, we conclude that for $t_0 < T \leqslant \tilde{\tau}(\tilde{s})$ also $s(T) > s_0$. This evidently means that the entire section $\tilde{M}\tilde{N}$ of $\tilde{L}$ lies inside the elementary quadrangle Γ_2. This completes the proof.

The next lemma is concerned with a single arc without contact, but the paths are assumed to meet it twice. Thus, let M_1 and M_2 be two interior points of the arc l; suppose that any path of the system (A) that meets the

section M_1M_2 of the arc l at $t = t_0$ crosses the arc l again at some $\tau = \tau(s) > t_0$. At the intermediate values of t, $t_0 < t < \tau(s)$, these paths, however, have no common points with the arc l.

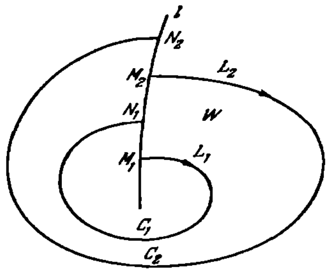

FIGURE 13

Let the paths L_1 and L_2 passing through the points M_1 and M_2, respectively, cross the arc l for the second time at the points N_1 and N_2, which are again different from the end-points of the arc l (if L_1 or L_2 is a closed path, M_1 and N_1, or respectively M_2 and N_2, coincide). We assume that the sections M_1M_2 and N_1N_2 of the arc l have common points, i.e., they intersect, since otherwise we could have reduced the treatment to the previous case of two distinct arcs without contact l_1 and l_2.

Let C_i $(i = 1, 2)$ be a simple closed curve which coincides with the path L_i if this path is closed; otherwise, this simple curve is made up of a section M_iN_i of the path L_i and a section M_iN_i of the arc l (Figure 13; compared QT, §3.9).

Let W be a region whose boundary is formed by two simple closed curves C_1 and C_2. This region, together with its boundary, is entirely contained in G^*. As before, let system $(\tilde{A})$ be δ-close in $\bar{G}^*$ to system (A) and let $\delta > 0$ be so small that the arc l remains an arc without contact for the paths of $(\tilde{A})$.

Let $\tilde{M}_1$ be a point of the arc l_1 sufficiently close to M_1, and $\tilde{M}_2$ a point of the arc l_1 sufficiently close to M_2; let $\tilde{L}_1$ and $\tilde{L}_2$ be the paths of system (A) which at $t = t_0$ pass through the points $\tilde{M}_1$ and $\tilde{M}_2$, respectively. By Lemma 5 we readily see that when the points $\tilde{M}_1$ and $\tilde{M}_2$ are sufficiently close to M_1 and M_2, and δ is sufficiently small, the paths $\tilde{L}_1$ and $\tilde{L}_2$ at $\tilde{\tau}_1$ and $\tilde{\tau}_2$ ($\tilde{\tau}_1 > t_0$ and $\tilde{\tau}_2 > t_0$), respectively, will again cross the arc l at points $\tilde{N}_1$ and $\tilde{N}_2$. The sections $\tilde{M}_1\tilde{N}_1$ and $\tilde{M}_2\tilde{N}_2$ of the paths $\tilde{L}_1$ and $\tilde{L}_2$ have no common points with the arc l, except the end points. Let $\tilde{C}_i$ $(i = 1, 2)$ be a simple closed curve which coincides with $\tilde{L}_i$ if this is a closed path and otherwise is made up from a section $\tilde{M}_i\tilde{N}_i$ of the path $\tilde{L}_i$ and a section $\tilde{M}_i\tilde{N}_i$ of the arc l. Let $\tilde{W}$ be the region formed by the simple closed curves $\tilde{C}_1$ and $\tilde{C}_2$.

Lemma 11. For any $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that if system $(\tilde{A})$ is δ-close in $\bar{G}^*$ to system (A) and the points $\tilde{M}_1$ and $\tilde{M}_2$ lie in $U_\eta(M_1)$ and $U_\eta(M_2)$, respectively, then

(a) the points $\tilde{N}_1$ and $\tilde{N}_2$ lie in $U_\varepsilon(N_1)$ and $U_\varepsilon(N_2)$, respectively;

(b) each path passing through a point in $\tilde{W}$ crosses the arc l both for increasing and decreasing t;

(c) $\tilde{W}$ is ε-close to W.

The validity of this lemma is readily established by drawing an auxiliary arc without contact which partitions $\tilde{W}$ into two elementary quadrangles, to which Lemma 7 is then applied.

Note that for W and $\tilde{W}$ the propositions of Lemma 8 in general do not hold true, i.e., without additional assumptions regarding the exact nature of the paths in W we cannot maintain that for any small η and δ the partitions of W and $\tilde{W}$ by the trajectories of systems (A) and $(\tilde{A})$ are ε-identical.

In conclusion of this chapter we will prove another lemma which is concerned with regions delimited by two cycles without contact.

Let C_1 and C_2 be two simple closed curves, which are cycles without contact for the paths of system $(\tilde{A})$; one of these curves, C_2 say, lies inside the other curve. Suppose that each path L of system (A) passing at $t = t_0$ through the point M of the cycle C_1 intersects for some $\tau = \tau(M) > t_0$ the cycle C_2 at the point N, so that the section MN of the path L has no common points with the cycles C_1 and C_2, other than its end-points (Figure 14). Let W be a closed annular region between the cycles C_1 and C_2, so that $W \subset G^*$.

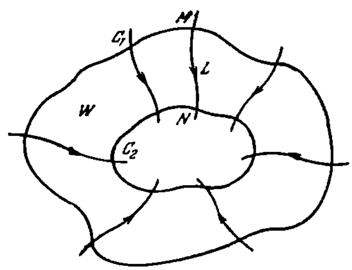

FIGURE 14

Lemma 12. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if system $(\tilde{A})$ is δ-close to system (A) in $\bar{G}^*$, then

(a) cycles C_1 and C_2 are cycles without contact for the trajectories of system $(\tilde{A})$;

(b) the partition of $\bar{W}$ by paths of system $(\tilde{A})$ is ε-identical to the partition of this region by paths of system (A).

Proof. Draw any two paths which intersect the cycle without contact C_1. They clearly partition $\bar{W}$ into two elementary quadrangles. Applying Lemma 8 to each of these quadrangles, we readily verify Lemma 12.

Chapter III

THE SPACE OF DYNAMIC SYSTEMS AND STRUCTURALLY STABLE SYSTEMS

INTRODUCTION

In this chapter we define the concepts of a structurally stable system and a structurally stable path and establish some of their elementary properties. The concept of a structurally stable dynamic system is in fact the cornerstone of this book. Exact definitions are given in §6 (Definitions 10 and 12). Roughly speaking, we say that system (A) is structurally stable in some two-dimensional region W if a sufficiently close system ($\tilde{A}$) partitions W (or some close region $\tilde{W}$) into paths in a manner which is topologically identical to the partition of W by system (A), and an infinitesimal translation suffices to change over from one partition to the other. It can be shown that structurally stable systems constitute, so to say, a majority in the set of all dynamic systems. Indeed, a given dynamic system is structurally stable as a rule, and structurally unstable systems are an exception. Structurally stable systems are of considerable importance in the analysis of physical problems.

Chapter III, §5 is of introductory nature. Its aim is to define a metric in the set of dynamic systems in a plane region or on a sphere, so as to convert this set into a metric space. The metric is introduced in the natural, and apparently the simplest, way. The aim of the metric-space approach to the set of dynamic systems is to permit operating with geometrical concepts, which are intrinsically less abstract in discussion.

In §6 some basic definitions are introduced: the definition of a structurally stable system in a plane region (§6.1) and on a sphere (§6.2), and the definition of relative structural stability.

In §7, structurally stable and structurally unstable paths are defined. A path L of system (A) is said to be structurally stable if system (A) is structurally stable in some neighborhood of L. It is proved that if a system is structurally stable in some region, all its paths in that region are structurally stable (Lemma 1). Therefore, if there is at least one structurally unstable path in some region, the system is structurally unstable in that region. It is proved (Theorem 10) that a structurally stable system may have only a finite number of states of equilibrium in a closed region. Finally, the multiplicity of an equilibrium state is defined and it is shown that a structurally stable equilibrium state $M_0(x_0, y_0)$ is of necessity simple, i.e.,

$$\Delta = \begin{vmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{vmatrix} \neq 0.$$

§5. THE SPACE OF DYNAMIC SYSTEMS

1. The space of dynamic systems in a plane region

In this section we will operate with metric spaces whose points are identified with dynamic systems. The introduction of these spaces lends a more graphic geometric form to the fundamental concepts and arguments of our theory.

We are concerned with dynamic systems in a plane region or on a sphere. In what follows, when dealing with dynamic systems on a plane, we invariably assume that all the relevant systems are defined in the same closed region $\bar{G}$ of the plane. If necessary, we will assume, without mentioning it explicitly, that the systems are also defined in a larger open domain (which contains $\bar{G}$), but the analysis will always be confined to $\bar{G}$. We will often have to consider dynamic systems (defined in $\bar{G}$) in some closed or open subregions of $\bar{G}$. In this case, we always assume that the closures of these subregions are entirely contained in G, i.e., they lie at a finite distance from the boundary of $\bar{G}$.

All dynamic systems of a given class k (k is a fixed natural number) or of the analytical class in $\bar{G}$ will be treated as points of some space. Let r be a given natural number, $r \leqslant k$. Let further M_1 and M_2 be two points in our space, i.e., two dynamic systems

$$\frac{dx}{dt} = P_1(x, y), \quad \frac{dy}{dt} = Q_1(x, y), \tag{M$_1$}$$

$$\frac{dx}{dt} = P_2(x, y), \quad \frac{dy}{dt} = Q_2(x, y). \tag{M$_2$}$$

Consider the maximum of the absolute value of the difference between the functions P_1 and P_2 in $\bar{G}$, i.e.,

$$\max_{(x, y)\in\bar{G}} |P_1(x, y) - P_2(x, y)|, \tag{1}$$

and also the maxima of the absolute values of the differences between the corresponding derivatives of these functions to order r, inclusive, i.e.,

$$\max_{(x, y)\in\bar{G}} |P^{(k+l)}_{1x^k y^l}(x, y) - P^{(k+l)}_{2x^k y^l}(x, y)| \qquad (k+l = 1, 2, \ldots, r), \tag{2}$$

and similar expressions for the functions Q_1 and Q_2 and their derivatives, i.e.,

$$\max_{(x, y)\in\bar{G}} |Q_1 - Q_2| \tag{3}$$

and

$$\max_{(x, y)\in G} |Q^{(k+l)}_{1x^k y^l} - Q_{2x^k y^l}| \qquad (k+l = 1, 2, \ldots, r). \tag{4}$$

The largest of the numbers (1)—(4) is taken as the distance between the points M_1 and M_2 in the space of dynamic systems. All the fundamental axioms of metric spaces are readily verified.

The space of dynamic systems of class k (or of analytical class) with the above maximum metric is designated $R_k^{(r)}$ (or $R_a^{(r)}$, respectively). Clearly

the set of all dynamic systems of class k which are δ-close to rank r in $\bar{G}$ to the system of class k

$$\frac{dx}{dt}=P_0(x, y), \quad \frac{dy}{dt}=Q_0(x, y)$$

constitutes a δ-neighborhood of the point M_0 in the space $R_k^{(r)}$. A similar proposition is true for points in the space $R_a^{(r)}$. For a fixed k, we may clearly consider the spaces

$$R_k^{(1)}, R_k^{(2)}, \ldots, R_k^{(k)}.$$

They consist of the same elements (dynamic systems of class k defined in $\bar{G}$), but have different metrics. The δ-neighborhoods defined by these metrics are thus different. Indeed, let $U_\delta(M \mid R)$ denote the δ-neighborhood of the point M in metric space R, and let $1 \leqslant r_1 < r_2 \leqslant k$. Consider a dynamic system M_0 of class k and the neighborhoods

$$U_\delta(M_0 \mid R_k^{(r_1)}) \text{ and } U_\delta(M_0 \mid R_k^{(r_2)}).$$

Clearly $U_\delta(M_0 \mid R_k^{(r_2)}) \subset U_\delta(M_0 \mid R_k^{(r_1)})$, but the reverse inclusion is not always true. The space $R_1^{(1)}$ wil be designated R_1 (without superscript). The space R_1 clearly contains each of the spaces $R_k^{(r)}$ $(k = 1, 2, \ldots)$. In the case of dynamic systems of the analytical class, we can consider an infinite sequence of spaces

$$R_a^{(1)}, R_a^{(2)}, \ldots, R_a^{(r)}, \ldots$$

As before, all these spaces consist of the same elements but have different metrics. Let now $k_1 < k_2$ and choose a fixed r. Since any system of class k_2 is also a system of class k_1, and an analytical system is a system of any class, we clearly have

$$R_{k_1}^{(r)} \supset R_{k_2}^{(r)} \supset R_a^{(r)}. \tag{5}$$

It is readily seen that any space in (5) is a subspace of all the preceding spaces in the sense that the distance between any two elements defined in this space coincides with the distance between these elements in the enclosing space.

2. The space of dynamic systems on a sphere

Dynamic systems on a sphere are generally defined in terms of open coverings of the sphere (see QT, Appendix, §7.3). However, for purposes of defining a metric in the set of dynamic systems on a sphere, it is better to consider closed coverings. We will define these closed coverings as follows: let S be a sphere (e.g., a sphere in the three-dimensional space R^3 described by the equation $x^2 + y^2 + z^2 = 1$), and

$$\Sigma = \{G_1, G_2, \ldots, G_N\}$$

some open covering of the sphere. If the closure $\bar{G}_i$ of each G_i ($i = 1, 2, \ldots, N$) is homeomorphic to a closed region H_i of the plane (u_i, v_i), we say that the covering

$$\bar{\Sigma} = \{\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_N\}$$

constitutes a closed covering of the sphere S. Under this definition, each closed covering of the sphere is related to some open covering, but not every open covering corresponds to a closed covering. Thus, none of the G_i can be a pierced sphere (i.e., a sphere with a single point removed). Note, however, that this restriction of the concept of a closed covering does not restrict the class of dynamic systems being considered.

A dynamic system on a sphere is defined in terms of closed coverings precisely in the same way as it is defined in terms of open coverings (see QT, §2.2). Consider a closed covering

$$\bar{\Sigma} = \{\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_N\}$$

of the sphere S. In each of the $\bar{G}_i$ we define a local system of coordinates u_i, v_i. To this end, we consider a mapping of some region $\bar{H}_i$ of the plane (u_i, v_i) onto the region $\bar{G}_i$ of the sphere S, defined by the equalities

$$x = \varphi_i(u_i, v_i), \quad y = \psi_i(u_i, v_i), \quad z = \chi_i(u_i, v_i). \tag{6}$$

The mapping (6) should satisfy the following conditions:

1) This is a topological mapping of the plane region $\bar{H}_i$ onto $\bar{G}_i$;

2) the functions $\varphi_i, \psi_i, \chi_i$ are functions of class $k + 1$, if the dynamic system is of class k, and analytical, if the dynamic system is analytical;

3) the functional determinants

$$\frac{D(\varphi_i, \psi_i)}{D(u_i, v_i)}, \quad \frac{D(\varphi_i, \chi_i)}{D(u_i, v_i)}, \quad \frac{D(\psi_i, \chi_i)}{D(u_i, v_i)}$$

do not vanish simultaneously anywhere in $\bar{H}_i$.

A dynamic system (A) of class k (or of the analytical class) on a sphere is defined by specifying in each $\bar{H}_i$ ($i = 1, 2, \ldots, N$) a dynamic system of class k (or of the analytical class, respectively)

$$\frac{du_i}{dt} = U_i(u_i, v_i), \quad \frac{dv_i}{dt} = V_i(u_i, v_i).$$

Moreover, in each region

$$\bar{W}_{jk} = \bar{H}_j \cap \bar{H}_k$$

the dynamic systems (A_j) and (A_k) are transformed into one another by the same transformation which transforms the coordinates u_j, v_j into u_k, v_k (see QT, §1.10, and also §2.2, Definition 1).

A dynamic system (A) defined in this way is described in terms of a system K of local coordinates on a sphere, which in its turn is described by the covering $\bar{\Sigma}$ and equations (6). The same dynamic system (A) can be specified using any other system K^* of local coordinates defined by the

closed covering

$$\overline{\Sigma}^* = \{\overline{G}_1^*, \overline{G}_2^*, \ldots, \overline{G}_{N^*}^*\}$$

and equations analogous to (6) (see QT, §2.2).

To define a metric in the set of dynamic systems on a sphere and thus convert this set into a metric space, we will describe all the dynamic systems using one fixed system K of local coordinates on a sphere. Suppose this system is defined by a closed covering $\overline{\Sigma} = \{\overline{G}_1, \ldots, \overline{G}_N\}$ and a set of equations (6). Consider two dynamic systems (A) and ($\tilde{\text{A}}$) of class k (or of the analytical class). Let system (A) be described by the equations

$$\frac{du_i}{dt} = U_i(u_i, v_i), \quad \frac{dv_i}{dt} = V_i(u_i, v_i) \qquad (\text{A}_i)$$

($i = 1, 2, \ldots, N$) and system ($\tilde{\text{A}}$) by the equations

$$\frac{du_i}{dt} = \tilde{U}_i(u_i, v_i), \quad \frac{dv_i}{dt} = \tilde{V}_i(u_i, v_i). \qquad (\tilde{\text{A}}_i)$$

Choose a natural number $r \geqslant 1$, such that $r \leqslant k$, if k is the class of the system (for analytical systems, any r can be chosen). Consider the numbers

$$\max_{(u_i, v_i) \in \overline{H}_i} |U_i(u_i, v_i) - \tilde{U}_i(u_i, v_i)| \qquad (i = 1, 2, \ldots, N),$$

and also the numbers

$$\max_{(u_i, v_i) \in \overline{H}_i} |U^{(k+l)}_{u_i^k v_i^l}(u_i, v_i) - \tilde{U}^{(k+l)}_{u_i^k v_i^l}(u_i, v_i)| \ (i = 1, 2, \ldots, N, \ k + l = 1, 2, \ldots, r),$$

as well as similar expressions for the functions V_i and $\tilde{V}_i$ and their partial derivatives to order r, inclusive. The largest of all these numbers is taken as the distance between the dynamic systems (A) and ($\tilde{\text{A}}$). All the axioms of a metric space are readily seen to hold true.

The space of dynamic systems of class k (or of the analytical class) on a sphere with this maximum metric is designated $R_k^{(r)}$ (or $R_a^{(r)}$). The remarks at the end of §5.1 pertaining to analogous spaces of dynamic systems on a plane are applicable to the spaces $R_k^{(r)}$ and $R_a^{(r)}$ of dynamic systems on a sphere.

Two dynamic systems (A) and ($\tilde{\text{A}}$) of class k (or the analytical class) on a sphere are said to be δ-close to rank r ($r \leqslant k$), if the distance between them in space $R_k^{(r)}$ ($R_a^{(r)}$) is less than δ.

R e m a r k. The metric defined in the set of dynamic systems essentially depends on the fixed system K of local coordinates on the sphere. The distances between the dynamic systems (A) and ($\tilde{\text{A}}$) defined in this way using different systems of local coordinates K and K^* are in general different. It can be seen, however, that the metrics defined by various systems of local coordinates are all equivalent, i.e., they induce the same topology in the space of dynamic systems.* This is obvious from the following proposition: for any $\delta > 0$, there exists $\delta^* > 0$ such that any dynamic system ($\tilde{\text{A}}$) which is δ^*-close to system (A) in the metric defined

* See Aleksandrov, P.S. Combinatorial Topology, Chapter 1, §2:3.

by the local coordinate system K^* is δ-close to (A) in the metric defined by the coordinate system K.

This proposition can be proved without difficulty, and is left as an exercise to the reader.

§6. DEFINITION OF A STRUCTURALLY STABLE DYNAMIC SYSTEM

1. Dynamic systems on a plane

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dT} = Q(x, y)$$

be a dynamic system defined in a bounded closed region $\bar{G}$, and W a closed or open subregion of G.*

Definition 10. A dynamic system (A) *is said to be structurally stable in* $W \subset G$ *if there exists an open domain* H *containing* $\bar{W}$,

$$\bar{W} \subset H \subset \bar{H} \subset G,$$

which satisfies the following condition: for any $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if system* $(\tilde{A})$ *is* δ*-close to system* (A) *in* $\bar{G}$, *one can find a region* $\tilde{H}$ *for which*

$$(\tilde{H}, \tilde{A}) \overset{\varepsilon}{\equiv} (H, A) \tag{1}$$

(see §4.1, Definition 9).

If system (A) is not structurally stable in region W, it is said to be structurally unstable in that region.

Evidently a dynamic system (A) is structurally unstable in W if for any H, $W \subset H \subset \bar{H} \subset G$, there exists $\varepsilon_0 > 0$ with the following property: for any $\delta > 0$ and any $\tilde{H}$, there exists a system $(\tilde{A})$ δ-close to (A) such that the partition of $\tilde{H}$ by the paths of $(\tilde{A})$ is not ε_0-identical to the partition of H by the paths of (A).

It follows from Definition 10 that if system (A) is structurally stable in W, the topological structure of the partition of some neighborhood H of W by the paths of system (A) does not change in a certain sense on passing to a sufficiently close system $(\tilde{A})$, or more precisely, an infinitesimal translation will transform H into $\tilde{H}$ so that the paths of (A) coincide with the paths of $(\tilde{A})$. This property explains the term structurally stable system. An alternative term used in the Russian literature is a coarse system, which implies that the topological structure of the partition of a given region by paths is not affected by small changes in system (A) or, in other words, the structure can resist small disturbances in system (A).

Examples of structurally stable systems are considered at a later stage. Here we will analyze a structurally unstable system.

* As we have noted in the previous section, it is implicitly assumed that the closures of the relevant subregions are entirely contained in G, i.e., they are at a finite distance from the boundary of G.

Example 3. Consider the system

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = +x. \tag{2}$$

It is defined on the entire plane, so that any closed region on the plane can be taken as $\bar{G}$. Let $\bar{G}$ be the region defined by the inequality

$$x^2 + y^2 \leqslant 100.$$

W is defined by the inequality

$$x^2 + y^2 \leqslant 16. \tag{3}$$

We will now show that system (2) is structurally unstable in region (3). Define H as

$$x^2 + y^2 \leqslant 25. \tag{4}$$

Any positive number can be taken as ε_0. The paths of system (2) are the point O (0, 0) and the circles

$$x = c\cos t, \quad y = c\sin t$$

centered at the origin (Figure 15).

Together with system (2) consider the modified system

$$\frac{dx}{dt} = -y + \mu x, \quad \frac{dy}{dt} = x + \mu y. \tag{5}$$

For small μ, system (5) is arbitrarily close in $\bar{G}$ to system (2), the point O (0, 0) is the focus of system (5), and all the other paths of this system are spirals (see QT, §1.14, Example 4). Regardless of what $\tilde{H}$ we choose, its partition by the paths of system (5) clearly cannot be ε_0-identical to the partition of region (4) by the paths of system (2). Indeed, all the paths of system (2) in region (4), except the equilibrium point O, are closed, whereas system (5) has no closed paths. Hence, there exists no mapping of region (4) transforming the paths of system (2) into the paths of system (5). This establishes the structural instability of system (2) in region (3).

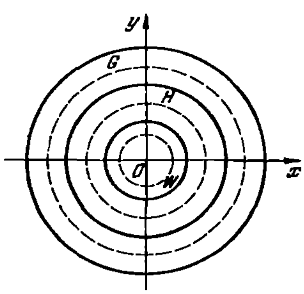

FIGURE 15

Before proceeding any further with our analysis, we would like to offer some background information on structurally stable systems. Structurally stable systems were first considered in 1937 by A. A. Andronov and L. S. Pontryagin /4/, who originally called them systèmes grossier or coarse systems. They considered, however, dynamic systems in a particular region W lying inside a cycle without contact. The definition

of a structurally stable system in such a region is very simple. Indeed, system (A) defined in a region W lying inside a cycle without contact Γ is said to be structurally stable in that region if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any system $(\tilde{A})$ δ-close to system (A)

$$(W, A) \stackrel{\varepsilon}{\equiv} (W, \tilde{A}).$$

Substantial simplification is achieved here because we do not have to consider either the region $H \supset W$ or the corresponding $\tilde{H}$: the entire analysis is confined to W. It can be shown that in region W lying inside a cycle without contact Γ both definitions — Definition 10 and the definition in /4/ — are equivalent. Unfortunately, in case of a general region W, a structurally stable system cannot be defined without introducing an auxiliary region H (see Appendix, subsection 5).

Structurally stable systems (in a given region) are in a sense the simplest dynamic systems, just as the simple roots of a function can be regarded as the most elementary among all the roots or simple (nonmultiple) intersection points of two curves are the most elementary among all the intersection points. These three concepts — a simple root of a function, a simple intersection point of two curves, and a structurally stable system in a certain region — are analogous in the sense that under small disturbances (of the function, the pair of curves, or the dynamic system) the object remains intrinsically unaffected and only a small translation or shift is observed (see remark to Theorem 5, §1, Remark 3 to Theorem 6, §2, and the definition of a structurally stable system).

In what follows (Chapter VI, §18.2, Theorem 23) we will derive the necessary and sufficient conditions for a dynamic system to be structurally stable in a given region W. These conditions, like the conditions of simplicity of a root or simplicity of an intersection point of two curves, are analytically expressed in the form of inequalities between certain quantities which are continuous functions of the right-hand sides of the dynamic system. Hence it follows that the systems which are structurally stable in W (see §5) form an open set in the metric space R_1.

The definition of a structurally stable system using the metric space R_1 can naturally be formulated as follows:

The system (A) *corresponding to a point $M \in R_1$ is said to be structurally stable in W if there exists a region H, $\overline{W} \subset H \subset \overline{H} \subset G$, such that the following condition is satisfied: for any $\varepsilon > 0$ we can choose $\delta > 0$ such that if $\tilde{M} \in U_\delta(M)$, the following relation holds true for the system* $(\tilde{A})$ *corresponding to the point $\tilde{M}$ and some region $\tilde{H} \in G$:*

$$(\tilde{H}, \tilde{A}) \stackrel{\varepsilon}{\equiv} (H, A).$$

Note that it does not follow directly from the definition of a structurally stable system that structurally stable systems (in W) form an open subset in R_1. This proposition follows, as we have remarked above, from the analytical conditions of structural stability.

We conclude this section with two simple, but highly important lemmas. The proof is self-evident and is therefore omitted.

Lemma 1. If system (A) *is structurally stable in W, it is structurally stable in any subregion W_1 of W.*

The next lemma deals with substitution of variables. Consider a regular mapping of class 2,

$$u=\varphi(x, y), \quad v=\psi(x, y), \tag{6}$$

defined in $\bar{G}$, which maps this region into some region $\bar{G}^*$ in the plane (u, v). Any system

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y) \tag{A}$$

is transformed by this mapping into some system

$$\frac{du}{dt}=P^*(x, y), \quad \frac{dv}{dt}=Q^*(x, y), \tag{A*}$$

defined in $\bar{G}^*$ (see §3.2). Let W be some subregion of G and W^* the image of this subregion under mapping (6).

Lemma 2. If system (A) is structurally stable in W, system (A) is stable in W^*.*

Lemma 2 follows from the uniform continuity of the functions φ and ψ.

Equations (6) can be considered as defining a certain substitution of variables. Lemma 2 thus signifies that the structural stability of a dynamic system is a property which is invariant under a substitution of variables of class 2.

2. Structurally stable systems on a sphere

The only case of interest among dynamic systems on a sphere is that when the system is defined on the entire sphere. Indeed, a dynamic system defined in a region whose closure does not coincide with the entire sphere evidently can be considered as a system defined in a plane region (see QT, §2.2, remark following equation (10)).

The definition of structural stability of a dynamic system on a sphere, on the one hand, is substantially simpler than the analogous definition on a plane, since one does not have to consider the auxiliary region H.* On the other hand, the analysis on a sphere involves certain difficulties, since the metric defined in the space of dynamic systems on a sphere depends on the particular system of local coordinates chosen on the sphere (see remark at the end of §5.2). This, however, does not constitute a fundamental difficulty.

First let us define the concept of ε-identity on a sphere, analogous to Definition 9.

Definition 11. Let (D) and ($\tilde{\text{D}}$) be two dynamic systems defined on a sphere S. The partition of the sphere S by the paths of system (D) is said to be ε-identical to the partition by the paths of system ($\tilde{\text{D}}$), or in symbols

$$(S, D) \stackrel{\varepsilon}{\equiv} (S, \tilde{D}),$$

if there exists a mapping of the sphere S onto itself which is an ε-translation and which transforms the paths of system (D) into the paths of system ($\tilde{\text{D}}$).

* Also see the remark in the previous subsection concerning the definition of structural stability of a system inside a cycle without contact.

When speaking of an ε-translation, we naturally assume that some metric is defined on the sphere, either internal or induced by the enclosing euclidean space.

To define a structurally stable system, we introduce some system of local coordinates on the sphere and measure the distances between two dynamic systems relative to this coordinate system (see §5.2).

Definition 12. A dynamic system (D) *on a sphere S is said to be structurally stable if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for any system* $(\tilde{D})$, δ*-close to system* (D), *we have*

$$(S,\ D) \overset{\varepsilon}{\equiv} (S,\ \tilde{D}).$$

Otherwise system (D) is said to be structurally unstable.

If a dynamic system (D) on a sphere is structurally unstable, there exists $\varepsilon_0 > 0$ with the following property: for any $\delta > 0$ there exists a system $(\tilde{D})$ δ-close to (D), such that partitions of the sphere (S) by the paths of (D) and $(\tilde{D})$ are not ε_0-identical.

Definition 12 makes use of a particular system of local coordinates K on the sphere. We now have to establish that the concept of a structurally stable system is in fact independent of the choice of the local system of coordinates on the sphere, or in other words, we have to show that if system (D) is structurally stable when the metric (in the space of dynamic systems) is defined in terms of some system of local coordinates K, it is also structurally stable in the metric defined using any other system of local coordinates K^*. This proposition clearly follows from a previous proposition formulated at the very end of §5 (in the remark to §5.2). Definition 12 is thus meaningful, and structurally stable systems on a sphere can be analyzed in a metric introduced using an arbitrary fixed system of local coordinates.

Let (A) be a dynamic system on a sphere and W some subregion of the sphere whose closure $\overline{W}$ does not coincide with the entire sphere. We choose a system K of local coordinates so that at least one of the components of the closed covering $\overline{\Sigma}$ corresponding to this system, $\overline{G}_1$ say, entirely contains $\overline{W}$, i.e., $\overline{W} \subset G_1$. From the definition of the system of local coordinates and of a dynamic system on a sphere (see §5.2) we see that $\overline{G}_1$ corresponds to some region $\overline{H}_1$ on the plane (u_1, v_1). Let W correspond to some W^* on the plane, $\overline{W}^* \subset H_1$. System (A) on the sphere thus corresponds to some dynamic system (A_1) in $\overline{H}_1$.

Lemma 3. If (A) *is a structurally stable system on a sphere, the dynamic system* (A_1) *is structurally stable in* W^*.

This lemma follows directly from the definitions of structural stability (Definitions 10 and 12) and the uniform continuity of the mapping of $\overline{G}_1$ onto H_1.

In simpler language, Lemma 3 can be formulated as follows: if system (D) is structurally stable on a sphere, it is structurally stable in any subregion of the sphere.

3. Structural stability of dynamic systems in $R_h^{(r)}$ and $R_a^{(r)}$

In the above definitions of structurally stable and structurally unstable dynamic systems, (A) and (D) were assumed to be systems of class 1, and

we have considered all systems δ-close to (A) and (D) to rank 1. In other words, we considered points in the space R_1 with their respective δ-neighborhoods (see §5). In some cases, however, we are not interested in all the possible systems of class 1, but in the systems of some narrower class, e.g., analytical systems or systems defined on a plane whose right-hand sides are polynomials.

Moreover, sometimes we have to consider δ-closeness to some higher rank, and not only to rank 1. This evidently means that the analysis is not confined to R_1, but to some other space $R_k^{(r)}$ or $R_a^{(r)}$. We thus naturally arrive at the concept of relative structural stability of a dynamic system, i.e., structural stability relative to some space in which the given dynamic system is a point.* The corresponding definitions are entirely analogous to Definitions 10 and 12, and we will therefore give only the definition of a structurally stable dynamic system on a plane relative to $R_k^{(r)}$. As before, we consider systems defined in a bounded plane region.

Definition 13. A dynamic system (A) *of class k is said to be structurally stable in W relative to $R_k^{(r)}$ ($W \subset \overline{W} \subset G$; $r \leqslant k$) if there exists an open domain H containing $\overline{W}$, $\overline{W} \subset H \subset \overline{H} \subset G$, which satisfies the following condition: for any $\varepsilon > 0$ one can choose $\delta > 0$ such that if* ($\tilde{A}$) *is a system of class k δ-close to rank r to* (A) *in $\overline{G}$ there exists $\tilde{H}$ for which*

$$(\tilde{H}, \tilde{A}) \stackrel{\varepsilon}{\equiv} (H, A)$$

(also see Definition 10).

Otherwise, system (A) of class k is said to be structurally unstable in W relative to $R_k^{(r)}$.

Using geometrical terminology, we say that a dynamic system (A) corresponding to point M in $R_k^{(r)}$ is structurally stable in W relative to this space if there exists a domain H, $\overline{W} \subset H \subset \overline{H} \subset G$, satisfying the following condition: for any $\varepsilon > 0$ we can choose $\delta > 0$ such that if $\tilde{M} \in U_\delta(M \mid R_k^{(r)})$, then for the system ($\tilde{A}$) corresponding to the point $\tilde{M}$ and some $\tilde{H} \subset G$ we have

$$(\tilde{H}, \tilde{A}) \stackrel{\varepsilon}{\equiv} (H, A).$$

"Simple" structural stability (in the sense of Definition 10) is clearly structural stability relative to the space R_1.

Let k_1 and k_2 be natural numbers, $k_1 < k_2$, (A) a dynamic system of class k_2 (and thus also of class k_1), and r a natural number, $r \leqslant k_1$. System (A) then belongs to both $R_{k_1}^{(r)}$ and $R_{k_2}^{(r)}$. We recall that $R_{k_2}^{(r)} \subset R_{k_1}^{(r)}$. It clearly follows from Definition 13 that if system (A) is structurally stable in W relative to $R_{k_1}^{(r)}$, it is also structurally stable relative to $R_{k_2}^{(r)}$. Similarly, if (A) is an analytical system, then for any natural k and $r \leqslant k$ system (A) belongs both to $R_k^{(r)}$ and to $R_a^{(r)} \subset R_k^{(r)}$. If (A) is structurally stable in W relative to $R_k^{(r)}$, it is also structurally stable relative to $R_a^{(r)}$. All these are particular cases of the following general proposition, which follows directly from the definition of structural stability: if a dynamic system (A) belongs to two spaces one of which is a subspace of the other and if the system is structurally stable in W relative to the enclosing space, it is structurally stable in W relative to the enclosed space.

* Instead we can speak of structural stability relative to a given class of dynamic systems. But then it should be explicitly stated what we mean by δ-closeness (i.e., to what rank).

Let further $r_1 < r_2 \leqslant k$, (A) is a dynamic system of class k, M is the point corresponding to (A) in $R_k^{(r_1)}$ (or in $R_k^{(r_2)}$; we recall that both these spaces are made up of the same points, but the corresponding metrics are different, see §5.1). Then, *if system (A) is structurally stable in W relative to $R_k^{(r_1)}$, it is also structurally stable in W relative to $R_k^{(r_2)}$.* This follows directly from Definition 13 and from the relation

$$U_\delta(M \mid R_k^{(r_2)}) \subset U_\delta(M \mid R_k^{(r_1)})$$

(see §5.1).

Is the reverse also true? In other words, can we maintain that if system (A) is structurally stable in W relative to $R_{k_2}^{(r)}$, it is also structurally stable relative to $R_{k_1}^{(r)}$ $(r \leqslant k_1 < k_2)$ or, alternatively, if system (A) is structurally stable in W relative to $R_k^{(r_2)}$, it is also structurally stable relative to $R_k^{(r_1)}$ $(r_1 < r_2 \leqslant k)$? These propositions clearly do not follow directly from the definition of relative structural stability. Moreover, if system (A) belongs to two spaces R and R^*, such that $R^* \subset R$, then in general system (A) may prove structurally stable in W relative to the enclosed space R^*, whereas it is structurally unstable relative to the enclosing space R.

However, if at the cost of generality we concentrate on the spaces $R_k^{(r)}$ and $R_a^{(r)}$, which are of the main interest in the analysis of dynamic systems, the situation is considerably simplified. It can be shown that if system (A) belongs to one of these spaces, the necessary and sufficient conditions for its structural stability (in W) relative to these spaces are at the same time the necessary and sufficient conditions of its simple structural stability (i.e., structural stability relative to R_1). This will be established in the derivation of the necessary and sufficient conditions of structural stability (§18.4, Remark d); additional proof (in connection with $R_k^{(r)}$ or $R_a^{(r)}$) is naturally required only for the necessary conditions of structural stability. Thus, if system (A) belongs to $R_k^{(r)}$ or $R_a^{(r)}$ and is structurally stable in W relative to the corresponding space, it is simply structurally stable. Therefore, we do not have to consider structural stability relative to these spaces and in what follows we can concentrate on structural stability in the sense of Definition 10.

The situation is clearly the same for dynamic systems on a sphere.

The above arguments pertaining to dynamic systems are analogous to the various arguments offered in §1.4 in connection with roots of functions and at the end of §2 in the analysis of intersection points of two curves. This analogy between dynamic systems, on the one hand, and functions or pairs of curves, on the other, unfortunately breaks down at one significant point. Let us discuss this aspect in some detail.

We will consider the analogy between dynamic systems in a plane region G and pairs of curves $F_1(x, y) = 0$, $F_2(x, y) = 0$. At the end of §2 we considered the multiplicity of the intersection point of two curves relative to a given class $\mathfrak{M}$ of functions. $\mathfrak{M}$ was identified in particular with the class $\mathfrak{M}_n$ of all polynomials in two variables not higher than of degree n and the necessary and sufficient condition of structural stability of the intersection point of two curves relative to this class was established to coincide $(\Delta \neq 0)$ with the condition of structural stability in the sense of Definition 5 (§2.1).

By analogy with the class of functions $\mathfrak{M}_n$ we can consider the class of dynamic systems (on a plane) whose right-hand sides are polynomials not higher than of degree n. We will designate this class of systems by $\mathfrak{A}_n$. The structural stability relative to $\mathfrak{A}_n$ is defined in the usual way (δ-closeness should be considered to rank 1 or to higher rank). The analogy between the pairs of curves in $\mathfrak{M}_n$ and the dynamic systems in $\mathfrak{A}_n$ is unfortunately incomplete. The point is that the attempts to derive the necessary and sufficient conditions of structural stability of dynamic systems relative to the class $\mathfrak{A}_n$ so far have remained unsuccessful. On the other hand, we do not know of any particular example of a system

$$\frac{dx}{dt}=P(x,\ y),\qquad \frac{dy}{dt}=Q(x,\ y)$$

(P and Q are polynomials of not higher than n-th degree) which is structurally stable in some region relative to the class $\mathfrak{A}_n$ and structurally unstable in the sense of Definition 10. The question of the possible existence of these systems and of the necessary and sufficient conditions of structural stability relative to the class $\mathfrak{A}_n$ thus remains open at this stage.

§7. STRUCTURALLY STABLE AND STRUCTURALLY UNSTABLE PATHS. NECESSARY CONDITION OF STRUCTURAL STABILITY OF AN EQUILIBRIUM

1. Structurally stable and structurally unstable paths

Our immediate problem is the derivation of necessary and sufficient conditions of structural stability of dynamic systems on a plane and on a sphere. This problem is discussed in this section and also in Chapters IV, V, and VI. The concepts of *structurally stable* and *structurally unstable paths* introduced in this section considerably simplify the approach to our problem. It is assumed throughout this chapter that all the systems are dynamic systems of first class defined in a fixed plane region $\overline{G}$, and δ-closeness is always interpreted as closeness to rank 1. In other words, the analysis is confined to R_1. When dealing with dynamic systems on a sphere, this is not stated explicitly.

Let (A) be a dynamic system which is structurally stable in a closed or an open region W, and L some complete path of system (A). From the definition of structural stability and Lemma 1, §6 we see that if L is entirely contained in W, then there is a certain neighborhood V of L where the system (A) is structurally stable. Moreover, $\overline{L}\subset V$.*

The reverse, naturally, is not always true: system (A) can be structurally stable in some neighborhood V of L, $V\subset W$, while it is structurally unstable in W.

These considerations lead to the concept of *structurally stable* and *structurally unstable paths*.

Definition 14. A path L of a dynamic system (A) *is said to be structurally stable if there exists a certain neighborhood V, $\overline{L}\subset V\subset\overline{V}\subset G$, where system* (A) *is structurally stable. Otherwise, L is said to be structurally unstable.*

* Any V satisfying the conditions $L\subset V\subset\overline{V}\subset H$, where H is as introduced in Definition 10, meets these requirements.

According to Definition 14, to establish structural stability of a path L it suffices to show that system (A) is structurally stable in some neighborhood V, $\overline{L} \subset V$. To establish structural instability, on the other hand, we have to show that system (A) is structurally unstable in any region which contains L. For this it is necessary and sufficient that system (A) be structurally unstable in any arbitrarily small neighborhood of L. We thus arrive at the following necessary and sufficient condition of structural instability of a path.

A path L is structurally unstable if for any $\varepsilon > 0$ there is a neighborhood V, $\overline{L} \subset V \subset \overline{V} \subset U_\varepsilon(\overline{L})$, where system (A) is structurally unstable.

Lemma 1. If system (A) is structurally stable in W, any path of this system which is entirely contained in W is structurally stable.

This follows directly from Lemma 1, §6 and from Definitions 10 and 14. As we have seen above, the proposition contained in this lemma was in fact the reason for introducing the concept of a structurally stable path. By Lemma 1, if there is at least one structurally unstable path of system (A) in W, the system is structurally unstable in W. The next question to ask is whether absence of structurally unstable paths in W necessarily leads to structural stability of the system as a whole in W. Only the potentially limiting paths are of importance in connection with this problem. As is known (QT, §4.6 and §15.6), these paths include

1) equilibrium states, 2) closed paths, 3) paths which are at the same time α- and ω-separatrices, i.e., α- and ω-orbitally-unstable paths approaching the equilibrium states.

We will now establish which of these three types of paths are structurally stable.

2. Finite number of equilibrium states in a structurally stable system

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system defined in $\overline{G}$, and let $\overline{W}$ be a closed region, $\overline{W} \subset G$.

Theorem 10. If system (A) is structurally stable in $\overline{W}$, it has only a finite number of equilibrium states in $\overline{W}$.

Proof. We will first show that for any (A) and any $\delta > 0$, there exists a dynamic system $(\tilde{A})$ δ-close to (A) which has only a finite number of equilibrium states. Choose some $\delta > 0$. From Weierstrass's theorem (§ 1.1, Theorem 1) there exist two polynomials $P^*(x, y)$, $Q^*(x, y)$, which are $\frac{\delta}{2}$-close in $\overline{G}$ to the functions $P(x, y)$, $Q(x, y)$.

If P^* and Q^* are irreducible, $(\tilde{A})$ can be chosen as the system

$$\frac{dx}{dt} = P^*(x, y), \quad \frac{dy}{dt} = Q^*(x, y).$$

Indeed, this system is $\frac{\delta}{2}$-close, and hence also δ-close, to system (A). Its equilibrium states are determined from the set of equations

$$P^*(x, y)=0, \; Q^*(x, y)=0,$$

and since the largest common divisor $(P^*, Q^*) = 1$, these equations have only a finite number of solutions according to the Bézout theorem (see /12/, Chapter III, §3.1).

Now suppose that P^* and Q^* are not irreducible. They can be written in the form

$$P^*(x, y)=P_1(x, y)R(x, y),$$
$$Q^*(x, y)=Q_1(x, y)R(x, y),$$

where $R(x, y)$ are polynomials of higher than zero degree, and P_1 and Q_1 are irreducible, $(P_1, Q_1)=1$.

Consider the polynomials

$$\tilde{P}(x, y)=P_1(x, y)[R(x, y)+\alpha],$$
$$\tilde{Q}(x, y)=Q_1(x, y)[R(x, y)+\beta],$$

where α and β are real numbers satisfying the following conditions:

1) α and β are sufficiently small, 2) $\alpha\neq\beta$, 3) the polynomial $P_1(x, y)$ is irreducible with $R+\beta$, and the polynomial Q_1 is irreducible with $R+\alpha$.

First we have to show that such real numbers always exist. Let $P_1(x, y) = p_1(x, y)p_2(x, y)\ldots p_s(x, y)$ be the factorization of the polynomial P_1 into irreducible factors. Consider the polynomials

$$R(x, y)+\beta_1, \; R(x, y)+\beta_2, \; \ldots, \; R(x, y)+\beta_s, \; R(x, y)+\beta_{s+1}, \tag{1}$$

where β_i are any sufficiently small different numbers. Suppose that none of the polynomials in (1) is irreducible with $P_1(x, y)$. Then each of these polynomials is reducible at least by one of the polynomials $p_1(x, y), \ldots, p_s(x, y)$. Since the number of polynomials in (1) is one higher than the number of polynomials $p_i(x, y)$, at least two polynomials in (1), $R(x, y)+\beta_k$ and $R(x, y)+\beta_l$, $k\neq l$, say, are reducible by the same polynomial $p_i(x, y)$. Then their difference $\beta_k-\beta_l$ is reducible by $p_i(x, y)$, which is absurd since $\beta_k\neq\beta_l$.

Thus at least one of the polynomials in (1) is irreducible with $P_1(x, y)$. We designate this polynomial as $R(x, y)+\beta$.

Exactly in the same way we can show that there exists a number α such that the polynomials $R+\alpha$ and Q_1 are irreducible.

These α and β clearly can be chosen arbitrarily small and, moreover, they can be taken different. Conditions 1, 2, 3 are thus all satisfied.

We choose α and β sufficiently small so that the polynomials $\tilde{P}$ and $\tilde{Q}$ are $\frac{\delta}{2}$-close to the polynomials P^* and Q^*, respectively, and thus δ-close to the functions P and Q.

Since $\alpha\neq\beta$, the polynomials $R+\alpha$ and $R+\beta$ are irreducible (otherwise, the difference $\alpha-\beta$ would be reducible by a zero degree polynomial, which

is impossible). From $(R+\alpha, R+\beta)=1$, $(P_1, Q_1)=1$ and condition 3 it follows, using a well-known algebraic theorem, that $(\tilde{P}, \tilde{Q})=1$. Then the system

$$\frac{dx}{dt}=\tilde{P}(x, y), \quad \frac{dy}{dt}=\tilde{Q}(x, y)$$

is δ-close to system (A) and has only a finite number of equilibrium states.

We have thus established that for any $\delta>0$ there exists a system $(\tilde{A})$ δ-close to (A) which has only a finite number of equilibrium states. Note that this lemma is true for any system (A), whether structurally stable or unstable.

Let now (A) be a structurally stable system in $\overline{W}$ with an infinite number of equilibrium states in $\overline{W}$. From the definition of structural stability, for any $\varepsilon>0$ there exists $\delta>0$ such that if $(\tilde{A})$ is δ-close to (A) then

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}), \tag{2}$$

where H and $\tilde{H}$ are some regions, $H \supset \overline{W}$.

We have seen that a system $(\tilde{A})$ δ-close to (A) can be chosen so that it has only a finite number of equilibria in a plane. Then, in virtue of (2), H and thus also $\overline{W}$ should contain only a finite number of equilibrium states of (A), which contradicts the starting assumption. Q. E. D.

C o r o l l a r y. If system (A) is structurally stable in some region $\overline{W}$, it has only isolated equilibrium states in that region.

Indeed, by Theorem 10 we conclude that an equilibrium state which is an inside point of $\overline{W}$ is necessarily i s o l a t e d. A boundary point of $\overline{W}$ is also inevitably an isolated equilibrium state. This follows from Theorem 10, since a system which is structurally stable in $\overline{W}$ is evidently structurally stable in some region containing $\overline{W}$.

From Theorem 10 and Definition 14 it follows that only isolated equilibrium states can be structurally stable.

Therefore, as we are concerned with structurally stable equilibrium states, we need only consider isolated points.

3. Multiplicity of an equilibrium state

Consider a dynamic system

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y),$$

defined in $\overline{G}$. Let $M_0(x_0, y_0)$ be an equilibrium state of this system, $M_0 \in G$. This M_0 is an intersection point of two curves

$$P(x, y)=0, \quad Q(x, y)=0. \tag{3}$$

The m u l t i p l i c i t y o f t h e e q u i l i b r i u m s t a t e M_0 is defined as the multiplicity of the intersection point M_0 of the two curves (3) (see §2.1).

D e f i n i t i o n 15. An equilibrium state $M_0(x_0, y_0)$ of a dynamic system is said to be of multiplicity r (or r-tuple) if M_0 is a common point of multiplicity r of curves (3).

An equilibrium state of multiplicity 1 is said to be **simple**.

An equilibrium state M_0 is said to be of infinite multiplicity if M_0 is an intersection point of infinite multiplicity of curves (3).

An equilibrium state M_0 is said to be of multiplicity higher than r if it is of finite multiplicity $r' > r$ or has infinite multiplicity.

Finally, an equilibrium state is said to be multiple if its multiplicity is > 1.

From Definition 5 (§2.1) and Definition 15 we see that if an equilibrium state M_0 is of multiplicity r, then system (A) is a system of class $k \geqslant r$ and the following are satisfied: a) there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any system ($\tilde{A}$) δ_0-close to rank r to system (A) has at most r equilibrium states in $U_{\varepsilon_0}(M_0)$; b) for any $\varepsilon < \varepsilon_0$ and $\delta > 0$ there exists a system ($\tilde{A}$) δ-close to rank r to system (A) which has at least r equilibrium states in $U_\varepsilon(M_0)$.

The following theorem establishes the necessary condition of structural stability of an isolated equilibrium state.

Theorem 11. An isolated equilibrium state $M_0(x_0, y_0)$ is structurally stable only if it is simple (of multiplicity 1), i.e., a necessary condition of structural stability of an equilibrium state is the inequality

$$\Delta = \begin{vmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{vmatrix} \neq 0. \tag{4}$$

Proof. Let an isolated equilibrium state M_0 be structurally stable and multiple. By the definition of structural stability of an equilibrium state, there exists a region H containing M_0 with the following property: for any $\varepsilon > 0$ there is $\delta > 0$, such that if system ($\tilde{A}$) is δ-close to system (A) then

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}), \tag{5}$$

where $\tilde{H}$ is some region. Clearly H can be identified with any sufficiently small neighborhood of M_0. We take $U_{\varepsilon_0}(M_0)$ as H, where $\varepsilon_0 > 0$ is so small that M_0 is the only equilibrium state of (A) inside $U_{\varepsilon_0}(M_0)$ (this ε_0 exists since M_0 is an isolated equilibrium state). Relation (5) is now written in the form

$$(U_{\varepsilon_0}(M_0), A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}). \tag{6}$$

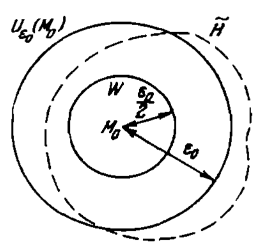

FIGURE 16

Let W be a neighborhood $U_{\varepsilon_0/2}(M_0)$ (Figure 16). Some positive number smaller than $\frac{\varepsilon_0}{4}$ is chosen as ε. System ($\tilde{A}$) is taken to be sufficiently close

to (A), so that relation (6) is satisfied; moreover ($\tilde{A}$) has at least two equilibrium states in W. Such system ($\tilde{A}$) exists because M_0 is a multiple (and not a simple) equilibrium state of (A) (see §2.2, Definition 5 and Theorem 6).

$\tilde{H}$ is generated from $U_{\varepsilon_0}(M_0)$ by an ε-translation, where $\varepsilon < \frac{\varepsilon_0}{4}$. Therefore, as is readily seen, $W = U_{\varepsilon_0/2}(M_0) \subset \tilde{H}$.* Hence, in $\tilde{H}$ there are at least two equilibrium states of ($\tilde{A}$). This contradicts relation (6), since in $U_{\varepsilon_0}(M_0)$ there is only one equilibrium state of (A). Thus, the assumption that M_0 is a structurally stable equilibrium state which is not a simple isolated equilibrium state leads to a contradiction. This completes the proof.

Theorem 11 can be alternatively stated as follows: an isolated equilibrium state $M_0(x_0, y_0)$ for which

$$\Delta = \begin{vmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{vmatrix} = 0$$

(i.e., a multiple state) is not structurally stable.

Thus, in our analysis of structurally stable equilibrium states, we need consider only simple equilibrium states, which is the subject of the next chapter.

* Indeed, take some point $M \in W$ outside $\tilde{H}$. Let f denote the ε-translation which transforms U_{ε_0} into $\tilde{H}$, and let Γ be the boundary of the neighborhood $U_{\varepsilon_0}(M)$, $\tilde{\Gamma}$ the boundary of $\tilde{H}$ ($\tilde{\Gamma} = f(\Gamma)$), and $\tilde{M} = f(M)$. By assumption $\rho(M, \tilde{M}) < \frac{\varepsilon_0}{4}$. The segment $M\tilde{M}$ contains at least one point $\tilde{S}$ of the boundary $\tilde{\Gamma}$. Therefore $\rho(M, \tilde{S}) < \frac{\varepsilon_0}{4}$. Let $\tilde{S} = f(S)$, where $S \in \Gamma$. Then $\rho(S, \tilde{S}) < \frac{\varepsilon_0}{4}$. This leads to the inequality $\rho(S, M) \leqslant \rho(M, \tilde{S}) + \rho(\tilde{S}, S) < \frac{\varepsilon_0}{2}$, which is impossible since $\rho(S, W) = \frac{\varepsilon_0}{2}$.

Chapter IV

EQUILIBRIUM STATES OF STRUCTURALLY STABLE SYSTEMS. SADDLE-TO-SADDLE SEPARATRIX

INTRODUCTION

We have shown in Chapter III (§7.3, Theorem 11) that if a dynamic system is structurally stable in some bounded region, it has only simple equilibrium states in that region. In this chapter we will establish which of these simple equilibrium states are structurally stable equilibria. The chapter is divided into four sections, §8 through §11. In §8 it is proved that a *simple node* (ordinary, dicritical, or confluent) and a *simple focus** are structurally stable states of equilibrium. The proof for these different cases is exactly the same, and we therefore consider only the ordinary node.

In §9 we prove that a *simple saddle point* (i.e., a simple equilibrium state which is a saddle point) is a structurally stable state of equilibrium.

In §10 we consider a simple equilibrium state with pure imaginary characteristic roots, and it is proved that these equilibria are structurally unstable. Incidentally a highly important theorem is proved concerning the creation of a closed path from a multiple focus (Theorem 14). According to this theorem, if O is a *multiple focus* of system (**A**) (i.e., a point with pure imaginary characteristic values which is neither a center nor a center-focus), infinitesimally small increments will convert the system (**A**) into a modified system which has a closed path in any arbitrarily small neighborhood of O.

Equilibrium states do not figure in §11. This section, however, is closely linked with §10, and hence its place in the present chapter. Indeed, §11 deals with a *saddle-to-saddle separatrix*, and it is proved that such a separatrix extending between two saddle points is a structurally unstable path of the dynamic system. For $t \to +\infty$ and $t \to -\infty$ the separatrix goes to saddle points which may be different or coincident.

§8. STRUCTURAL STABILITY OF A NODE AND A SIMPLE FOCUS

1. Canonical system

In this subsection we will review the basic propositions concerning simple states of equilibrium. Detailed proof will be found in QT, Chapter IV.

* A simple node is a simple equilibrium state which is a node (the characteristic values are real numbers of equal sign). A simple focus is an equilibrium state with complex characteristic values, which are not pure imaginary.

Without loss of generality, we consider a simple equilibrium state at the origin, i.e., at the point $O\,(0,0)$. The dynamic system in this case can be written in the form

$$\frac{dx}{dt}=ax+by+\varphi(x,\ y),\qquad \frac{dy}{dt}=cx+dy+\psi(x,\ y), \tag{1}$$

where the functions $\varphi(x,y)$ and $\psi(x,y)$ are continuous and continuously differentiable to first order in x and y in $\bar{G}$; at the point $O\,(0,0)$, the functions φ and ψ and their partial derivatives all vanish:

$$\varphi(0,\ 0)=\psi(0,\ 0)=\varphi'_x(0,\ 0)=\varphi'_y(0,\ 0)=\psi'_x(0,\ 0)=\psi'_y(0,\ 0)=0. \tag{2}$$

Since O is a simple equilibrium,

$$\Delta=\begin{vmatrix} a & b \\ c & d \end{vmatrix}\neq 0. \tag{3}$$

Applying a non-singular linear transformation to system (1) we can reduce it to a canonical form described by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in a normal Jordan form. Let λ_1 and λ_2 be the characteristic values, i.e., the roots of the characteristic equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=0 \tag{4}$$

or

$$\lambda^2-\sigma\lambda+\Delta=0, \tag{5}$$

where

$$\sigma=a+d. \tag{6}$$

We should distinguish between the following cases:

I. The characteristic roots λ_1 and λ_2 are real, different, and of the same sign. System (1) is then reduced to the canonical form

$$\frac{dx}{dt}=\lambda_1 x+\varphi(x,\ y),\qquad \frac{dy}{dt}=\lambda_2 y+\psi(x,\ y), \tag{7}$$

where $\lambda_1\lambda_2>0$. The equilibrium point $O\,(0,0)$ is then called a n o d e (an ordinary node).

II. λ_1 and λ_2 are equal, i.e., $\lambda_1=\lambda_2=\lambda$. System (1) reduces either to the form

$$\frac{dx}{dt}=\lambda x+\varphi(x,\ y),\qquad \frac{dy}{dt}=\lambda y+\psi(x,\ y), \tag{8}$$

or

$$\frac{dx}{dt}=\lambda x+\varphi(x,\ y),\qquad \frac{dy}{dt}=\mu x+\lambda y+\psi(x,\ y), \tag{9}$$

where $\mu\neq 0$.

In case (8), the equilibrium O is called a *dicritical node*, and in case (9) a *confluent node*.

III. λ_1 and λ_2 are real, different, and of opposite signs. In this case, the canonic form of the system is as in case I, i.e., (7), but $\lambda_1\lambda_2 = \Delta < 0$. The point O is then called a *saddle point*.

IV. λ_1 and λ_2 are complex numbers, which are not pure imaginary. The canonical form of system (1) is then

$$\frac{dx}{dt} = \alpha x - \beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \alpha y + \psi(x, y), \tag{10}$$

where $\lambda_{1,2} = \alpha \pm \beta i$, $\alpha \neq 0$ $\beta > 0$. The equilibrium state O is then a *focus* (a *simple focus*).

V. λ_1 and λ_2 are pure imaginary numbers, $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$, $\beta \neq 0$. The canonical form of system (1) is then

$$\frac{dx}{dt} = -\beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \psi(x, y). \tag{11}$$

In this case O is called an equilibrium state with pure imaginary characteristic roots.

We will show that in cases I through IV the point O is a structurally stable equilibrium state, and in case V it is structurally unstable.

The behavior of the paths of the dynamic system in the neighborhood of O (0, 0) in each of these five cases is investigated in QT, §7 and §8.

We will now review some of the properties which are used in establishing the structural stability (for detailed proof, see QT, §7).

In case I (a node), all the circles

$$x^2 + y^2 = r^2 \tag{12}$$

of sufficiently small radius r are contact-free cycles for the trajectories of the system. Let O be a *stable node*,* i.e., $\lambda_1 < 0$, $\lambda_2 < 0$. Then any path with O as its limiting point goes to O for $t \to +\infty$. Let

$$x = x(t), \quad y = y(t) \tag{13}$$

be one of these paths, $M(t)$ a point of this path with the coordinates $x(t)$, $y(t)$, and let $\rho(t) = \sqrt{x(t)^2 + y(t)^2}$ be the distance of M from the origin. For $t \to +\infty$, $\rho(t)$ goes monotonically to zero. In the case of an *unstable node* ($\lambda_1 > 0$, $\lambda_2 > 0$), $\rho(t) \to 0$ for $t \to -\infty$.

In the case of a *dicritical node* (case II, canonical system (8)) or a *simple focus* (case IV), the situation is precisely the same as for an ordinary node. Indeed, circles (12) of sufficiently small radius r are without contact cycles, and any path (13) which for $t = t_0$ crosses one of these circles goes to O for $t \to +\infty$ if the node or the focus is stable ($\lambda < 0$, or $\alpha < 0$, respecitvely) and to $t \to -\infty$ if the node or the focus is unstable ($\lambda > 0$, or $\alpha > 0$, respectively). In either case, $\rho(t) \to 0$ monotonically.

In the case of a *confluent node* (case II, canonical system (9)) the circles (12) are replaced by ellipses 326

$$x^2 + ky^2 = r^2, \tag{14}$$

* [Not to be confused with *structurally* stable or *structurally* unstable nodes.]

where k is some positive number.* For sufficiently small r, all these ellipses are without contact cycles for the paths of system (9). For a stable (unstable) node, i.e., for $\lambda < 0$ $(\lambda > 0)$, each path initially crossing such a cycle without contact approaches the point O for $t \to +\infty$ $(t \to -\infty)$.

2. Structural stability of a simple node and a focus

We will prove that a simple state of equilibrium which is a node (ordinary, dicritical, or confluent) or a simple focus is a structurally stable path of the dynamic system. The main part of the proof is contained in Lemma 1 below.

Let

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system defined in some region $\overline{G}_1$, and $O\,(0, 0)$ an equilibrium state of this system which is a simple node or a simple focus $(O \subset G_1)$. System (A) is given in canonical form.

Lemma 1. There exists a neighborhood H of the equilibrium state O with the following property: for any $\varepsilon > 0$ there exists $\sigma > 0$ such that if system (B) *defined in $\overline{G}_1$ is σ-close in $\overline{G}_1$ to system* (A) *and $O\,(0, 0)$ is an equilibrium state of system* (B), *then*

$$(H, A) \overset{\varepsilon}{\equiv} (H, B).$$

Proof. First consider the case when O is an ordinary node (case I). System (A) has the form (7):

$$\frac{dx}{dt} = \lambda_1 x + \varphi(x, y), \qquad \frac{dy}{dt} = \lambda_2 y + \psi(x, y). \tag{A}$$

We take $\lambda_1 < 0$, $\lambda_2 < 0$, i.e., a stable node.

Let us summarize some of the results from QT, Chapter IV.

In QT, §6.3 it is shown that the functions φ and ψ can be represented in a certain neighborhood of O in the form

$$\varphi(x, y) = xg_1(x, y) + yg_2(x, y), \qquad \psi(x, y) = xf_1(x, y) + yf_2(x, y), \tag{15}$$

where

$$g_1(x, y) = \int_0^1 \varphi'_x(tx, ty)\, dt, \qquad g_2(x, y) = \int_0^1 \varphi'_y(tx, ty)\, dt, \tag{16}$$

and f_1 and f_2 are similarly expressed in terms of the derivatives of $\psi(x, y)$. The functions g_1, g_2, f_1, f_2 are continuous and

$$g_1(0, 0) = g_2(0, 0) = f_1(0, 0) = f_2(0, 0) = 0. \tag{17}$$

Let

$$x = x(t), \qquad y = y(t) \tag{18}$$

* For k we can take any positive number, satisfying the inequality $k < 4\frac{\lambda^2}{\mu^2}$. See QT, §7.1.

be a path of system (A). Its equation in polar coordinates is

$$\rho = \rho(t), \quad \theta = \theta(t). \tag{19}$$

Since

$$\rho^2(t) = x^2(t) + y^2(t), \tag{20}$$

we have from (A) and (15), using the standard relations $x = \rho\cos\theta$, $y = \rho\sin\theta$,

$$\frac{d\rho^2(t)}{dt} = 2\rho^2(t)\,[\lambda_1\cos^2\theta + \lambda_2\sin^2\theta + \cos^2\theta g_1 + \cos\theta\sin\theta\,(g_2 + f_1) + \sin^2\theta f_2], \tag{21}$$

where f_1, f_2, g_1, g_2 are functions of $\rho\cos\theta$ and $\rho\sin\theta$. The expression $\lambda_1\cos^2\theta + \lambda_2\sin^2\theta$ is negative for all real θ and is periodic in θ. Its maximum value is therefore $-m$, where $m > 0$. Since g_1, g_2, f_1, f_2 are continuous functions which vanish at the origin $O\,(0, 0)$, there exists some $r_0 > 0$ such that if $\rho(t) \leqslant r_0$, then

$$|\cos^2\theta g_1 + \cos\theta\sin\theta\,(g_2 + f_1) + \sin^2\theta f_2| < \frac{m}{3}.$$

The expression in square brackets in (21) is thus definitely less than $-\frac{2m}{3} < -\frac{m}{2}$, so that

$$\frac{d\rho^2(t)}{dt} < -m\rho^2(t). \tag{22}$$

From (22) it follows immediately that all the circles

$$x^2 + y^2 = r^2, \tag{23}$$

where $r \leqslant r_0$, are cycles without contact for paths of system (A). Indeed, for a path (18) to be tangent to one of these circles at the point $(x(t_0), y(t_0))$ we should have $x(t_0)x'(t_0) + y(t_0)y'(t_0) = 0$, i.e., $\frac{d\rho^2(t_0)}{dt} = 0$, which contradicts (22). Now separating the variables in (22) and integrating from t_0 to $t > t_0$, we see that

$$\rho^2(t) < \rho^2(t_0)\,e^{-m(t-t_0)},$$

whence it follows that for $t \to +\infty$, $\rho(t)$ goes to zero (monotonically, as we see from (22)). We thus obtain, as required, that any path (18) which for $t = t_0$ passes through the circle

$$x^2 + y^2 = r_0^2, \tag{24}$$

crosses all the concentric circles of smaller radii as t increases and approaches the point O for $t \to +\infty$.

Let C_0 be the circle (24) and H the region inside this circle (i.e., $U_{r_0}(O)$). We will show that the H defined in this way satisfies the lemma.

Let (B) be a dynamic system defined in $\overline{G}_1$, which is sufficiently close to system (A) and for which O is an equilibrium state. System (B) can be

written in the form

$$\frac{dx}{dt}=(\lambda_1+\alpha_1)x+\alpha_2 y+\widetilde{\varphi}(x, y), \quad \frac{dy}{dt}=\alpha_3 x+(\lambda_2+\alpha_4)y+\widetilde{\psi}(x, y), \tag{B}$$

where α_i are sufficiently small numbers, and the functions $\widetilde{\varphi}$ and $\widetilde{\psi}$ are sufficiently close to the functions φ and ψ, respectively; moreover, $\widetilde{\varphi}$ and $\widetilde{\psi}$ and their first derivatives vanish at O.

$\widetilde{\varphi}$ and $\widetilde{\psi}$ can be represented, like φ and ψ, in the form

$$\widetilde{\varphi}(x, y)=x\widetilde{g}_1(x, y)+y\widetilde{g}_2(x, y), \quad \widetilde{\psi}(x, y)=x\widetilde{f}_1(x, y)+y\widetilde{f}_2(x, y), \tag{25}$$

where $\widetilde{g}_i$ and $\widetilde{f}_i$ are expressed in the form (16) in terms of the first derivatives of $\widetilde{\varphi}$ and $\widetilde{\psi}$.

Let

$$\rho=\widetilde{\rho}(t), \quad \theta=\widetilde{\theta}(t)$$

be a path of system (B) defined by the equations in polar coordinates. Using (B) and (25) to calculate $\frac{d\widetilde{\rho}(t)}{dt}$, we obtain

$$\begin{aligned}\frac{d\widetilde{\rho}^2(t)}{dt}= 2\widetilde{\rho}^2 [(\lambda_1\cos^2\widetilde{\theta}+\lambda_2\sin^2\widetilde{\theta}) + (\alpha_1\cos^2\widetilde{\theta}+(\alpha_2+\alpha_3)\cos\widetilde{\theta}\sin\widetilde{\theta} +\\ +\alpha_4\sin^2\widetilde{\theta})+\widetilde{g}_1\cos^2\theta+(\widetilde{g}_2+\widetilde{f}_1)\cos\widetilde{\theta}\sin\widetilde{\theta}+\widetilde{f}_2\sin^2\widetilde{\theta}].\end{aligned} \tag{26}$$

If system (B) is sufficiently close to system (A) in $\overline{G}_1$, the expressions in brackets in the right-hand sides of (21) and (26) are sufficiently close to each other in $\overline{G}_1$, and hence also in $\overline{H}$.* In virtue of the particular choice of H, the expression in brackets in (26) is less than $-\frac{2}{3}m<-\frac{m}{2}$ in $\overline{H}$. Therefore, if system (B) is sufficiently close to system (A), the expression in brackets in (26) is also less than $-\frac{m}{2}$ in $\overline{H}$. Then relation (22) with all the consequences is satisfied for system (B) in $\overline{H}$. Hence it follows that there exists $\sigma_1>0$ with the following property: if system (B) is σ_1-close to system (A) in $\overline{G}_1$, all the circles (23) are cycles without contact for the paths of system (B) and any path of this system which for $t=t_0$ passes through the circle (24) intersects all the concentric circles of smaller radii as t increases, approaching the point O for $t\to+\infty$.**

Let r_1 be a positive number such that $r_1<r_0$, $r_1<\frac{\varepsilon}{2}$. C_1 is a circle of radius r_1 centered at O, H_1 is the region inside this circle, and W the ring between the circles C_0 and C_1 (Figure 17).

We can now apply Lemma 12 (§4.2). By this lemma, there exists $\sigma_2>0$ such that if system (B) is σ_2-close to system (A) the partition of $\overline{W}$ by the paths of system (B) is ε-identical to the partition of $\overline{W}$ by the paths of

* We compare the values of these expressions in the same point (x, y), i.e., we take $\widetilde{\rho}=\rho$, $\widetilde{\theta}=\theta$. The closeness of the functions $\widetilde{f}_i$ and $\widetilde{g}_i$ to f_i and g_i, respectively, follows from equations (16) and the analogous relations for $\widetilde{f}_i$ and $\widetilde{g}_i$. Here closeness is to be understood to rank 0.

** Hence it follows, in particular, that the point O is a node or a focus of system (B). It can be shown with much less effort, using the continuous dependence of the characteristic roots on the system coefficients, that for sufficiently close systems (B), O is a node. This is not enough for our purposes, however. It is clear from the proof that, among other things, system (B) has no limit cycles in $\overline{H}$.

system (A). The mapping T of $\overline{W}$ on itself corresponding to this ε-identity can be chosen so that each point $M \in C_0$ is mapped into itself:

$$T(M) = M.$$

Let σ be any number satisfying the inequalities $\sigma > 0$, $\sigma < \sigma_1$, $\sigma < \sigma_2$. Let system (B) be σ-close to system (A) in $\overline{G}_1$. Since $\sigma < \sigma_2$, there exists, as we have just pointed out, a mapping of the ring $\overline{W}$ onto itself which is an ε-translation, which transforms the paths of system (A) into paths of system (B), and which leaves unchanged all the points of the boundary circle C_0. Let this mapping be T. It is defined on $\overline{W}$, and hence on the circle C_1. Let $N \in C_1$, L a path of system (A) passing through N, M an intersection point of L with C_0, $\tilde{L}$ a path of system (B) through M, and $\tilde{N}$ an intersection point of $\tilde{L}$ with the circle C_1 (Figure 17). Clearly $T(N) = \tilde{N}$.

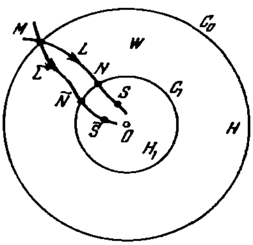

FIGURE 17

The mapping T is originally defined in the ring $\overline{W}$. We will now continue it to the entire $\overline{H}$ in the following way. Let $T(0) = 0$. Now let S be any point in H_1, which is not 0. The path L of system (A) passing through S at $t = t_1$ clearly crosses the circle C_1 at the point N for $t = t_0 < t_1$. Let $T(S) = \tilde{S}$, where $\tilde{S}$ is the point which corresponds to the time t_1 on the path $\tilde{L}$ of system (B), if at $t = t_0$ the path $\tilde{L}$ passes through the point $\tilde{N} = T(N)$ ("time reflection," see Figure 17). Since the radius of C_1 is less than $\frac{\varepsilon}{2}$, we have $\rho(S, \tilde{S}) < \varepsilon$.

The mapping T continued in this way is now defined in the entire $\overline{H}$. It is evidently an ε-translation which transforms the paths of (A) into the paths of (B). We have thus established that if (B) is δ-close to (A), then $(\overline{H}, A) \overset{\varepsilon}{\equiv} (\overline{H}, B)$, and also $(H, A) \overset{\varepsilon}{\equiv} (H, B)$. This completes the proof of the lemma for the case when O is an ordinary node.

If O is a dicritical node or a focus, the lemma is proved precisely in the same way. If O is a confluent node, the proof proceeds along the same lines, and the only change is that the concentric circles (12) are replaced by a family of ellipses (14), and H is taken as the region lying inside one of these ellipses. The proof of the lemma is thus complete.

We now proceed with the fundamental theorem of structural stability of a simple node or focus.

Theorem 12. The equilibrium state $M_0(x_0, y_0)$ of the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

for which $\Delta > 0$, $\sigma \neq 0$ (i.e., a node or a focus) is structurally stable.

Proof. Without loss of generality, we assume that the state of equilibrium is at the origin $O(0, 0)$, and system (A) is given in canonical form. This evidently can be achieved with the aid of a linear transformation. By Lemma 2, §6.1 and from the definition of a structurally stable system (Definition 14, §7.1) M_0 is a structurally stable equilibrium state of the original system if and only if O is a structurally stable equilibrium state of the transformed system.

The proof will be carried out for the case when O is an ordinary node. In the other cases (confluent and dicritical node, focus) the proof is exactly the same. System (A) is taken in canonical form (7).

Let H be the region considered in Lemma 1, i.e., the interior of a circle C_0 of sufficiently small radius r_0 centered at O. Let G_1 be some region satisfying the condition $\overline{H} \subset G_1$, $\overline{G}_1 \subset G$ (Figure 18; $\overline{G}$ is the original region used to define the closeness of systems). $\overline{G}_1$ is at a positive distance from the boundary of G. Let this distance be d.

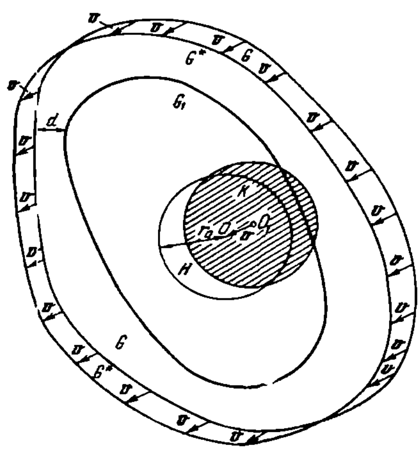

FIGURE 18

Let ε be some positive number. By Lemma 1, there exists $\sigma > 0$ such that if system (B) is defined in $\overline{G}_1$, where it is σ-close to system (A) and has an equilibrium state at O (0, 0), then

$$(H,\ A) \overset{\frac{\varepsilon}{2}}{\equiv} (H,\ B). \tag{27}$$

Let δ be a positive number, $\delta < \frac{\sigma}{2}$, with the following property: if system $(\widetilde{A})$ is δ-close to system (A) in $\overline{G}$, then $(\widetilde{A})$ has only one equilibrium state O_1 $(\xi_0,\ \eta_0)$ in $\overline{H}$ and

$$\rho(O,\ O_1) = \sqrt{\xi_0^2 + \eta_0^2} < \rho_0, \tag{28}$$

where ρ_0 is a fixed number satisfying the inequalities

$$\rho_0 < \frac{\varepsilon}{2}, \quad \rho_0 < d \tag{29}$$

and an additional condition which will be formulated at a later stage. This δ exists in virtue of Remark 3 to Theorem 6 (§2.2).

Consider the transformation

$$x = u + \xi_0, \quad y = v + \eta_0, \tag{30}$$

where $O_1(\xi_0, \eta_0)$ is the previously mentioned equilibrium state of system $(\tilde{A})$ δ-close to system (A). Let system $(\tilde{A})$ be

$$\frac{dx}{dt} = \tilde{P}(x, y), \quad \frac{dy}{dt} = \tilde{Q}(x, y).$$

Transformation (30) reduces it to the system

$$\frac{du}{dt} = \tilde{P}(u + \xi_0, v + \eta_0), \quad \frac{dv}{dt} = \tilde{Q}(u + \xi_0, v + \eta_0),$$

which, changing over from u and v to x and y, respectively, takes the form

$$\frac{dx}{dt} = \tilde{P}(x + \xi_0, y + \eta_0), \quad \frac{dy}{dt} = \tilde{Q}(x + \xi_0, y + \eta_0). \tag{B}$$

System (B) is defined in $\bar{G}^*$, which is obtained when $\bar{G}$ is translated by a vector $\mathbf{v}(-\xi_0, -\eta_0)$. Since by assumption $\sqrt{\xi_0^2 + \eta_0^2} < \rho_0 < d$, G^* contains $\bar{G}_1$, i.e., $\bar{G}_1 \subset G^*$. Therefore both $(\tilde{A})$ and (B) are defined in $\bar{G}_1$. Clearly if ρ_0 is sufficiently small, $(\tilde{A})$ and (B) are sufficiently close in $\bar{G}_1$. The third condition imposed on ρ_0 in addition to (29) is the following: ρ_0 is sufficiently small so that if $\sqrt{\xi_0^2 + \eta_0^2} < \rho_0$, system (B) is $\frac{\sigma}{2}$-close to system $(\tilde{A})$ in $\bar{G}_1$.

Note that $O(0, 0)$ is the equilibrium state of system (B).

Let K be the inside of a circle of radius r_0 centered at O_1 (Figure 18). K is obtained when H is translated by a vector $-\mathbf{v}$ (i.e., a vector with the coordinates ξ_0, η_0). Hence $K \subset \bar{G}$, and system $(\tilde{A})$ is therefore defined in K. Transformation (30) transforms region K into region H, system $(\tilde{A})$ into system (B), and the paths of system $(\tilde{A})$ in K are transformed into the paths of system (B) in H. Since $\sqrt{\xi_0^2 + \eta_0^2} < \rho_0 < \frac{\varepsilon}{2}$, and (30) is a topological mapping, we conclude that

$$(K, \tilde{A}) \overset{\frac{\varepsilon}{2}}{\equiv} (H, B). \tag{31}$$

Let now $(\tilde{A})$ be some system δ-close to (A) in $\bar{G}$. Then, from the definition of δ and in virtue of the conditions imposed on ρ_0, system (B) is $\frac{\sigma}{2}$-close to system (A) in $\bar{G}_1$ and relation (31) is satisfied. Now, since $\delta < \frac{\sigma}{2}$, system (B) is σ-close to system (A) in $\bar{G}_1$. Moreover, the point $O(0, 0)$ is an equilibirum state of system (B). Therefore relation (27) is also satisfied, and from (31) and (27) we clearly have

$$(H, A) \overset{\varepsilon}{\equiv} (K, \tilde{A}). \tag{32}$$

We have thus shown that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if system $(\tilde{A})$ is δ-close to (A) in $\bar{G}$, relation (32) applies. This implies in its turn that system (A) is structurally stable in H, i.e., O is a structurally stable state of equilibrium of system (A).

As we have noted above, the structural stability of a dicritical or a confluent node and of a simple focus is proved exactly in the same way; in the case of a confluent node, H is the inside of the corresponding ellipse. This completes the proof of the theorem.

Remark 1. Lemma 1 can be strengthened by omitting condition 2 from its statement. It will be used in this form in what follows, and we therefore give here the altered formulation and the corresponding proof.

There exists a neighborhood H^* of a simple node or a focus O enclosed in a cycle without contact which has the following property: for any $\varepsilon > 0$ there exists $\delta > 0$ such that if system $(\tilde{\mathrm{A}})$ *is δ-close to system* (A), *then* $(\overline{H}^*, A) \stackrel{\varepsilon}{\equiv} (\overline{H}^*, \tilde{A})$.

Note that the structural stability of the equilibrium state O follows immediately from this proposition. We will, conversely, prove this lemma proceeding from the previously established structural stability of the point O.

Proof. Consider the case when O is an ordinary node and system (A) is given in canonical form. Take a sufficiently small circle H of radius r centered at O, which contains no other equilibrium states except O and no closed paths of the system. System (A) is structurally stable in this circle. Let H^* be a circle of radius $\frac{r}{2}$ centered at O. We take a sufficiently small $\varepsilon^* > 0$, e.g., $\varepsilon^* < \frac{r}{10}$. Because of structural stability there exists $\delta^* > 0$ such that if system $(\tilde{\mathrm{A}})$ is δ^*-close to (A), then

$$(H, A) \stackrel{\varepsilon^*}{\equiv} (\tilde{H}, \tilde{A}). \tag{33}$$

Since $\varepsilon^* < \frac{r}{10}$, we conclude that $H^* \subset \tilde{H}$. Hence, and by (33), it follows that $\overline{H}^*$ contains no closed paths of system $(\tilde{\mathrm{A}})$. The boundary circle of H^* is clearly a cycle without contact for system (A).

$\overline{H}^*$ is thus enclosed by a cycle without contact of system (A) and all systems sufficiently close to (A) have no closed paths in $\overline{H}^*$. The rest of the proof proceeds along the same lines as the proof of Lemma 1, but the paths of system (A) and of the close system $(\tilde{\mathrm{A}})$ in general tend to different points O and $\tilde{O}$.

Remark 2. A simple state of equilibrium $M_0(x_0, y_0)$ of the system

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y) \tag{A}$$

is a node or a focus if

$$\Delta = \begin{vmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{vmatrix} > 0,$$
$$\delta = P'_x(x_0, y_0) + Q'_y(x_0, y_0) \neq 0.$$

If system $(\tilde{\mathrm{A}})$ is sufficiently close to system (A), there is precisely one equilibrium state $\tilde{O}(\tilde{x}_0, \tilde{y}_0)$ of system $(\tilde{\mathrm{A}})$ in a sufficiently small neighborhood of the equilibrium state M_0. The quantities $\tilde{\Delta}$ and $\tilde{\delta}$ corresponding to the point $\tilde{O}$ are not markedly different from Δ and δ, so that $\tilde{\Delta} > 0, \tilde{\delta} \neq 0$, and the point $\tilde{O}$ is a simple node or focus. It is readily seen that if O is an ordinary node, $\tilde{O}$ is also an ordinary node, and if O is a simple focus, $\tilde{O}$ is also a simple focus. If, however, O is a confluent node, $\tilde{O}$ is either an ordinary

or a confluent node; if O is a dicritical node, $\tilde{O}$ is either a focus or any of the three different types of node: ordinary, confluent, or dicritical. In all these cases, the equilibrium state $\tilde{O}$ is a structurally stable path of system (A).

§9. STRUCTURAL STABILITY OF A SADDLE POINT

1. Reduction of the system to canonical form by a nearly identical transformation

In our proof of the structural stability of a simple equilibrium state which is a saddle point, we assume as before that the saddle point coincides with $O(0,0)$ and the dynamic system has the canonical form

$$\frac{dx}{dt}=\lambda_1 x+\varphi(x, y), \quad \frac{dy}{dt}=\lambda_2 y+\psi(x, y), \tag{A}$$

where $\lambda_1\lambda_2<0$. Without loss of generality, we may take

$$\lambda_1>0, \quad \lambda_2<0. \tag{1}$$

We will show that any system sufficiently close to system (A) can be reduced to canonical form by a transformation which is as close to the identity transformation as desired.

Lemma 1. If the system

$$\frac{dx}{dt}=\tilde{P}(x, y), \quad \frac{dy}{dt}=\tilde{Q}(x, y) \tag{$\tilde{A}$}$$

is sufficiently close in $\bar{G}$ to system (A), ***a nearly identical linear transformation will reduce system** ($\tilde{A}$) **to the form***

$$\frac{dx}{dt}=(\lambda_1+\varepsilon_1)x+\tilde{\varphi}(x, y), \quad \frac{dy}{dt}=(\lambda_2+\varepsilon_2)y+\tilde{\psi}(x, y), \tag{B}$$

where ε_1 and ε_2 are infinitesimal, and the functions $\tilde{\varphi}$ and $\tilde{\psi}$ together with their first derivatives vanish at the point $O(0,0)$ and are arbitrarily close to the corresponding functions φ and ψ.

Proof. Let ε_0 and δ_0 be two positive numbers with the following property: for any system ($\tilde{A}$) δ_0-close in $\bar{G}$ to system (A), there is precisely one equilibrium state $\tilde{O}(\xi_0, \eta_0)$ in $U_{\varepsilon_0}(O)$, and this equilibrium state is a saddle point. The existence of these ε_0 and δ_0 follows from Theorem 6 and Definition 5 (§2.1), and also from the fact that small changes in system (A) leave the determinant $\Delta=\lambda_1\lambda_2$ negative.

Let system ($\tilde{A}$) be δ-close to system (A), where $\delta<\delta_0$, and let $\tilde{O}(\xi_0, \eta_0)$ be the equilibrium state of ($\tilde{A}$) lying in $U_{\varepsilon_0}(O)$. System ($\tilde{A}$) clearly can be written in the form

$$\begin{aligned}\frac{dx}{dt}&=(\lambda_1+\alpha_1)(x-\xi_0)+\alpha_2(y-\eta_0)+\varphi_1(x, y),\\ \frac{dy}{dt}&=\beta_1(x-\xi_0)+(\lambda_2+\beta_2)(y-\eta_0)+\psi_1(x, y),\end{aligned} \tag{2}$$

and if $\delta > 0$ is sufficiently small, the numbers α_i, β_i, ξ_0, η_0 can be made as small as desired and φ_1 and ψ_1 will be arbitrarily close to the corresponding functions $\widetilde{\varphi}$ and $\widetilde{\psi}$, vanishing together with their derivatives at the point $\widetilde{O}(\xi_0, \eta_0)$. Applying the transformation

$$x = X + \xi_0, \qquad y = Y + \eta_0, \tag{3}$$

we obtain the system

$$\frac{dX}{dt} = (\lambda_1 + \alpha_1) X + \alpha_2 Y + \varphi_2(X, Y), \quad \frac{dY}{dt} = \beta_1 X + (\lambda_2 + \beta_2) X + \psi_2(X, Y), \tag{4}$$

where φ_2 and ψ_2 vanish at $O(0, 0)$ together with their derivatives.

If the numbers α_i and β_i are sufficiently small, the matrix

$$\begin{pmatrix} \lambda_1 + \alpha_1 & \alpha_2 \\ \beta_1 & \lambda_2 + \beta_2 \end{pmatrix} \tag{5}$$

is sufficiently close to the matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{6}$$

and the characteristic roots of the matrix (5) are therefore close to λ_1 and λ_2. We write these characteristic roots in the form $\lambda_1 + \varepsilon_1$, $\lambda_2 + \varepsilon_2$.

We will now establish the existence of a non-singular matrix S which is close to the unit matrix and satisfies the relation

$$S \begin{pmatrix} \lambda_1 + \alpha_1 & \alpha_2 \\ \beta_1 & \lambda_2 + \beta_2 \end{pmatrix} S^{-1} = \begin{pmatrix} \lambda_1 + \varepsilon_1 & 0 \\ 0 & \lambda_2 + \varepsilon_2 \end{pmatrix}. \tag{7}$$

We seek S in the form

$$S = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Right-multiplying (7) by S, we write it in the form

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \lambda_1 + \alpha_1 & \alpha_2 \\ \beta_1 & \lambda_2 + \beta_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \varepsilon_1 & 0 \\ 0 & \lambda_2 + \varepsilon_2 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{8}$$

Comparing the elements in the left- and the right-hand sides of (8), we obtain four equations:

$$p(\lambda_1 + \alpha_1) + q\beta_1 = (\lambda_1 + \varepsilon_1) p, \qquad p\alpha_2 + q(\lambda_2 + \beta_2) = (\lambda_1 + \varepsilon_1) q \tag{9}$$

and

$$r(\lambda_1 + \alpha_1) + s\beta_1 = (\lambda_2 + \varepsilon_2) r, \qquad r\alpha_2 + s(\lambda_2 + \beta_2) = (\lambda_2 + \varepsilon_2) s. \tag{10}$$

Let us first consider equations (9). They can be written in the form

$$[(\lambda_1 + \alpha_1) - (\lambda_1 + \varepsilon_1)] p + \beta_1 q = 0, \qquad \alpha_2 p + [(\lambda_2 + \beta_2) - (\lambda_1 + \varepsilon_1)] q = 0. \tag{11}$$

The determinant of this system of homogeneous equations in p and q is equal to the characteristic polynomial of the matrix (5) for $\lambda=\lambda_1+\varepsilon_1$. But since $\lambda_1+\varepsilon_1$ is a characteristic root of the matrix (5), this determinant vanishes. Hence, equations (11) are equivalent, and it suffices to find a solution satisfying only one of these equations. Let $p=1$, and the second equation in (11) gives $q=\frac{\alpha_2}{\lambda_1-\lambda_2+\varepsilon_1-\beta_2}$.

Similarly, seeing that $\lambda_2+\varepsilon_2$ is a characteristic root of the matrix (5), we take $s=1$ in (10) and find $r=\frac{\beta_1}{\lambda_2-\lambda_1+\varepsilon_2-\alpha_1}$.

These numbers p, q, r, s satisfy equations (9) and (10), and therefore the matrix

$$S=\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} 1 & \frac{\alpha_2}{\lambda_1-\lambda_2+\varepsilon_1-\beta_2} \\ \frac{\beta_1}{\lambda_2-\lambda_1+\varepsilon_2-\alpha_1} & 1 \end{pmatrix} \tag{12}$$

satisfies the matrix equation (7). Since $\lambda_1\neq\lambda_2$, the matrix S is arbitrarily close to the unit matrix for sufficiently small $\alpha_i, \beta_i, \varepsilon_i$ $(i=1, 2)$, i.e., when system $(\tilde{\mathrm{A}})$ is sufficiently close to system (A), and it is therefore a nonsingular matrix. Applying the transformation

$$u=pX+qY, \qquad v=rX+sY, \tag{13}$$

where p, q, r, s are the matrix elements from (12), to system (4) and writing x and y for u and v, respectively, we obtain the system

$$\frac{dx}{dt}=(\lambda_1+\varepsilon_1)x+\varphi_3(x, y), \qquad \frac{dy}{dt}=(\lambda_2+\varepsilon_2)y+\psi_3(x, y) \tag{14}$$

(see QT, §6.2), i.e., system (B). If system $(\tilde{\mathrm{A}})$, i.e., system (2), is sufficiently close to system (A), transformation (3) and (13), as we have seen, are arbitrarily close to the identity transformation. But then the inverse transformations are also arbitrarily close to the identity transformation, and the same applies to their product. Now, this product transforms system $(\tilde{\mathrm{A}})$ into system (14). We have thus shown that if system $(\tilde{\mathrm{A}})$ is sufficiently close in $\bar{G}$ to system (A), a linear transformation arbitrarily close to the identity transformation will reduce system $(\tilde{\mathrm{A}})$ to the form (14), i.e., to the form (B). The numbers ε_1 and ε_2 in this transformation are arbitrarily small, and the functions φ_3 and ψ_3, as is readily seen, are arbitrarily close to the functions φ and ψ, respectively. This completes the proof of the lemma.*

2. Proof of the structural stability of a saddle point

Before proceeding with the actual proof, we briefly review the procedure for investigating the pattern of the paths of system (A) near a saddle point $O(0, 0)$. For more details, see QT, §7.3.

* In our proof of Lemma 1 we only made use of the fact that $\lambda_1\neq\lambda_2$, without resorting to the different signs of λ_1 and λ_2. The lemma therefore remains valid when $O(0, 0)$ is an ordinary node, and not only a saddle point. This proposition, however, was not needed in our proof of the structural stability of the node.

System (A), in accordance with relations (15) from §8, is written in the form

$$\frac{dx}{dt}=\lambda_1 x+xg_1(x,\ y)+yg_2(x,\ y),$$
$$\frac{dy}{dt}=\lambda_2 y+xf_1(x,\ y)+yf_2(x,\ y), \tag{15}$$

where f_1, f_2, g_1, g_2 are continuous functions which vanish at the point $O\ (0, 0)$. Without loss of generality, we again take

$$\lambda_1>0, \quad \lambda_2<0. \tag{1}$$

The entire analysis is confined to a neighborhood of the point O where 0 is the only equilibrium state of system (A).

Let k be a fixed positive number. We draw the straight lines $y=\pm kx$ and focus our attention on the rectangle with its vertices at $A\ (x_0, kx_0)$, $B\ (-x_0, kx_0)$, $B_1\ (-x_0, -kx_0)$, $A_1\ (x_0, -kx_0)$. These straight lines are the diagonals of this rectangle (Figure 19). Let $\overline{R}$ correspond to the entire rectangle, and R to its interior. x_0 is understood to be a sufficiently small positive number. Consider the intercepts of horizontal and vertical lines between the diagonals of the rectangle R and the diagonals also.

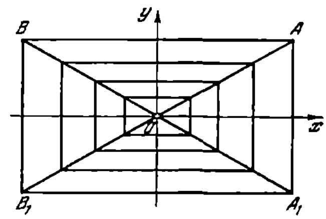

FIGURE 19

Let L be a path of system (A) corresponding to the solution

$$x=x(t), \quad y=y(t).$$

If at $t=t_0$, the path L crosses the diagonal $y=kx$ (or $y=-kx$) at some point other than the origin, we have

$$\left[\frac{d\left(\frac{y(t)}{x(t)}\right)}{dt}\right]_{t=t_0}=[\pm k(\lambda_2-\lambda_1)+f_1\pm k(f_2-g_1)-g_2k^2]_{t=t_0}. \tag{16}$$

If at $t=t_0$, the path L crosses an intercept of the vertical line between the diagonals of the rectangle R, we have

$$\left[\frac{dx}{dt}\right]_{t=t_0}=\left[x(t)\left(\lambda_1+g_1+\frac{y(t)}{x(t)}g_2\right)\right]_{t=t_0}, \tag{17}$$

where

$$\left|\frac{y(t_0)}{x(t_0)}\right|\leqslant k. \tag{18}$$

Finally, if at $t = t_0$, the path L crosses the intercept of the horizontal line between the diagonals of the rectangle R, we have

$$\left[\frac{dy(t)}{dt}\right]_{t=t_0} = \left[y(t)\left(\lambda_2 + f_1\frac{x(t)}{y(t)} + f_2\right)\right]_{t=t_0}, \tag{19}$$

and

$$\frac{x(t_0)}{y(t_0)} \leqslant \frac{1}{k}. \tag{20}$$

Since the functions f_i and g_i are continuous and vanish at the point $O(0, 0)$, relations (16), (17), and (19) for sufficiently small x_0 lead to the equalities

$$\left[\frac{d\left(\frac{y(t)}{x(t)}\right)}{dt}\right]_{t=t_0} = \pm\,\chi_1 k(\lambda_2 - \lambda_1), \tag{21}$$

$$\left[\frac{dx}{dt}\right]_{t=t_0} = \chi_2 x(t_0)\lambda_1, \tag{22}$$

$$\left[\frac{dy}{dt}\right]_{t=t_0} = \chi_3 y(t_0)\lambda_2, \tag{23}$$

where χ_1, χ_2, χ_3 are some numbers which satisfy the inequalities, say,

$$\frac{3}{4} < \chi_i < \frac{5}{4} \tag{24}$$

($i = 1, 2, 3$. The sign + in (21) corresponds to the diagonal AB_1, and the sign − to the diagonal BA_1).

It follows from (21)—(23) (see QT, §7.3) that the paths of system (A) in the rectangle R make the pattern shown in Figure 20. On each of the sides AB and A_1B_1 of the rectangle R there is one point — D and D_1, respectively — through which an ω-separatrix of the saddle O passes, and on each of the sides AA_1 and BB_1 there are the points C and C_1 through which pass α-separatrices, which are the continuation of the ω-separatrices. The sides of the rectangle $\bar{R}$ are segments without contact for the paths of system (A). Through any point inside R, which is not the origin O and does not belong to one of the above separatrices, passes a path of the system (A) which emerges from the rectangle R through one of the sides AB, A_1B_1 as t decreases or through one of the sides AA_1, BB_1 as t increases.

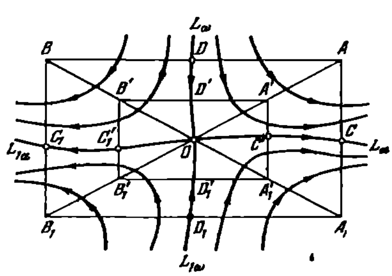

FIGURE 20

All the above relations and propositions clearly remain valid if $\bar{R}$ is replaced by any smaller "concentric" rectangle $\bar{R}'$ with the same diagonals $y = \pm kx$, whose sides are parallel to the coordinate axes (the rectangle $A'B'B_1'A_1'$ in Figure 20).

It is assumed in what follows that the number $x_0 > 0$ is sufficiently small, so that equalities (21)—(23) are applicable. We hence consider the rectangle $\bar{R}$ corresponding to this x_0.

Lemma 2. If a dynamic system $(\tilde{A})$ *has the canonical form*

$$\frac{dx}{dt} = \tilde{\lambda}_1 x + \tilde{\varphi}(x, y), \qquad \frac{dy}{dt} = \tilde{\lambda}_2 x + \tilde{\psi}(x, y)$$

and is sufficiently close to system (A), *the pattern of paths of system* $(\tilde{A})$ *in rectangle* $\bar{R}$ *is similar to the pattern of paths of system* (A). *Specifically, on each of the sides* AB *and* A_1B_1 *of the rectangle* R *there is one point* — $\tilde{D}$ *and* $\tilde{D}_1$ — *through which passes an* ω-*separatrix of the saddle* O *of system* $(\tilde{A})$, *and on the sides* AA_1 *and* BB_1 *there are points* $\tilde{C}$ *and* $\tilde{C}_1$ *through which pass* α-*separatrices that constitute the continuation of the* ω-*separatrices. The sides of the rectangle* $\bar{R}$ *are segments without contact for the paths of system* $(\tilde{A})$, *and each path of system* $(\tilde{A})$ *passing through an interior point of the rectangle* R *which is neither a separatrix nor the point* O *emerges from* $\bar{R}$ *through one of the sides* AB, A_1B_1 *as* t *decreases and through one of the sides* AA_1, BB_1 *as* t *increases.*

Proof. If system $(\tilde{A})$ is sufficiently close to system (A), $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are sufficiently close to λ_1 and λ_2, respectively. Therefore $\tilde{\lambda}_1 > 0$, $\tilde{\lambda}_2 < 0$, and O is a saddle point of system $(\tilde{A})$. The point $O(0, 0)$ is a simple equilibrium state of system (A) (§7.3, Definition 15 and Theorem 11). Hence it readily follows that all dynamic systems $(\tilde{A})$ which are sufficiently close to (A) have precisely one equilibrium state in $\bar{R}$, namely the point $O(0, 0)$. Further analysis of (16)—(20) shows that relations analogous to (21)—(24) apply to the paths $x = \tilde{x}(t)$, $y = \tilde{y}(t)$ of any system $(\tilde{A})$ which is sufficiently close to (A).* The situation with regard to the pattern of paths of system $(\tilde{A})$ in rectangle $\bar{R}$ is therefore exactly the same as for system (A) (QT, §7.3), and the proof of the lemma is complete.

In the following two lemmas, $(\tilde{A})$ is a dynamic system in canonical form which is sufficiently close to (A) for Lemma 2 to apply. Let L_ω, $L_{1\omega}$, L_α, $L_{1\alpha}$ be the separatrices of system (A) passing through the points D, D_1, C, C_1, respectively (Figure 20), and $\tilde{L}_\omega$, $\tilde{L}_{1\omega}$, $\tilde{L}_\alpha$, $\tilde{L}_{1\alpha}$ the separatrices of system $(\tilde{A})$ passing through the points $\tilde{D}$, $\tilde{D}_1$, $\tilde{C}$, $\tilde{C}_1$, respecitvely, which lie on the sides of the rectangle $\bar{R}$ ($\tilde{D}$ is on the side AB, $\tilde{D}_1$ on the sides A_1B_1, etc.).

Lemma 3. For any $\varepsilon > 0$, *there is* $\delta > 0$ *such that if system* $(\tilde{A})$ *is* δ-*close to system* (A) *and the separatrices* L_ω *and* $\tilde{L}_\omega$ *pass through the points* D *and* $\tilde{D}$ *at* $t = t_0$, *any two points of these separatrices corresponding to the same* $t \geqslant t_0$ *are distant less than* ε *from each other. A similar proposition applies to any of the separatrices* $L_{1\omega}$, L_α, $L_{1\alpha}$.

Proof. The proof is carried out for the separatrix L_ω. The same proof applies to the other separatrices.

Let $\varepsilon > 0$ be given. Consider a rectangle R', "concentric" with R, which is entirely contained in $U_{\varepsilon/2}(O)$. Let A', B', B_1', and A_1' be the vertices of this

* This is so because if system $(\tilde{A})$ is sufficiently close to system (A), the functions $\tilde{f}_i$ and $\tilde{g}_i$ are arbitrarily close to f_i and g_i, respectively ($i = 1, 2$), as we see from equations (15), §8 and the corresponding relations for f_i, $\tilde{f}_i$, $\tilde{g}_i$.

rectangle; D' is the intersection point of the separatrix L_ω with the side $A'B'$ of the rectangle (Figure 21). Suppose the separatrix L_ω passes through the point D for $t = t_0$ and through the point D' for $t = T > t_0$. Let T_1 be some number, $T_1 > T$.

By Theorem 8 (§3) there exists a segment K_1K_2 of the side AB of the rectangle $\bar{R}$ containing the point D and a number $\delta_1 > 0$ with the following property: if system $(\tilde{A})$ is δ_1-close to system (A), $\tilde{M}_0$ is a point of the segment K_1K_2, $\tilde{L}$ is the path of system $(\tilde{A})$ which passes through $\tilde{M}_0$ at $t = t_0$, $M(t)$, $\tilde{M}(t)$ the points of the paths L_ω and $\tilde{L}$ corresponding to the time t, then $\rho(M(t), \tilde{M}(t)) < \varepsilon$ for all t, $t_0 \leqslant t \leqslant T_1$ and all the points $\tilde{M}(T_1)$ lie inside the rectangle R'.

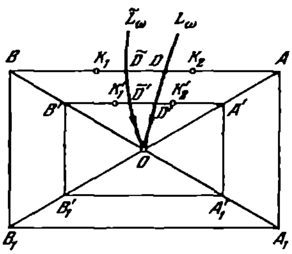

FIGURE 21

By Lemma 5, §4 there exists a segment $K_1'K_2'$ of the side $A'B'$ of the rectangle R' containing the point D' (Figure 21) and a number $\delta_2 > 0$ with the following property: as t decreases, all the paths of any system $(\tilde{A})$ which is δ_2-close to (A) passing through the points of the segment $K_1'K_2'$ cross the side AB of the rectangle R at the points of the segment K_1K_2.

Consider the paths L_1 and L_2 of system (A) which at some t pass through the points K_1' and K_2', respectively. Since these points lie on different sides of the separatrix L_ω, with the increase in t the paths L_1 and L_2 will cross the diagonals OB' and OA' of the rectangle R', respectively. By Lemma 5, §4, there exists $\delta_3 > 0$, such that if system $(\tilde{A})$ is δ_3-close to system (A), the paths $\tilde{L}_1$ and $\tilde{L}_2$ of $(\tilde{A})$ passing through the points K_1' and K_2' will also meet the diagonals OB' and OA', respectively, with increasing t. Then by Lemma 2 the intersection point $\tilde{D}'$ of the separatrix $\tilde{L}_\omega$ with the side $A'B'$ of the rectangle R' lies between K_1' and K_2'.

Let δ be the smallest of the three numbers $\delta_1, \delta_2, \delta_3$. If system $(\tilde{A})$ is δ-close to system (A), its separatrix $\tilde{L}_\omega$ evidently crosses the segment $K_1'K_2'$ (since $\delta < \delta_3$), and hence also the segment K_1K_2 (since $\delta < \delta_2$). Let $\tilde{D}$ be the intersection point of the separatrix $\tilde{L}_\omega$ with K_1K_2. Suppose that the separatrices L_ω and $\tilde{L}_\omega$ pass through the points D and $\tilde{D}$, respectively, at the same time t_0. Then, since $\delta < \delta_1$ for any t, $t_0 \leqslant t \leqslant T_1$, we have $\rho(M(t), \tilde{M}(t)) < \varepsilon$. All the points $M(t)$, $\tilde{M}(t)$ of these separatrices corresponding to $t > T_1$ lie inside the rectangle R', i.e., in $U_{\varepsilon/2}(O)$. The distance between any pair of such points is therefore less than ε. We have thus established that if system $(\tilde{A})$ is δ-close to system (A), $\rho(M(t), \tilde{M}(t)) < \varepsilon$ for any $t \geqslant t_0$. This completes the proof of the lemma.

Remark. Lemma 3 can be generalized. Let l be any arc without contact that meets the separatrix L_ω at a single point S, which does not coincide with the end-points of l. Then for any $\varepsilon > 0$ there exists $\delta > 0$, such that if system $(\tilde{A})$ is δ-close to system (A), the separatrix $\tilde{L}_\omega$ of system $(\tilde{A})$ meets the arc l at a single point $\tilde{S}$, and $\rho(S, \tilde{S}) < \varepsilon$; if in addition to the above, L_ω and $\tilde{L}_\omega$ pass through S and $\tilde{S}$ at $t = t_0$, any two points of these separatrices corresponding to the same time $t > t_0$ are also distant less than ε from each other. This proposition can be proved without difficulty. In what follows we will only make use of the fact that if δ is sufficiently small, the points S and $\tilde{S}$ (or D and $\tilde{D}$) are arbitrarily close.

We now proceed to the next lemma. Retaining the notation of Lemma 3, we again consider the rectangle $\bar{R}$ and the separatrices L_ω, $L_{1\omega}$, L_α, $L_{1\alpha}$ of the saddle point O.

On the side AB of the rectangle R we choose two points S and S_1 lying on the two sides of the point D, and on A_1B_1 we choose two points S_2 and S_3 lying on the two sides of D_1 (Figure 22). Let L, L_1, L_2, L_3 be the paths passing through these points. As the parameter t increases, these paths meet the sides AA_1 and BB_1 of the rectangle R at the points T, T_1, T_2, T_3, respectively. Let H be the region delimited by the segments SS_1, S_2S_3, TT_3, T_1T_2 and the arcs ST and S_iT_i ($i = 1, 2, 3$) of the paths L and L_i. H is a canonical neighborhood of the equilibrium state O (QT, §19.2). The separatrices L_ω, $L_{1\omega}$, L_α, and $L_{1\alpha}$ of system (A) partition the canonical neighborhood H into four regular saddle-point regions, which are designated σ, σ_1, σ_2, σ_3 (Figure 22). The region σ is delimited by the arcs OD and OC of the separatrices, the point O, the arc ST of the path L, and the segments DS and CT; the boundaries of the other regions σ_i are similar to the boundary of σ.

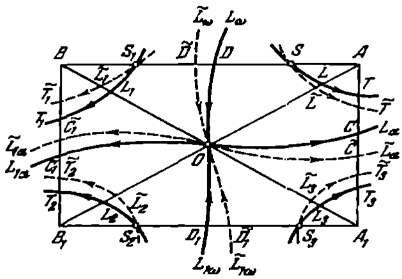

FIGURE 22

Consider system $(\tilde{A})$ which is δ-close to system (A) and is also given in canonical form. Let $\delta > 0$ be so small that, first, Lemma 2 holds true and, second, the points $\tilde{D}$ and $\tilde{D}_1$ (the intersection points of the separatrices $\tilde{L}_\omega$ and $\tilde{L}_{1\omega}$ with the segments AB and A_1B_1) lie inside the segments S_1S and S_2S_3, respectively. Let $\tilde{L}$, $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{L}_3$ be the paths of system $(\tilde{A})$ which pass through the points S, S_1, S_2, S_3, respectively (in Figure 22, these paths and the separatrices of system $(\tilde{A})$ are marked by dashed lines).

Let $\tilde{H}$ be the canonical neighborhood of the saddle point O of system $(\tilde{A})$ analogous to H. Like H, the neighborhood $\tilde{H}$ is partitioned by the separatrices of the saddle point O of system $(\tilde{A})$ into four regular saddle-point regions $\tilde{\sigma}$ and $\tilde{\sigma}_i$ ($i = 1, 2, 3$), which are analogous to the regions σ and σ_i. All the regions σ and $\tilde{\sigma}$ are assumed to be closed.

Lemma 4. For any $\varepsilon > 0$, there exists $\delta > 0$, such that if system $(\tilde{A})$ is δ-close to system (A), we have

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}). \qquad (25)$$

Proof. We will show that if δ is sufficiently small, there exists a mapping θ (or θ_i, $i = 1, 2, 3$, respectively) which maps σ (and σ_i) onto $\tilde{\sigma}$ ($\tilde{\sigma}_i$); the mappings θ and θ_i (a) are ε-translations, (b) map paths into paths, and (c) θ and θ_1, θ_1 and θ_2, θ_2 and θ_3, θ_3 and θ coincide on the arcs of the separatrices OD, OC_1, OD_1, OC, respecitvely.

Lemma 4 clearly follows directly from this proposition.

To fix ideas, let us consider the region σ and the corresponding $\tilde{\sigma}$ (Figure 23).

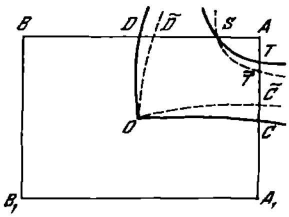

FIGURE 23

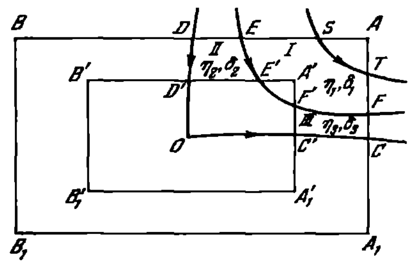

FIGURE 24

Let $\varepsilon > 0$ be given. As in Lemma 3, we consider the rectangle R' with the vertices A', B', B_1', A_1', which is "concentric" with R and is entirely contained in $U_{\varepsilon/2}(O)$. In σ we draw the path EF which meets the sides AB, AA_1, $A'B'$, $A'A_1'$ of the rectangles R and R' at the points E, F, E', F', respectively (Figure 24). Let I, II, III be the elementary quadrangles shown in Figure 24 (their vertices are $TSEF$, $E'EDD'$, $FF'C'C$, respectively). By Lemma 9, §4.2, for any given ε, there exists a pair of numbers η_1, δ_1 (η_2, δ_2; η_3, δ_3) for the quadrangle I (II, III) with the following property: if system ($\tilde{A}$) is δ_1 (δ_2, δ_3)-close to system (A), $\tilde{\text{I}}$ ($\tilde{\text{II}}$, $\tilde{\text{III}}$) is the corresponding elementary quadrangle of system ($\tilde{A}$), and mappings are given of the sides FE, ES, and ST of the quadrangle I onto the respective sides $\tilde{F}\tilde{E}$, $\tilde{E}\tilde{S}$, and $\tilde{S}\tilde{T}$ of the quadrangle $\tilde{\text{I}}$ (or of the sides DD', DE, and EE' of the quadrangle II and the sides CC', $C'F'$, and $F'F$ of the quadrangle III), which are η_1-translations (or η_2-, η_3-translations, respectively), there exists a path-conserving mapping of the quadrangle I onto $\tilde{\text{I}}$ (II onto $\tilde{\text{II}}$, III onto $\tilde{\text{III}}$) which is an ε-translation and coincides with the given mapping on the three sides of the quadrangle.

Let $\delta_4 > 0$ be so small that if system ($\tilde{A}$) is δ_4-close to (A), the segments EE', $E'F'$, and $F'F$ of the path of system (A) can be mapped onto the respective segments $\tilde{E}\tilde{E}'$, $\tilde{E}'\tilde{F}'$, and $\tilde{F}'\tilde{F}$ of the path of system ($\tilde{A}$) through the point E, and the segment ST can be mapped onto the segment $\tilde{S}\tilde{T}$ by mappings which are η_4-translations, where $\eta_4 < \min\{\eta_1, \eta_2, \eta_3\}$. The existence of such δ_4 is self-evident.

Draw the path KL' of system (A) which crosses the side AB of the rectangle R at the point K lying between D and E (Figure 25), and the segment $C'F'$ at the point L'. From QT, §7.3 it follows that if point K is sufficiently close to D, L' is arbitrarily close to C'. We choose the point K so that if the length of the segment DK is less than $< \frac{\eta_2}{10}$, the length of $C'L'$ is less than $< \frac{\eta_3}{10}$. Let $\delta_5 > 0$ be so small that the following condition is satisfied: if system ($\tilde{A}$) is δ_5-close to system (A), we have $|D\tilde{D}| < \frac{\eta_2}{10}$, $|C'\tilde{C}'| < \frac{\eta_3}{10}$, the arc DD' of the separatrix L_ω can be mapped onto the arc $\tilde{D}\tilde{D}'$ of the separatrix $\tilde{L}_\omega$ by an η_2-translation, and the arc $C'C$ of the separatrix L_α can be mapped onto the arc $\tilde{C}'\tilde{C}$ of the separatrix $\tilde{L}_\alpha$ by an η_3-translation.

The existence of such δ_5 follows from Lemma 3 and from general considerations concerning the phase portraits of close systems (see Lemma 5, §4.1). Let finally $\delta_6 > 0$ be such that if system $(\tilde{A})$ is δ_6-close to system (A), the paths of (A) and $(\tilde{A})$ passing through any point M of the segment KE will intercept with increasing t the side $A'A_1'$ of the rectangle R' at two points which are distant less than η_3 from each other.

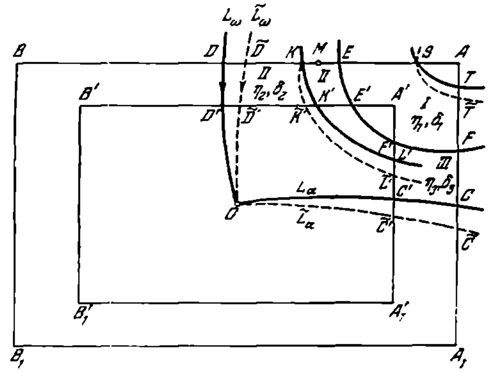

FIGURE 25

We will now show that if δ is a positive number less than any of our δ_i $(1 \leqslant i \leqslant 6)$ and system $(\tilde{A})$ is δ-close to system (A), then

$$(\sigma, A) \stackrel{\varepsilon}{\equiv} (\tilde{\sigma}, \tilde{A}). \tag{26}$$

To prove this we will establish the existence of a mapping θ which realizes the relation (26). The mapping θ is constructed in several stages. We will describe these successive stages by indicating what is mapped onto what and to what closeness. When the mappings are extended, care is taken to ensure that paths are still mapped into paths.

1) ST onto $\tilde{S}\tilde{T}$, EE' onto $E\tilde{E}'$, $E'F'$ onto $\tilde{E}'\tilde{F}'$, $F'F$ onto $\tilde{F}'\tilde{F}$, all these to closeness $\eta_4 < \min\{\eta_1, \eta_2, \eta_3\}$.

2) ES onto ES identically (to zero closeness).

3) Mappings 1 and 2 are extended to I: I onto $\tilde{\text{I}}$ to closeness ε.

4) DK onto $\tilde{D}K$ (closeness automatically $< 2\frac{\eta_2}{10} < \eta_2$), KE onto KE identically, DD' onto $\tilde{D}\tilde{D}'$ to closeness η_2.

5) Mappings 1 and 4 are extended to II: II onto $\tilde{\text{II}}$ to closeness ε.

6) $K'L'$ onto $\tilde{K}'\tilde{L}'$ arbitrarily, $K'E'$ onto $\tilde{K}'\tilde{E}'$ using the mapping which is induced by the identical mapping of KE onto itself.

7) Mappings 1 and 6 are extended to a mapping of the elementary quadrangle $K'E'F'L'$ onto the quadrangle $\tilde{K}'\tilde{E}'\tilde{F}'\tilde{L}'$. Since both quadrangles lie in $U_{\varepsilon/2}(O)$, the closeness is automatically less than ε.

8) O into itself, $D'O$ onto $\tilde{D}'O$, OC' onto $O\tilde{C}'$, $D'K'$ onto $\tilde{D}'\tilde{K}'$ by the mapping induced by mapping 4, DK onto $\tilde{D}K$.

9) Mappings 6 and 8 are extended to a mapping of the regular saddle-point region $L'K'D'OC'$ onto the analogous region $\tilde{L}'\tilde{K}'\tilde{D}'O\tilde{C}'$. This is feasible by QT, §18.4, Lemma 11. This mapping is an ε-translation, since both saddle-point regions are in $U_{\varepsilon/2}(O)$.

10) $C'C$ onto $\tilde{C}'\tilde{C}$ to closeness η_3, $C'F'$ onto $\tilde{C}'\tilde{F}'$ using the mappings induced by the mapping of $D'K'$ in 8 and the mapping of $K'E'$ in 6. The closeness is $<\eta_3$.

11) Mappings 1 and 10 are extended to III: III onto $\widetilde{\text{III}}$ to closeness ε.

Mappings 3, 5, 7, 9, and 11 jointly define a mapping θ of the regular saddle-point region σ onto $\tilde{\sigma}$. It is readily seen that θ is an ε-translation and maps paths into paths, i.e., it has the properties (a) and (b). The mappings θ_i $(i = 1, 2, 3)$ of σ_i onto $\tilde{\sigma}_i$ are constructed in the same way.

Note that on the arc OD of the separatrix L_ω the mapping θ_1 should coincide with the given mapping θ. Therefore, in constructing the mapping of DD' onto $\tilde{D}\tilde{D}'$ in 4, closeness to η_2 is not enough: we should ensure closeness to $\eta_2^* < \eta_2$, and correspondingly replace δ_2 by $\delta_2^* < \delta_2$. This is clearly always feasible. A similar remark applies to the mappings θ_2 and θ_3. The set of four mappings $\theta, \theta_1, \theta_2, \theta_3$ can be treated as a mapping of H onto $\tilde{H}$. This mapping realizes the relation (25). The proof of the lemma is completed.

Theorem 13. An equilibrium state $M_0(x_0, y_0)$ of the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y),$$

for which $\Delta < 0$ (a saddle point) is structurally stable.

Proof. Theorem 13 is proved in the same way as Theorem 12 (§8). However, instead of Lemma 1 and transformation (30), used in §8 in our proof of Theorem 12, we should use Lemma 4 and a linear transformation which reduces the modified system

$$\frac{dx}{dt} = \tilde{P}(x, y), \quad \frac{dy}{dt} = \tilde{Q}(x, y)$$

to the canonical form

$$\frac{dx}{dt} = \tilde{\lambda}_1 x + \tilde{\varphi}(x, y), \quad \frac{dy}{dt} = \tilde{\lambda}_2 y + \tilde{\psi}(x, y).$$

If system $(\tilde{\mathrm{A}})$ is sufficiently close to (A), this transformation, by Lemma 1, is arbitrarily close to the identity transformation. The arguments used in the proof of Lemma 12 therefore still apply. Q. E. D.

Remark 1. From Lemma 4 and the proof of Theorem 13 we see that if H is a sufficiently small canonical neighborhood of the saddle point O, then for any $\varepsilon^* > 0$ there exists $\delta^* > 0$ with the following property: if system (A^*) is δ^*-close to system (A), then

$$(H, A) \overset{\varepsilon^*}{\equiv} (H^*, A^*), \tag{27}$$

where H^* is the canonical neighborhood of the saddle point O^* of system (A^*).

Indeed, relation (27) follows from relation (25) of Lemma 4 and from the relation $(\tilde{H}, \tilde{A}) \overset{\frac{\varepsilon^*}{2}}{\equiv} (H^*, A^*)$ which is obtained in the proof of Theorem 13. But $\tilde{H}$ is a canonical neighborhood, and H^* and A^* are obtained from $\tilde{H}$ and $\tilde{A}$,

respectively, by a linear transformation (specifically, by the transformation which reduces (A^*) to the canonical form ($\tilde{A}$)). Therefore H^* is also a canonical neighborhood of the saddle point O^*.

Remark 2. As we have already noted, if $M_0(x_0, y_0)$ is a simple saddle point of system (A), any system sufficiently close to (A) has precisely one state of equilibrium in a sufficiently close neighborhood of M_0, which is also a saddle point.

§10. STRUCTURAL INSTABILITY OF AN EQUILIBRIUM STATE WITH PURE IMAGINARY CHARACTERISTIC ROOTS

1. Investigation of an equilibrium state with complex characteristic roots (a review)

We will show in this section that an equilibrium state with pure imaginary characteristic roots is structurally unstable. As in the previous sections, we will study, without loss of generality, canonical systems of the form

$$\frac{dx}{dt} = -\beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \psi(x, y), \tag{1}$$

where $\beta \neq 0$. We always assume that $\beta > 0$. φ and ψ are functions of class $k \geq 1$ or analytical in $\bar{G}$; they vanish at the point $O(0, 0)$ together with their first-order partial derivatives.

System (1) is a particular case of the system

$$\frac{dx}{dt} = \alpha x - \beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \alpha y + \psi(x, y). \tag{2}$$

The phase portrait of system (2) near the point $O(0, 0)$ is studied in detail in QT, §8. We will only summarize the corresponding results, which are needed in what follows. We use the same notation as in QT.

System (2) is investigated in polar coordinates, introduced by the relations $x = \rho\cos\theta$, $y = \rho\sin\theta$. The transformation to polar coordinates gives the set of equations

$$\frac{d\rho}{dt} = F(\rho, \theta), \quad \frac{d\theta}{dt} = \beta + \Phi(\rho, \theta), \tag{3}$$

where

$$\begin{gathered} F(\rho, \theta) = \alpha\rho + \varphi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta, \\ \Phi(\rho, \theta) = \frac{\psi(\rho\cos\theta, \rho\sin\theta)}{\rho}\cos\theta - \frac{\varphi(\rho\cos\theta, \rho\sin\theta)}{\rho}\sin\theta. \end{gathered} \tag{4}$$

It is further assumed that

$$\Phi(0, \theta) \equiv 0 \tag{5}$$

for $-\infty < \theta < +\infty$. This condition ensures the continuity of the function Φ.

System (3) can be reduced to a single equation

$$\frac{d\rho}{d\theta}=\frac{F(\rho,\theta)}{\beta+\Phi(\rho,\theta)}, \tag{6}$$

which is obtained by dividing the first equation in (3) through the second equation. The right-hand side of equation (6) is written in abbreviated form as $R(\rho,\theta)$:

$$R(\rho,\theta)=\frac{F(\rho,\theta)}{\beta+\Phi(\rho,\theta)}. \tag{7}$$

System (3) and equation (6), and hence the function $R(\rho,\theta)$, are considered in the strip

$$-\rho^*<\rho<\rho^* \tag{8}$$

of the (ρ,θ) plane, where ρ^* is a sufficiently small positive number.

Elementary calculations show that the function $R(\rho,\theta)$ has a continuous partial derivative with respect to ρ in the entire strip (8), and

$$\left.\frac{\partial R(\rho,\theta)}{\partial\rho}\right|_{\rho=0}=\frac{\alpha}{\beta}, \tag{9}$$

for any θ. The existence of a continuous derivative of R with respect to ρ indicates that the existence and uniqueness theorem and the theorem of continuous dependence on the initial conditions both apply to equation (6) in the strip (8) (QT, Appendix, §8.2). Therefore, for any θ_0 and ρ_0, $|\rho_0|<\rho^*$, there exists a unique solution of equation (6)

$$\rho=f(\theta,\theta_0,\rho), \tag{10}$$

satisfying the condition

$$f(\theta_0,\theta_0,\rho_0)\equiv\rho_0. \tag{11}$$

Solution (10) is defined in some (maximum) interval (θ_1,θ_2) which contains the point θ_0. Moreover,

$$f(\theta;\theta_0,0)\equiv 0, \tag{12}$$

so that $\rho\equiv 0$ (the axis θ in the (θ,ρ) plane) is a solution of equation (6). This solution is defined for all θ, $-\infty<\theta<\infty$.

The family of paths of system (3) in strip (8) coincides with the family of integral curves of equation (6). If

$$\rho=\rho(t),\quad \theta=\theta(t)$$

is a solution of system (3), $\hat{L}$ is the corresponding path, and (ρ_0,θ_0) is a point on this path,

$$\rho=f(\theta,\theta_0,\rho_0) \tag{13}$$

is the equation of this path.

The relation between the paths of system (2) in the (x, y) plane, on the one hand, and the paths of system (3) in the (ρ, θ) plane (or, equivalently, the integral curves of equation (6)), on the other, amounts to the following: the path $\rho = 0$ of system (3) in the (ρ, θ) plane corresponds to the equilibrium state O (0, 0) of system (2) in the (x, y) plane. Let now $\dot{L}$ be a path of system (3), other than the axis θ, which lies in the strip (8) and corresponds to the solution $\rho = \rho(t)$, $\theta = \theta(t)$; let (13) be the equation of this path. In the (x, y) plane, $\dot{L}$ corresponds to a path L of system (2) lying inside a circle of radius ρ^* centered at the point O, which describes the solution

$$x = \rho(t)\cos\theta(t), \quad y = \rho(t)\sin\theta(t) \tag{14}$$

of this system. Equation (13) can be considered as the equation of the path L in polar coordinates. Note that if L is a closed path, there is only one path $\dot{L}$ of system (3) corresponding to it; if, however, L is not closed, there is an infinity of such paths corresponding to it, with the parametric equations

$$\rho = \rho(t), \quad \theta = \theta(t) + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots; \text{ see QT, § 8.3}).$$

The investigation of the paths of system (2) in the neighborhood of O (0, 0) is based on the following proposition:

For any $\varepsilon > 0$ there exists $\eta > 0$, such that any path of system (2) which at $t = t_0$ passes through some point M_0 in $U_\eta(O)$, other than the point O itself, will cross with the increase and decrease in t every half-line θ = const without leaving $U_\varepsilon(O)$ (Figure 26) (QT, §8.4, Lemma 3).

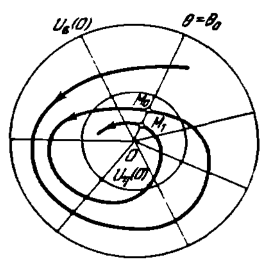

FIGURE 26.

Take any ray $\theta = \theta_0$, $\rho > 0$ (or $\rho < 0$). By the last proposition, if ρ_0 is sufficiently small (e.g., $|\rho_0| < \delta_0$), solution (13)

$$\rho = f(\theta, \theta_0, \rho_0)$$

is defined for all θ, $\theta_0 \leqslant \theta \leqslant \theta_0 + 2\pi$.

The function

$$\rho_1 = f(\theta_0 + 2\pi, \theta_0, \rho_0) = f_{\theta_0}(\rho_0)$$

is known as the succession function on the ray $\theta = \theta_0$, since the points $M_0(\rho_0, \theta_0)$ and $M_1(\rho_1, \theta_0)$ are two successive intersection points of a path of system (2) with this ray obtained for increasing t (for $\beta > 0$; for $\beta < 0$, $\frac{d\theta}{dt} < 0$ and increase in θ corresponds to a decrease in t). Since θ_0 is any number, we may take without loss of generality $\theta_0 = 0$, and for the corresponding succession function $f_0(\rho_0)$ we simply write $f(\rho_0)$. Thus,

$$f(\rho_0) = f(2\pi;\, 0, \rho_0). \tag{15}$$

Since $f(\theta, \theta_0, 0) \equiv 0$, we have

$$f(0) = 0. \tag{16}$$

To determine the behavior of the path L of system (2) which passes through the point M_0 with the polar coordinates ρ_0, 0 $(\rho_0 > 0)$, consider the function

$$d(\rho_0) = f(\rho_0) - \rho_0. \tag{17}$$

If $d(\rho_0) = 0$, the path L through M_0 is a closed path. If, however, $d(\rho_0) < 0$ $(d(\rho_0) > 0)$, L is a spiral which for $t \to +\infty$ (or $t \to -\infty$) either approaches the equilibrium state O or tends to a closed path enclosing the point O.

The equilibrium state O of system (2) is a stable (unstable) focus if and only if for all sufficiently small $\rho_0 > 0$, $d(\rho_0) < 0$ $(d(\rho_0) > 0)$.

2. Calculation of the first focal value

Since the right-hand side $R(\rho, \theta)$ of equation (6) has a continuous partial derivative with respect to ρ, the solution of this equation $f(0; \theta_0, \rho_0)$ is continuously differentiable with respect to ρ_0 (QT, Appendix, §8.3). The succession function $f(\rho_0)$ therefore has a continuous first derivative, whose value at $\rho_0 = 0$ we will now calculate.

Note that by definition $f(\theta;\, 0, \rho_0)$ satisfies the differential equation

$$\frac{df(\theta;\, 0, \rho_0)}{d\theta} = R(f(\theta;\, 0, \rho_0), \theta). \tag{18}$$

Differentiation with respect to ρ_0 gives

$$\frac{\partial}{\partial\rho_0}\left(\frac{df(\theta;\, 0, \rho_0)}{d\theta}\right) = \frac{\partial R(f(\theta;\, 0, \rho_0), \theta)}{\partial\rho}\,\frac{\partial f(\theta;\, 0, \rho_0)}{\partial\rho_0}. \tag{19}$$

As we know (/13/, §24, Theorem 16), the mixed partial derivative in the left-hand side is continuous and is thus independent of the order of differentiation. Therefore

$$\frac{d}{d\theta}\left(\frac{\partial f(\theta;\, 0, \rho_0)}{\partial\rho_0}\right) = \frac{\partial R(f(\theta;\, 0, \rho_0), \theta)}{\partial\rho}\,\frac{\partial f(\theta;\, 0, \rho_0)}{\partial\rho_0}. \tag{20}$$

This is a variational equation relative to the initial value ρ_0. It is clearly a linear differential equation with the unknown $\frac{\partial f}{\partial \rho_0}$. From (12) and (9) we see that for $\rho_0=0$ equation (20) takes the form

$$\frac{d}{d\theta}\left(\frac{\partial f(\theta; 0, 0)}{\partial \rho_0}\right)=\frac{\alpha}{\beta}\frac{\partial f(\theta; 0, 0)}{\partial \rho_0}. \tag{21}$$

Let us determine the corresponding initial condition. By (11), $f(0; 0, \rho_0)\equiv\rho_0$. Therefore

$$\left.\frac{\partial f(\theta; 0, \rho_0)}{\partial \rho_0}\right|_{\theta=0}=1. \tag{22}$$

Integrating (21) with the initial condition (22), we obtain

$$\frac{\partial f(\theta; 0, 0)}{\partial \rho_0}=e^{\frac{\alpha}{\beta}\theta}. \tag{23}$$

Hence, using (17) and (15), we find

$$d'(0)=e^{2\pi\frac{\alpha}{\beta}}-1. \tag{24}$$

The number $d'(0)$ is called the first focal value of the equilibrium state O.* We see from (24) that for an equilibrium state with pure imaginary characteristic roots, in particular for the equilibrium state $O(0, 0)$ of system (1), the first focal value is zero.

3. The theorem of the creation of a closed path from a multiple focus

An equilibrium state with pure imaginary characteristic roots is called a focus (either stable or unstable), a center, or a center-focus (QT, §8.6).

Definition 16. An equilibrium state of a dynamic system which has pure imaginary characteristic roots and is a focus will be called a multiple focus.

Theorem 14 (theorem of the creation of a closed path from a multiple focus). If the equilibrium state $M_0(x_0, y_0)$ of a dynamic system (A) is a multiple focus, then for any $\varepsilon>0$ and $\delta>0$, there exists a system $(\bar{A})$ δ-close to (A) which has at least one closed path in the ε-neighborhood of M_0.

We will use this theorem in our proof of the structural instability of an equilibrium state with pure imaginary roots. It is, however, also of considerable independent significance.

Proof. As before, it suffices to consider the equilibrium state $O(0, 0)$ of the canonical system

$$\frac{dx}{dt}=-\beta y+\varphi(x, y), \quad \frac{dy}{dt}=\beta x+\psi(x, y). \tag{A}$$

* The i-th focal value is the number $d^{(i)}(0)$ (if it exists, of course).

Let $\beta > 0$ and suppose that $O(0, 0)$ is a stable focus. As we have noted at the end of §10.1, there exists $r_0 > 0$, such that for all ρ_0, $0 < \rho_0 < r_0$, the function $d(\rho_0)$ is defined and $d(\rho_0) < 0$.

Let r_1 be some fixed number, $0 < r_1 < r_0$. Then

$$d(r_1) < 0. \tag{25}$$

Consider the modified system

$$\begin{aligned} \frac{dx}{dt} &= \tilde{\alpha}x - \beta y + \varphi(x, y), \\ \frac{dy}{dt} &= \beta x + \tilde{\alpha}y + \psi(x, y) \end{aligned} \tag{$\tilde{\mathrm{A}}$}$$

with the corresponding functions $\tilde{F}$, $\tilde{\Phi}$, etc., which are the analogs of the functions F, Φ, etc., relating to system (A). If $\tilde{\alpha}$ is sufficiently small in magnitude, system $(\tilde{\mathrm{A}})$ is arbitrarily close to system (A). It is readily seen that the functions $\tilde{F}$ and $\tilde{\Phi}$, and therefore also $\tilde{R}$, are arbitrarily close to the respective functions F, Φ, and R (§10.1, (4) and (6)), at least to rank 0. Then by Theorem 1, Appendix, subsection 1, the solutions $f(\theta, \theta_0, \rho_0)$ and $\tilde{f}(\theta, \theta_0, \rho_0)$ of the equations $\frac{d\rho}{d\theta} = R(\rho, \theta)$ and $\frac{d\rho}{d\theta} = \tilde{R}(\rho, \theta)$ are arbitrarily close to each other over a finite range of values of θ, and hence the numbers $d(r_1)$ and $\tilde{d}(r_1)$ are also arbitrarily close. Using (25) we thus conclude that if $\tilde{\alpha}$ is sufficiently small, we have

$$\tilde{d}(r_1) < 0. \tag{26}$$

Let $\tilde{\alpha}$ be so small that inequality (26) is satisfied; we choose this number positive, $\tilde{\alpha} > 0$. Then, by (24)

$$\tilde{d}'(0) = e^{2\pi\frac{\tilde{\alpha}}{\beta}} - 1 > 0. \tag{27}$$

Since $d'(\rho_0)$ is continuous, $\tilde{d}'(\rho_0) > 0$ for all sufficiently small ρ_0; in other words, $\tilde{d}(\rho_0)$ is an increasing function in a certain range. Hence, using the equality $\tilde{d}(0) = 0$ (see (15), (16), (17)), we conclude that for all sufficiently small positive ρ_0, $\tilde{d}(\rho_0) > 0$. In particular, for some r_2, $0 < r_2 < r_1$,

$$\tilde{d}(r_2) > 0. \tag{28}$$

From inequalities (26) and (28) and the continuity of $\tilde{d}$ it follows that there exists at least one r_3, $r_2 < r_3 < r_1$, such that $\tilde{d}(r_3) = 0$. This signifies that the path $\tilde{L}_0$ of system $(\tilde{\mathrm{A}})$ passing through the point with polar coordinates $(r_3, 0)$ is a closed path.

It is readily seen that if $\tilde{\alpha}$ and r_1 are sufficiently small, the closed path $\tilde{L}_0$ is entirely contained in $U_\varepsilon(O)$. Indeed, let $\rho = \tilde{f}(\theta; 0, \rho)$ be the general solution of the equation $\frac{d\rho}{d\theta} = \tilde{R}(\rho, \theta)$ corresponding to system $(\tilde{\mathrm{A}})$. The solution corresponding to the closed path $\tilde{L}_0$ is $\rho = \tilde{f}(\theta; 0, r_3)$. The general solution of the equation $\frac{d\rho}{dt} = \tilde{R}(\rho, \theta)$ is $\rho = f(\theta; 0, \rho_0)$, and $f(\theta; 0, 0) \equiv 0$ (see (12)). If $\tilde{\alpha}$ and r_1 are sufficiently small, $(\tilde{\mathrm{A}})$ is arbitrarily close to

system (A), the function $\tilde{R}(\rho, \theta)$ is arbitrarily close to $R(\rho, \theta)$, and r_3 is arbitrarily close to zero. Therefore, by the theorem of continuous dependence on the right-hand side and the initial conditions (Appendix, subsection 1, Theorem 2), for all θ, $0 \leqslant \theta \leqslant 2\pi$, the difference $\tilde{f}(\theta; 0, r_3) - f(\theta; 0, 0)$, equal to $\tilde{f}(\theta; 0, r_3)$, is less than ε, i.e.,

$$0 < \tilde{f}(\theta; 0, r_3) < \varepsilon.$$

This indicates that the closed path $\tilde{L}_0$ is entirely contained in $U_\varepsilon(O)$. Q. E. D.

Remark 1. Since by assumption $O(0, 0)$ is a focus of system (A), there are no closed paths in a sufficiently small neighborhood U of O. On the other hand, any system $(\tilde{A})$ sufficiently close to (A) with $\tilde{\alpha} > 0$ has at least one closed path in U. We will say in what follows that this closed path is created from the multiple focus.

Remark 2. Clearly, if r_3 is the least of numbers satisfying the conditions $r_2 < r_3 < r_1$, $\tilde{d}(r_3) = 0$, the closed path $\tilde{L}_0$ is a priori a limit-continuum from the inside (it may, however, also be a limit-continuum from the outside, and thus a limit cycle). Similarly, if r_3 is the largest of these numbers, $\tilde{L}_0$ is a priori a limit continuum from the outside.

Remark 3. In our proof of Theorem 14, system $(\tilde{A})$ can be replaced by the system

$$\begin{aligned} \frac{dx}{dt} &= P - \mu Q = -\mu\beta x - \beta y + \ldots, \\ \frac{dy}{dt} &= Q + \mu P = \beta x - \mu\beta y + \ldots \end{aligned} \qquad (\tilde{\tilde{A}})$$

whose vector field is obtained from the field of system (A) by rotating through an angle $\tan^{-1}\mu$ (see end of §3). If $\beta > 0$, the point $O(0, 0)$ is a simple focus of system $(\tilde{\tilde{A}})$, which is stable for $\mu > 0$ and unstable for $\mu < 0$. If $O(0, 0)$ is a stable focus of system (A), and $\tilde{\tilde{d}}$ is the function corresponding to system $(\tilde{\tilde{A}})$, we can again find r_1 and r_2, such that $0 < r_2 < r_1 < r_0$ and $d(r_1) < 0$, $\tilde{\tilde{d}}(r_1) < 0$, $\tilde{\tilde{d}}(r_2) > 0$.

The last inequality is satisfied for sufficiently small r_2 if $\mu < 0$. We thus have the following proposition: if $O(0, 0)$ is a multiple focus of system (A), a sufficiently small rotation of the vector field through a positive angle or through a negative angle creates a closed path in an arbitrarily small neighborhood of O.

4. Proof of structural instability

Theorem 15. The state of equilibrium $M_0(x_0, y_0)$ of the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y),$$

for which

$$\Delta = \begin{vmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{vmatrix} > 0,$$

$$\sigma = P'_x(x_0, y_0) + Q'_y(x_0, y_0) = 0,$$

i.e., a state of equilibrium with pure imaginary characteristic roots, is structurally unstable.

Proof. We will carry out the proof for an equilibrium state $O(0, 0)$ of system (A) in canonical form (1) (by Lemmas 1 and 2, §6.1, this can be done without loss of generality). Let O be a structurally stable equilibrium state of system (A). According to Definition 14 (§7.1) this implies that system (A) is structurally unstable in some neighborhood H of the point O, where H can be made arbitrarily small.

From the definition of structural stability, for any $\varepsilon > 0$ there is $\delta > 0$ such that for each $(\tilde{A})$ which is δ-close to (A) we have

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}), \tag{29}$$

where H is some region. The neighborhood H is made so small that O is the only equilibrium state of system (A) in it. The next step is to make ε so small that the region $\tilde{H}$, obtained from H by ε-translation, contains the point O. Finally, $(\tilde{A})$ is identified with system (2) with sufficiently small non-zero α.

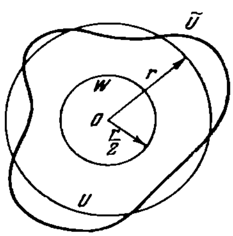

FIGURE 27

Under these conditions $O \in \tilde{H}$ and it is an equilibrium state of $(\tilde{A})$. It follows from (29) that, as H contains only one equilibrium state of (A), system $(\tilde{A})$ has only one equilibrium state in $\tilde{H}$, namely the point O. Since $\alpha \neq 0$, the point O is a focus of system $(\tilde{A})$. Hence, in a sufficiently small neighborhood of O system $(\tilde{A})$ has no closed paths. Under the ε-translation realizing relation (29), point O is mapped into itself (since this is a path-conserving translation). Therefore system (A) does not have any closed paths in a sufficiently small neighborhood of O, either. This indicates that O is neither a center nor a center-focus for (A), i.e., it is inevitably a focus.

We have shown so far that if $O(0, 0)$ is a structurally stable state of equilibrium of system (A), O is a multiple focus.

Let U be the neighborhood $U_r(O)$, where r is so small that system (A) has no closed paths in U and the only equilibrium state in this neighborhood is O; moreover, system (A) is structurally stable in U. For a small r, these conditions are satisfied since O is a structurally stable multiple focus.

Let W denote $U_{r/2}(O)$ (Figure 27). Fix $\varepsilon > 0$, $\varepsilon < \frac{r}{4}$, and let $\delta > 0$ be so small that if system $(\tilde{A})$ is δ-close to (A), we have

$$(U, A) \overset{\varepsilon}{\equiv} (\tilde{U}, \tilde{A}), \tag{30}$$

where $\tilde{U}$ is some region. System $(\tilde{A})$ is a system δ-close to (A) which has a closed path $\tilde{L}_0$ in W; this system $(\tilde{A})$ exists in virtue of Theorem 14 (i.e., the theorem of creation of a limit-cycle from a multiple focus).

Note that $W \subset \tilde{U}$.* Therefore $\tilde{U}$ contains the closed path $\tilde{L}_0$ of system $(\tilde{A})$, and then, by (30), U contains a closed path of system (A). This contradicts our choice of U. The contradiction establishes the fallacy of our original

* All the points of W are distant more than $\frac{r}{2}$ from the boundary of U. Therefore a $\frac{r}{4}$-translation cannot move any point of W outside the "translated" region $\tilde{U}$. Also see footnote on p. 67.

assumption, namely that the equilibrium state O of system (A) is structurally stable. We have thus proved that this equilibrium state is structurally unstable. Q. E. D.

Remark 1. If (A) is a system of class N, but not of class $N+1$ ($N \geqslant 1$), it can be treated as a point in any of the spaces $R_k^{(r)}$, $1 \leqslant k \leqslant N$, $1 \leqslant r \leqslant k$ (see §5.1). If (A) is an analytical system, it can be treated as a point of any space $R_k^{(r)}$ (k is any natural number, $1 \leqslant r \leqslant k$) or of any space $R_a^{(r)}$ (r is any natural number). Theorem 15 of structural instability remains valid relative to any of these spaces containing system (A). This is so because the modified system $(\tilde{A})$ used in our proof of structural instability differs from system (A) by the analytical increments $\tilde{\alpha}x$ and $\tilde{\alpha}y$, and is therefore part of any space containing (A).

Remark 2. Theorems 11, 12, 13, and 15 show that structurally stable equilibrium states are simple nodes, foci, and saddle points. From Remark 2 to Theorem 12 and Remark 2 to Theorem 13 it follows that if point O is a structurally stable equilibrium state of system (A), any system $(\tilde{A})$ sufficiently close to (A) has precisely one equilibrium state in a sufficiently small neighborhood of the point O, which is also structurally stable and is a point of the same type as that of system (A) (i.e., a node, a focus, or a saddle point, respectively).

§11. A SADDLE-TO-SADDLE SEPARATRIX

As is known (QT, §4.6 and §23.1), if a dynamic system considered in a bounded closed region has a finite number of equilibrium states, the α- or ω-limit-set of any path of this system is either (a) an equilibrium state, (b) a closed path, or (c) a limit continuum comprising a finite number of separatrices, which are continuations of one another from the same direction, and a finite number of equilibrium states.

We would like to establish what limit sets the paths of structurally stable systems have. In the preceding sections we have identified the structurally stable equilibrium states. The structural stability of a closed path is discussed in the next chapter. At this stage, consider a structurally stable system with a limit continuum of the form (c). This continuum comprises the separatrices of structurally stable saddle points (since other structurally stable equilibrium points have no separatrices), and each of these separatrices tends to a saddle point for both $t \to -\infty$ and $t \to +\infty$. Such a separatrix is said to *go from one saddle point to another*, and for brevity we will refer to it as a *saddle-to-saddle separatrix*. The two saddle points of such a separatrix are either distinct or coincident. We will show that a *saddle-to-saddle separatrix is structurally unstable* (Theorem 16). Hence it follows that the limit set of the paths of a structurally stable system is either an equilibrium state or a closed path.

1. The behavior of the separatrix under vector field rotation

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system considered in a bounded planar region $\bar{G}$, O a simple saddle point of this system, $O \in G$, L_0 the separatrix of the saddle point O. Let further C be a point of the separatrix L_0, and l an arc without contact through C which has no common points with L_0 (except the point C) or with other separatrices of the saddle point O. The arc l is defined by the parametric equations

$$x = f(s), \quad y = g(s), \quad a \leqslant s \leqslant b,$$

and the point C on l corresponds to the value s_0 of the parameter s $(a < s_0 < b)$. The positive direction along the arc l is the direction of increasing s.

To fix ideas, let the angles between the paths of system (A) and the arc l be positive (Figure 28).

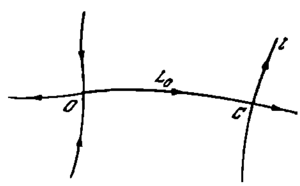

FIGURE 28

Consider a modified system (A*) of the form

$$\frac{dx}{dt} = P - \mu Q = P^*, \quad \frac{dy}{dt} = Q + \mu P = Q^*, \tag{A*}$$

where μ is a parameter. Clearly, if μ is sufficiently small in magnitude, (A*) is arbitrarily close to (A). System (A*) was considered at the end of §3 (§3.2, Lemma 3). It was shown that the sets of the equilibrium states of (A) and (A*) coincide and that at any point which is not a state of equilibrium the field of system (A) makes a constant angle θ with the field of system (A*), such that

$$\sin\theta = \frac{\mu}{\sqrt{1+\mu^2}}.$$

For $\mu > 0$ this angle is positive. According to §9 (see §9, Lemmas 2 and 3 and remark to Lemma 3), if (A*) is sufficiently close to system (A), i.e., if μ is sufficiently small, O is a saddle point of system (A*), whereas system (A) has a single separatrix L_0^* of the saddle point O; this separatrix meets the arc l, and there is only one intersection point C^*. If L_0 is an α (ω-)-separatrix of the saddle point O of system (A), L_0^* is an α (ω-)-separatrix of the saddle point O of system (A*). Let s_0^* be the value of the parameter s corresponding to the point C^*.

*L e m m a 1. If $\mu > 0$ and L_0 is an α-separatrix of the saddle point O, then $s_0^* > s_0$; if, however, $\mu > 0$ and L_0 is a ω-separatrix, then $s_0^* < s_0$.**

* We naturally assume that the above condition is satisfied, namely that the paths of system (A) make positive angles with the arc without contact l.

Proof. Without loss of generality we may take the saddle point O of system (A) to coincide with the origin and write (A) in the canonical form

$$\begin{aligned}\frac{dx}{dt}&=\lambda_1 x+\varphi(x, y)=P(x, y),\\ \frac{dy}{dt}&=\lambda_2 y+\psi(x, y)=Q(x, y),\end{aligned}\tag{1}$$

where $\lambda_1>0$, $\lambda_2<0$.

As in §9, consider a sufficiently small rectangle $\bar{R}$ with vertices A, B, B_1, A_1 (Figure 20). The paths of system (A) in this rectangle are shown in Figure 20. As the separatrix L_0 we take L_α. We will first prove a particular case, when the arc without contact l is the side AA_1 of the rectangle $\bar{R}$.* The positive direction along l is defined as the direction from A_1 to A, and the parameter s is identified with y. Under these conditions, the paths of system (A) form positive angles with A_1A.

Note that if $(\tilde{A})$ is a system sufficiently close to system (1), then

1) system $(\tilde{A})$ has a single equilibrium state $\tilde{O}$ in the rectangle $\bar{R}$, which is also a saddle point and is arbitrarily close to $O\,(0, 0)$;

2) the sides of the rectangle $\bar{R}$ are arcs without contact for the paths of system $(\tilde{A})$, the paths leaving R through the sides AA_1 and BB_1 and entering the rectangle R through AB and A_1B_1;

3) the separatrices $\tilde{L}_\alpha$ and $\tilde{L}_{1\alpha}$ of system $(\tilde{A})$ cross the sides AA_1 and BB_1 at the points $\tilde{C}$ and $\tilde{C}_1$, respectively, which are arbitrarily close to C and C_1, and the separatrices $\tilde{L}_\omega$ and $\tilde{L}_{1\omega}$ cross the sides AB and A_1B_1 at the points $\tilde{D}$ and $\tilde{D}_1$, respectively, which are arbitrarily close to D and D_1.

The validity of propositions 1 and 2 above is self-evident.

The validity of proposition 3 follows from propositions 1 and 2, the theorem of continuous dependence on the right-hand sides, and from the fact that a closed path cannot enclose only one equilibrium state of system $(\tilde{A})$ if it is a saddle point (see QT, §11.2, Corollary 1 of Theorem 29 and §11.4, Theorem 30), so that B contains no closed paths of system $(\tilde{A})$.**

For system $(\tilde{A})$ we takes a system (A^*) of the form

$$\frac{dx}{dt}=P-\mu Q=P^*,\qquad \frac{dy}{dt}=\mu P+Q=Q^*.\tag{A*}$$

Let $\mu>0$; this number is assumed to be so small that conditions 1, 2, 3 above are satisfied. Then system (A^*) has a single equilibrium state in $\bar{R}$, viz., the point O, which is a saddle point, and the separatrix L_α^* of this saddle point crosses the segment A_1A at a point C^*. Let y_0 be the ordinate of the point C, and y_0^* the ordinate of the point C^*. To prove the lemma, we have to establish that $y_0^*>y_0$. Consider the path L^* of system (A^*) which passes at $t=t_0$ through the point C. Since the separatrix L_α makes a positive angle with L^*, L^* enters into the region W, limited by the simple closed line OD_1A_1CO, as t increases (Figure 29). Take some point M^* from W which lies on the path L^*. As t increases, the path L of system (A) passing through M^* crosses the segment A_1A at the point K which is below the point C, and as t decreases it crosses the segment B_1A_1 at the point N. Consider the region W_1 limited by the simple closed curve NA_1KN.

* It is readily seen that if the rectangle $\bar{R}$ is sufficiently small, each of its sides, and in particular AA_1, satisfies the previous conditions imposed on the arc without contact l.

** The reader is advised to work out the complete proof of proposition 2.

The path L makes a positive angle with L^* at the point M^*, and therefore as t decreases, L^* penetrates into W_1. As t decreases further, the path L^* should emerge from this region. The path L^*, however, may cross neither the segment KA_1 nor the arc NK of the path L. Therefore, as t decreases, L^* will cross the segment NA_1. But then all the paths of system (A^*) passing through various points of the segment A_1C will cross the segment NA_1 as t decreases, i.e., none of these paths is the separatrix L^*_α of system (A^*). Hence it follows that the separatrix L^*_α crosses the segment A_1A at a point C^* above C, as originally stated. The corresponding proposition for the ω-separatrix L_ω is proved along the same lines. Our lemma is thus proved for a particular case, when the arc without contact is the segment A_1A (or BA). Now, by Lemma 10, §4.2, we see that the lemma also holds true for any arc without contact l, which satisfies the relevant conditions. Q. E. D.

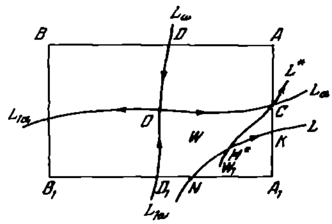

FIGURE 29

Remark. It is clear from the above proof that the lemma remains valid if system (A^*) is given in the form

$$\frac{dx}{dt} = P - \mu f(x, y) Q, \quad \frac{dy}{dt} = Q + \mu f(x, y) P,$$

with $f(x, y) > 0$ everywhere in $\bar{G}$, except the saddle point, where it may vanish.

2. Proof of structural instability

Theorem 16. A saddle-to-saddle separatrix is a structurally unstable path.

Proof. Let the path L_0 be also an α-separatrix of the saddle point O and a ω-separatrix of the saddle point O'. To fix our ideas, let O and O' be two distinct points (the same proof will apply if these two points coincide).

Let the separatrix L_0 be structurally stable, i.e., system (A) is structurally stable in any sufficiently small neighborhood H of the separatrix L_0. Then for any $\varepsilon > 0$ there exists $\delta > 0$, such that if system ($\tilde{A}$) is δ-close to system (A), we have

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}), \tag{2}$$

where $\tilde{H}$ is some region. Let d be the distance from the boundary of H to L_0 and choose $\varepsilon < d$. Let T be the mapping which realizes relation (2) (i. e., T is an ε-translation which transforms H into $\tilde{H}$ and maps paths into paths). Under the mapping T, the separatrix L_0 of system (A) is clearly mapped into a saddle-to-saddle separatrix $\tilde{L}_0$ of system ($\tilde{A}$) which lies in H (the latter follows from the inequality $\varepsilon < d$, so that the ε-translation does not move the path L_0 out of H).

We thus see that if L_0 is a structurally stable path of system (A), then, for any neighborhood H of this path, every system $(\tilde{A})$ sufficiently close to (A) has a saddle-to-saddle separatrix which is entirely contained in H. We will show that the last condition is not satisfied in a certain neighborhood H. This contradiction will evidently establish the validity of our theorem.

Let L_1, L_2, L_3 be separatrices of the saddle point O, and L_1', L_2', L_3' separatrices of the saddle point O', none of them coinciding with L_0. On the separatrix L_i (L'_i) we choose a point M_i (M_i') and draw through this point an arc without contact l_i (l_i'), whose end points do not coincide with M_i (M_i') ($i = 1, 2, 3$). Let further the point M_i (M_i') be the only common point of the arc l_i (l_i') with the separatrices of the saddle points O and O' (Figure 30).

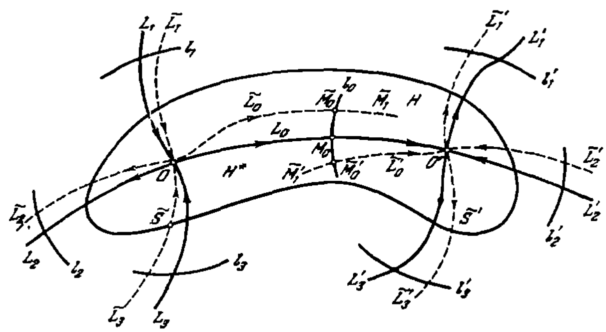

FIGURE 30

H is taken to be a sufficiently small neighborhood of the path L_0; it is so small that O and O' are the only equilibrium states of system (A) in this neighborhood, and the arcs without contact l_i and l_i' ($i = 1, 2, 3$) lie outside H.

Let M_0 be a point on the separatrix L_0. Through this point we draw an arc without contact l_0, which is entirely contained in H and which has no common points with the separatrices of the saddle points O and O', other than M_0.

Consider the system

$$\frac{dx}{dt} = P - \mu Q, \quad \frac{dy}{dt} = Q + \mu P. \qquad (\tilde{A})$$

Let $\mu > 0$. If μ is sufficiently small, the only equilibrium states of system $(\tilde{A})$ in H are O and O'; these equilibrium states are saddle points, and their separatrices $\tilde{L}_i$ and $\tilde{L}_i'$ ($i = 1, 2, 3$) cross the respective arcs without contact l_i and l_i' (in virtue of the remark to Lemma 3, §9). Moreover, the saddle point O of system $(\tilde{A})$ has a separatrix $\tilde{L}_0$ which crosses the arc l_0 at the point $\tilde{M}_0$, and the saddle point O' has a separatrix $\tilde{L}_0'$ which crosses the arc l_0 at the point $\tilde{M}_0'$. By the lemma of the previous subsection, $\tilde{M}_0$ and $\tilde{M}_0'$ lie on the arc l_0 on different sides of the point M_0 (Figure 30). We will prove that system $(\tilde{A})$ does not have in H a separatrix going from saddle point O to saddle point O'.

Indeed, suppose that such a separatrix exists (call it γ). Clearly γ cannot be one of the separatrices $\tilde{L}_i$ or $\tilde{L}_i'$ ($i = 1, 2, 3$), since these separatrices

cross the arcs l_i (l_i') and emerge from H. Hence γ should coincide both with $\tilde{L}_0$ and $\tilde{L}_0'$, i.e., at some $t=\tilde{t}_0$ it should pass through the point $\tilde{M}_0$ and at some $t=\tilde{t}_0'$ it should pass through the point $\tilde{M}_0'$. Clearly, $\tilde{t}_0<\tilde{t}_0'$. Let $\tilde{S}$ ($\tilde{S}'$) be the intersection point of one of the separatrices $\tilde{L}_i$ ($\tilde{L}_i'$), the separatrix $\tilde{L}_3$ ($\tilde{L}_3'$), say, with the boundary of region H.* We choose $\tilde{S}$ ($\tilde{S}'$) as the first intersection point of $\tilde{L}_3$($\tilde{L}_3'$) with the boundary, i.e., all the points of this separatrix between O and S' (between O' and $\tilde{S}'$) are entirely contained in H. Consider a simple closed line $O\tilde{M}_0\tilde{M}_0'O'\tilde{S}'\tilde{S}O$ made up of arcs of the paths $\tilde{L}_3$ and $\tilde{L}_3'$, the section $\tilde{M}_0\tilde{M}_0'$ of the arc l_0, and the sections $O\tilde{M}_0$ and $\tilde{M}_0'O'$ of the separatrix γ. This line delimits a certain subregion H^* of H.

Let $\tilde{M}_1$ be the point of γ corresponding to the time $\tilde{t}_0+\tau$, and $\tilde{M}_1$ the point corresponding to the time $\tilde{t}_0'-\tau$, where τ is a sufficiently small positive number and $\tilde{t}_0+\tau<\tilde{t}_0'-\tau$. The points $\tilde{M}_1'$ and $\tilde{M}_1$, as is readily seen, lie on two different sides of the arc without contact l_0. Therefore, one of these points lies outside H^* ($\tilde{M}_1$ in Figure 30), and the other lies inside this region. But then the path γ, moving from $\tilde{M}_1$ to $\tilde{M}_1'$ with the increase in t, should cross the boundary of H^*. This is impossible, however, since for $\tilde{t}_0+\tau<t<\tilde{t}_0'-\tau$ the separatrix cannot cross itself somewhere along $O\tilde{M}_0$ or $\tilde{M}_0'O'$, nor can it corss the arc without contact l_0. Moreover, for any t, the separatrix γ cannot have common points with the arcs $O\tilde{S}$ and $O'\tilde{S}'$ of the separatrices $\tilde{L}_3$ and $\tilde{L}_3'$, nor with the boundary of region H. We have thus proved that there is no separatrix in H going from O to O'. Q.E.D.

Corollary. If a dynamic system (A) is structurally stable in a bounded region, the α- and ω-limit sets of any path of this system are either equilibrium states or closed paths.

Indeed, if system (A) is structurally stable in a bounded region, it only has a finite number of equilibrium states (Theorem 10, §7.2). Therefore, by QT, §4.6 and §23.1, each limit set of the system is either an equilibrium state, or a closed path, or a limit continuum comprising saddle-to-saddle paths. The last possibility, however, is ruled out in virtue of Theorem 16.

Remark 1. System ($\tilde{\text{A}}$)

$$\frac{dx}{dt}=P-\mu Q, \quad \frac{dy}{dt}=Q+\mu P$$

is a system of the same class as the original system (A), and for sufficiently small μ it is arbitrarily close to (A) to any rank. Hence and from the proof of Theorem 16 it follows that a saddle-to-saddle separatrix of system (A) is a structurally unstable path relative to any space $R_N^{(r)}$, $R_a^{(r)}$ containing (A) as a point.

Remark 2. In the proof of Theorem 16, system (A) can be replaced with

$$\frac{dx}{dt}=P-\mu f(x, y)Q, \quad \frac{dy}{dt}=Q+\mu f(x, y)P, \qquad (\tilde{\text{A}}_f)$$

where the function $f(x, y)$ maintains a constant sign everywhere in $\bar{G}$, except the saddle points, where it may vanish. This follows from the remark to Lemma 1, §11.1. Thus, if $\mu\neq0$ is sufficiently small, the saddle-to-saddle separatrix disappears on passing from (A) to ($\tilde{\text{A}}_f$).

* It is assumed, without loss of generality, that the boundary of H is a simple smooth closed curve.

Chapter V

CLOSED PATHS IN STRUCTURALLY STABLE SYSTEMS

INTRODUCTION

In the present chapter we consider closed paths and classify them into structurally stable and structurally unstable. The chapter is made up of four sections (§12 through §15). In §12 we investigate the configuration of paths in the neighborhood of a closed path L_0. To this end, we draw an arc without contact l which intersects L_0 and on this arc consider the succession function $f(n)$ and the function $d(n) = f(n) - n$, where n is the parameter defined on l. It is established that either 1) a certain neighborhood of L_0 contains no closed paths other than L_0 itself, i.e., L_0 is a limit cycle, or 2) all the paths passing through the points of some neighborhood of L_0 are closed paths, or finally 3) any arbitrarily small neighborhood of L_0 contains both open and closed paths, other than L_0 itself. If the system is analytical, only cases 1 and 2 are possible. The particular topological structure of the dynamic system in a neighborhood of the path L_0 depends on the properties of the function $d(n)$; particularly significant is the quantity $d'(n_0)$, where n_0 is the value of the parameter on the arc l corresponding to the intersection of l with L_0.

In §13, a relatively simple system of curvilinear coordinates is introduced in the neighborhood of the closed path L_0 which makes it possible to calculate $d'(n_0)$. This system is introduced as follows: through every point $M(s)$ of L_0, corresponding to the time s, a segment of the normal is passed, and to the points on this segment we assign the coordinates s and n, where n is the value of the parameter on the normal. The arc l is identified with one of these normal segments. It is established that if the equations of the close path L_0 are $x = \varphi(t)$, $y = \psi(t)$, where φ and ψ are periodic functions of period τ, then

$$d'(0) = e^{\int_0^\tau [P'_x(\varphi(s),\, \psi(s)) + Q'_y(\varphi(s),\, \psi(s))]\, ds} - 1 = e^J - 1.$$

If the integral J in this equality does not vanish, L_0 is an isolated closed path, i.e., a limit cycle, and it is called a *simple limit cycle*. It is proved that if $J < 0$, L_0 is a stable limit cycle, and if $J > 0$, L_0 is an unstable limit cycle.

If L_0 is a limit cycle and $J = 0$, L_0 is called a *multiple limit cycle*.

In §14 it is proved that any simple limit cycle is a structurally stable path of a dynamic system (Theorem 18). Finally, in §15 we consider a

closed path L_0 with $J = 0$ and prove that this path is structurally unstable. The proof uses Theorem 19, which is of considerable independent interest (this is the theorem of creation of a closed path from a multiple limit cycle).

Note that the investigation of a structurally unstable closed path has much in common with the investigation of an equilibrium state with pure imaginary characteristic roots, presented in §10.

§12. A CLOSED PATH AND ITS NEIGHBORHOOD. SUCCESSION FUNCTION

1. Introduction of the succession function

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system of class N or an analytical system, defined in $\overline{G}$. Suppose that (A) has a closed path L_0 in G. Let

$$x = \varphi(t), \quad y = \psi(t) \tag{1}$$

be the motion corresponding to this path. $\varphi(t)$ and $\psi(t)$ are periodic functions of the same period, which we denote τ ($\tau > 0$). We are interested in the configuration of the paths of system (A) in the neighborhood of L_0. Let $\varepsilon_0 > 0$ be so small that $U_\varepsilon(L_0)$ does not contain any equilibrium states of (A) (this ε_0 exists since L_0 is a closed path).

On L_0, we choose some point M_0 and pass through this point an arc without contact l which is contained in $U_{\varepsilon_0}(L_0)$, ensuring that M_0 remains an inner point of l. On the arc l, we define some parameter n. This can be done, say, by specifying the parametric equations of the arc l:

$$x = g_1(n), \quad y = g_2(n). \tag{2}$$

The parameter $n = n_0$ corresponds to the point M_0 on the arc l.

Let $\eta > 0$ be so small that all the paths crossing at $t = t_0$ the part of the arc l corresponding to

$$n_0 - \eta \leqslant n \leqslant n_0 + \eta \tag{3}$$

cross the arc l again for t greater than t_0, without leaving $U_{\varepsilon_0}(L_0)$ prior to the second intersection. This η exists by QT, §3.8, Lemma 13. In this way, on the part of the arc l corresponding to the values (3) of the parameter n, we have defined a *succession function*

$$\bar{n} = f(n), \tag{4}$$

constructed in the direction of increasing t (QT, §3.8).

Let M' be the point of l corresponding to $n = n_0 - \eta$, L' the path of (A) passing through M', N' the point at which L crosses the arc l the second time with increasing t. The point N' evidently corresponds to $n = f(n_0 - \eta)$. N' is either different from M' or coincides with it. If the two points coincide, L' is a closed path, and we denote by Γ' the annular region between the paths L_0 and L'. If M' and N' are different points, L' is not a closed path, and Γ' is the annular region limited by the path L_0 and the closed curve consisting of the turn $M'N'$ of the path L' and the part $M'N'$ of the arc l (Figure 31).

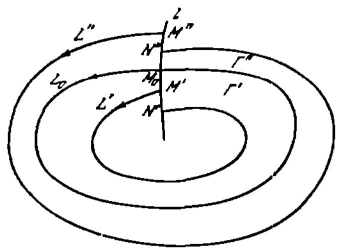

FIGURE 31

Similarly, taking M'' to be the point of the arc l corresponding to $n = n_0 + \eta$, we introduce a path L'', a point N'', and a region Γ'', analogous to Γ' (Figure 31). Regions of this type were considered in QT, §3.9. From QT, §3.9, Lemma 14 and Remark 1 to the lemma, we clearly see that

1) regions Γ' and Γ'' together with their boundaries are contained in $U_{\varepsilon_0}(L_0)$;

2) every path passing through a point of Γ' crosses the segment M_0N' (or M_0M', if N' lies on l between M_0 and M') of the arc l for both increasing and decreasing time t. Similarly, every path passing through a point of Γ'' crosses the segment M_0N'' (or M_0M'') of the arc l for both increasing and decreasing t.

2. The configuration of paths in the neighborhood of a closed path

Let us first establish the possible configuration of a single path passing near a closed path. To this end, we take a path L^* which for $t = t_0$ passes through a point M^* of the arc l corresponding to the value of the parameter n^*, $|n^* - n_0| < \eta$. Let M^{**} be the succeeding intersection point (the successor of M^*) of the path L^* (with the arc l). M^{**} corresponds to $n = f(n^*)$.

We introduce an auxiliary function

$$d(n) = f(n) - n, \tag{5}$$

which is analogous to the function $d(\rho)$ used in the investigation of a multiple focus (§10.1, (17)). The function $d(n)$ is a priori defined for all the values of n satisfying inequality (3), $|n - n_0| \leqslant \eta$. As we show in QT, §3.8, Remark 1 to Lemma 13, if system (A) and function (2) belong to class N (or to the analytical class), the function $f(n)$, and hence $d(n)$, are functions of the class N (or analytical).

The following cases should be considered:

1) $d(n^*) = 0$, i.e., $n^* = f(n^*)$. In this case M^* and M^{**} coincide, and L^* is a closed path completely contained in Γ' and Γ''.

2) $n^* > 0$, $d(n^*) < 0$, or $n^* < 0$, $d(n^*) > 0$. To fix ideas, let us consider the case $n^* > 0$, $d(n^*) < 0$. In this case, L^* is not a closed path, M^{**} is a point

of the arc l between M_0 and M^*, and the path L^* for $t > t_0$ is completely contained in Γ'' (Figure 32). Since Γ'' is contained in $U_{\varepsilon_0}(L)$ and therefore does not contain any equilibrium states, the path L^* for $t \to +\infty$ goes to some closed path which either coincides with L_0 or is completely contained in Γ''. For $t < t_0$, the path L^* may either emerge from Γ'' through the section $M''N''$ of the arc l, or remain entirely in Γ''. In the latter case, the path L^* is defined for all $t < t_0$, and its limit α-continuum is a closed path completely contained in $\bar{\Gamma}''$ which does not coincide with L_0. Note that if the α-limit or the ω-limit closed path of L^* is completely contained in Γ'', it is homotopic to zero in Γ'' (i.e., the internal region limited by this closed curve is not contained entirely in Γ'').

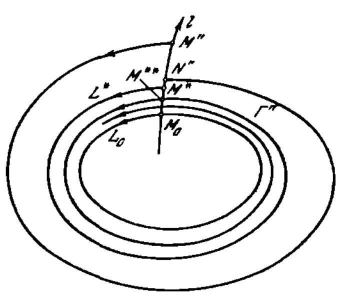

FIGURE 32

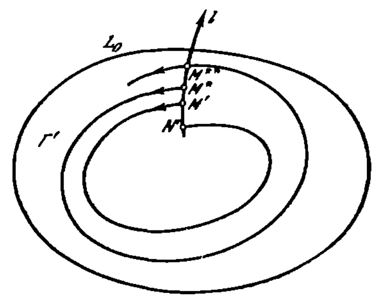

FIGURE 33

For $n^* < 0$, $d(n^*) > 0$, the situation is precisely the same, with the only difference that L^* is completely contained in Γ'. For $t \to +\infty$, the path L^* does not leave Γ', going either to L_0 or to a closed path in Γ'. For $t \to -\infty$, L^* either leaves Γ' or goes to a closed path in $\bar{\Gamma}'$ which is different from L_0 (Figure 33).

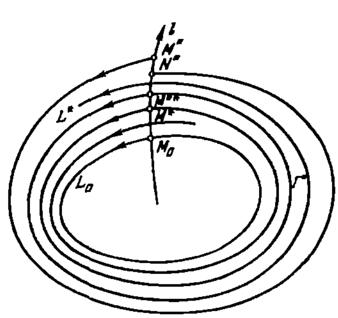

FIGURE 34

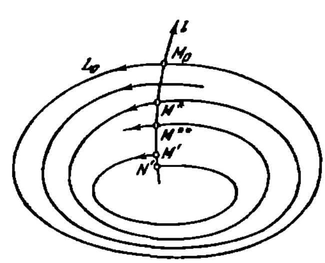

FIGURE 35

3) $n^* > 0$, $d(n^*) > 0$ or $n^* < 0$, $d(n^*) < 0$. This case is analogous to the preceding. L^* is not a closed path, M^* is a point of the arc l between M_0 and M^{**} (Figures 34 and 35), and the path L^* for $t \to -\infty$ goes either to a closed

path contained completely in Γ'' (Γ'), or to L_0. As increases ($t > t_0$), the path L^* either leaves Γ'' (Γ') or goes to a closed path in $\bar{\Gamma}''$ ($\bar{\Gamma}'$) which is different from L_0.

Cases 1, 2, 3 above clearly cover all the types of paths which cross the section $M'M''$ of the arc without contact l. Every path passing through a point of Γ' and Γ'' crosses this section $M'M''$ of the arc l either with increasing or with decreasing t. We have thus established the possible configuration of the paths which pass through points of Γ' and Γ''.

Let us now consider the topological structure of the dynamic system (A) in the neighborhood of a closed path L_0. To this end, we again introduce the function $d(n) = f(n) - n$. Since $n = n_0$ corresponds to the point M_0 of the closed path L_0 on the arc without contact, we have $d(n_0) = 0$. The following two cases should be considered:

a) there exists $m > 0$, $m \leqslant \eta$, such that $d(n) \neq 0$ for all $n, 0 < |n - n_0| \leqslant m$;

b) for any $m > 0$, there exists n, $0 < |n - n_0| < m$, such that $d(n_0) = 0$.

Let us consider case a in some detail. All the paths passing through the points of the arc l which correspond to the values of the parameter n, $0 < |n - n_0| \leqslant m$, are not closed.

The closed path L_0 in this case is isolated, i.e., it is a limit cycle (QT, §4.9). Because of the continuity of $d(n)$, we conclude that in case a, $d(n)$ has the same sign for all positive n and the same sign for all negative n ($|n| < m$). The following four subcases are therefore possible:

a₁) $d(n) < 0$ for $n > n_0$, $d(n) > 0$ for $n < n_0$;
a₂) $d(n) > 0$ for $n > n_0$, $d(n) < 0$ for $n < n_0$;
a₃) $d(n) < 0$ for $n > n_0$, $d(n) < 0$ for $n < n_0$;
a₄) $d(n) > 0$ for $n > n_0$, $d(n) > 0$ for $n < n_0$.

In case a_1, all the paths passing for a sufficiently small δ through the points of $U_\delta(L_0)$ which do not coincide with L_0 go to L_0 for $t \to +\infty$, whereas for decreasing t, they leave $U_\delta(L_0)$. The limit cycle L_0 is s t a b l e, and we shall say that all the paths sufficiently close to L_0, which do not coincide with L_0, w i n d o n t o L_0.

In case a_2, all the paths sufficiently close to L_0 (which do not coincide with L_0) go to L_0 for $t \to -\infty$, and for increasing t, they leave $U_\delta(L_0)$. In this case, the limit cycle L_0 is u n s t a b l e. We shall say that all these paths paths u n w i n d f r o m L_0.

In cases a_3 and a_4, all the paths which pass through points of $U_\delta(L_0)$ and lie on one side of L_0 go to L_0 for $t \to +\infty$ and leave $U_\delta(L_0)$ as t decreases; the paths lying on the other side of L_0, go to L_0 for $t \to -\infty$ and leave $U_\delta(L_0)$ as t increases.

The limit cycle in cases a_3 and a_4 is said to be s e m i s t a b l e. The paths close to the semistable cycle L_0 wind onto L_0 on one side and unwind from L_0 on the other side of the cycle.

Let us now consider case b. Any neighborhood of L_0 contains an infinite number of closed paths in this case. Without going into a detailed analysis, we will only note that this case, like the analogous case of an equilibrium state with pure imaginary characteristic values (see QT, §8.6), covers an infinity of various possible topological structures of the neighborhood of the closed path L_0. Any particular topological structure is completely determined by the properties of the function $d(n)$ and depends on the

structure of the set of its roots and on the signs of the function $d(n)$ whenever it does not vanish. In particular, if for sufficiently small n, $d(n) = 0$, all the paths through the points of a sufficiently small neighborhood of L_0 are closed.

3. The case of an analytical dynamic system

We will now consider in greater detail the case of an analytical system (A) and establish all the possible topological structures of the neighborhood of a closed path. We will derive the conditions to be satisfied by the succession function in the neighborhood of a closed path L_0 in order for this closed path to be a limit cycle of a certain type (stable, unstable, or semistable). The arc l (i.e., the functions (2)) is assumed analytical. Then, if (A) is analytical, the succession function $\bar{n} = f(n)$ is also analytical (QT, §3.8, Remark 1 to Lemma 13). As before, we assign to the closed path L_0 the parameter $n = n_0$, so that

$$f(n_0) = n_0 \quad \text{for} \quad d(n_0) = 0. \tag{6}$$

In the case of an analytical dynamic system (A), we have to consider two possibilities:

1) At least one of the derivatives $d^{(i)}(n_0)$ does not vanish, i.e., there exists a natural number $k \geqslant 1$ such that

$$d'(n_0) = d''(n_0) = \quad = d^{(k-1)}(n_0) = 0, \quad d^{(k)}(n_0) \neq 0. \tag{7}$$

In this case, expanding the function $d(n)$ around $n = n_0$ in powers of $n - n_0$, we get

$$\begin{aligned} d(n) = (n-n_0)^k [d^{(k)}(n_0) + (n-n_0)\, d^{(k+1)}(n_0) + \cdots] = \\ = (n-n_0)^k [d^{(k)}(n_0) + (n-n_0)\, \Phi(n)], \end{aligned} \tag{8}$$

where $\Phi(n)$ is some analytical function. It follows from (8) that the sign of $d(n)$ in the neighborhood of n_0 coincides with the sign of the number $(n - n_0)^k d^{(k)}(n_0)$. Hence it readily follows that for an even k, we have case a_3 or a_4 (according as $d^{(k)}(n_0)$ is negative or positive), i.e., a semistable limit cycle is obtained. For an odd k and $d^{(k)}(n_0) < 0$, we have case a_1, i.e., a stable limit cycle, and finally for odd k and $d'(n_0) > 0$, we have case a_2, i.e., an unstable limit cycle.

Let us consider separately the case $k = 1$, i.e., when

$$d'(n_0) = f'(n_0) - 1 \neq 0. \tag{9}$$

The closed path L_0 in this case is called a *simple limit cycle*. A simple limit cycle is evidently either stable or unstable, according as $d'(n_0)$ is positive or negative, or, equivalently, according as $f'(n_0) < 1$ or $f'(n_0) > 1$.

If $k > 1$, the closed path L_0 is called a *multiple limit cycle of multiplicity* k.* The multiplicity of the cycle evidently coincides with the multiplicity of the root $n = n_0$ of the function $d(n)$.

* These definitions of a *simple limit cycle* and a *multiple limit cycle of multiplicity* k are linked with the choice of the arc without contact l. For these definitions to be meaningful, we have to show that they are independent of the particular choice of the arc. No proof of this will be given here, but later (§13) an invariant definition of a simple limit cycle will be formulated.

2) All the derivatives $d^{(i)}(n_0)$ are zero,

$$d'(n_0) = d''(n_0) = \ldots = d^{(i)}(n_0) = \ldots = 0.$$

Then, by analyticity,

$$d(n) \equiv 0,$$

i.e., all the paths through the points of a sufficiently small neighborhood of the path L_0 are closed.

We have thus shown that any closed path of an analytical dynamic system is a simple or a multiple limit cycle, or else all the paths through the points of a sufficiently small neighborhood of the path L_0 are closed. In the case of an analytical dynamic system, the necessary and sufficient condition for a closed path L_0 to be a limit cycle is that at least one of the derivatives $d^{(i)}(n_0)$ $(i \geqslant 1)$ vanishes.

4. The case of a nonanalytical dynamic system

Let us now consider the case when a dynamic system (A) is not analytical and belongs to the class N. The arc l (i.e., the functions (2)) is also assumed to belong to the class N. Then, by QT, §3.8, Remark 1 to Lemma 13, the succession function $\bar{n} = f(n)$ is also a function of class N.

In this case, proceeding along the same lines as before, we can find the sufficient conditions for a closed path to be a limit cycle. Indeed, suppose that not all the derivatives of the function $d(n)$

$$d'(n_0), \ldots, d^{(N)}(n_0)$$

vanish at $n = n_0$, and there is a number k, $1 \leqslant k \leqslant N$, such that

$$d'(n_0) = \ldots = d^{(k-1)}(n_0) = 0, \quad d^{(k)}(n_0) \neq 0. \tag{10}$$

From this condition and from the equality $d(n_0) = 0$, we obtain using Taylor expansion

$$d(n) = \frac{d^{(k)}(n_0 + \theta(n - n_0))}{k!}(n - n_0)^k, \tag{11}$$

where $0 < \theta < 1$.

Since the k-th derivative is continuous, the sign of the coefficient of $(n - n_0)^k$ in (11) for $n - n_0$ of sufficiently small absolute value coincides with the sign of $d^{(k)}(n_0)$. Then it follows from (11), as in the previous section from (8), that we obtain one of the cases a_1, a_2, a_3, and a_4 according as k is even or odd and $d^{(k)}(n_0)$ is positive or negative.

As for an analytical dynamic system, we consider the case $k = 1$ separately.

In this case, the path is called a **simple limit cycle**, as before.*

The condition that at least one of the numbers $d^{(k)}(n_0)$, $k = 1, 2, \ldots, N$ does not vanish is sufficient, but not necessary, for the closed path L_0 of a system (A) of class N to be a limit cycle.

* See footnote to the definition of simple and multiple limit cycles in §12.3.

If for a system of class N

$$d'(n_0) = d''(n_0) = \ldots = d^{(N)}(n_0) = 0,$$

additional information regarding the function $d(n)$ is required in order to draw definite conclusions about the topological structure of the dynamic system near the closed path L_0.

§13. CURVILINEAR COORDINATES IN THE NEIGHBORHOOD OF A CLOSED PATH. SUCCESSION FUNCTION ON A NORMAL TO A PATH

1. Curvilinear coordinates in the neighborhood of a closed path

In the previous section we demonstrated that the succession function $f(n)$ on an arc without contact l (or the function $d(n) = f(n) - n$) and the values of its derivatives at the point n_0 corresponding to a closed path L_0 are of the greatest importance for investigating the topological structure of the dynamic system in the neighborhood of L_0. To calculate the derivatives of the succession function, we introduce an auxiliary system of curvilinear coordinates in the neighborhood of L_0. This system is analogous to the polar system of coordinates, and the treatment that follows is not unlike the analysis of the paths near a focus (§10).

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system (of class N or analytical), L_0 a closed path of this system,

$$x = \varphi(t), \quad y = \psi(t) \tag{1}$$

the motion corresponding to the path L_0, $\tau > 0$ the period of the functions φ and ψ. Note that the functions φ and ψ, being solutions of a system (A) of class N (or an analytical system), are functions of class $N+1$ (or analytical). In particular, they are a priori known to have continuous second derivatives. In what follows, the particular choice of the arc without contact for the construction of the succession function near the closed path is immaterial. We therefore choose l as the simplest of these arcs, namely a segment of the normal to the path L_0. We will construct a system of curvilinear coordinates in the neighborhood of L_0 which is the most convenient for our purpose.

Through every point $M(\varphi(s), \psi(s))$ of the path L_0 we draw a normal to the path at that point and lay off segments of length $\delta\sqrt{\varphi'(s)^2 + \psi'(s)^2}$ on either side of L_0 along the normal.

Lemma 1. If $\delta > 0$ is sufficiently small, no two segments of normals drawn through different points of L_0 have any points in common.

The proof of this lemma using the usual compactness considerations and the existence and continuity of the second derivatives of the functions φ and ψ is given in the Appendix, subsection 4.

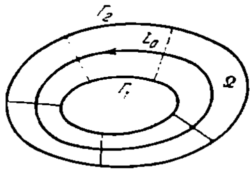

FIGURE 36

We will take $\delta > 0$ to be so small that Lemma 1 is satisfied. Then the ends of the segments of length $\delta \sqrt{\varphi'(s)^2 + \psi'(s)^2}$ laid off the normal on the positive (negative) side of the path L_0 form a simple closed curve Γ_1 (Γ_2). The curves Γ_1 and Γ_2 are "concentric" with the path L_0 and enclose a region Ω of the plane (x, y). Ω is clearly homomorphic to an open circular ring (Figure 36).

For a sufficiently small δ, system (A) has no equilibrium states in Ω. We will assume that this condition is also satisfied.

We introduce the functions

$$\bar{\varphi}(s, n) = \varphi(s) + n\psi'(s), \quad \bar{\psi}(s, n) = \psi(s) - n\varphi'(s), \tag{2}$$

where s stands for the time t. These functions are defined on the entire plane (s, n), but we will only consider them in the strip

$$-\infty < s < +\infty, \qquad -\delta < n < \delta. \tag{3}$$

The functions $\bar{\varphi}$ and $\bar{\psi}$ clearly have the following properties:

1) if (A) is an analytical system, $\bar{\varphi}$ and $\bar{\psi}$ are analytical functions; if (A) is a system of class N, $\bar{\varphi}$ and $\bar{\psi}$ are also functions of class N, and moreover $\bar{\varphi}$ and $\bar{\psi}$ have continuous derivatives with respect to n of all orders and continuous mixed derivatives $\frac{\partial^{k+l}\bar{\varphi}}{\partial n^k \partial s^l}$, $\frac{\partial^{k+l}\bar{\psi}}{\partial n^k \partial s^l}$ for any k and $s \leqslant N$.

2) The functions $\bar{\varphi}$ and $\bar{\psi}$ with their partial derivatives are periodic functions of s of period τ.

3) $\bar{\varphi}(s, 0) \equiv \varphi(s)$, $\bar{\psi}(s, 0) \equiv \psi(s)$, i.e., for $n = 0$ the equations $x = \bar{\varphi}(s, n)$, $y = \bar{\psi}(s, n)$ are the parametric equations of the path L_0 where the parameter s coincides with t.

4) The functional determinant

$$\Delta(s, n) = \frac{D(\bar{\varphi}, \bar{\psi})}{D(s, n)} = \begin{vmatrix} \bar{\varphi}'_s(s, n) & \bar{\varphi}'_n(s, n) \\ \bar{\psi}'_s(s, n) & \bar{\psi}'_n(s, n) \end{vmatrix} \tag{4}$$

for $n = 0$ is equal to

$$\Delta(s, 0) = -\varphi'(s)^2 - \psi'(s)^2,$$

i.e., for $n = 0$ it does not vanish for any $s, -\infty < s < +\infty$.

From property 4 and the compactness of the segment $0 \leqslant s \leqslant \tau$, using the periodicity of the functions $\bar{\varphi}$ and $\bar{\psi}$ in s, we see that for all sufficiently small n,

$$\Delta(s, n) \neq 0. \tag{5}$$

Finally, it is readily seen that if δ is sufficiently small, all the normal segments are arcs without contact for the paths of (A).

Let δ be so small that all the above conditions are satisfied. To help the reader, we list them here again.

1) The segments of normals of length $2\delta\sqrt{\varphi'(s)^2+\psi'(s)^2}$ drawn through different points of the path L_0 do not intersect. Hence it follows, in particular, that the region Ω is homomorphic to a circular ring.

2) All these normal segments are arcs without contact for the paths of (A).

3) There are no equilibrium states of (A) in Ω.

4) The determinant $\Delta(s, n)$ does not vanish in the strip (3).

Consider the mapping defined by the equalities

$$x=\bar{\varphi}(s, n)\equiv\varphi(s)+n\psi'(s),\qquad\qquad (6)$$
$$y=\bar{\psi}(s, n)\equiv\psi(s)-n\varphi'(s).$$

It has the following properties:

a) Mapping (6) maps the strip (3) of the (s, n) plane onto the annular region Ω of the (x, y) plane.

b) The axis $n=0$ is mapped by (6) into the path L_0, the lines $n = c$, $0<|c|<\delta$ parallel to the axis s are mapped into nonintersecting simple closed curves which lie one inside the other in Ω (these curves are "concentric" with L_0).

c) The segments $s=\text{const}$, $-\delta<n<\delta$, of the strip (3) are mapped into normal segments to L_0 (Figure 37).

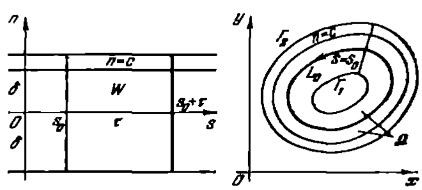

FIGURE 37

d) All the points (s, n) with the same value of n and s differing by multiples of the period τ, i.e., all the points of the form $(s+k\tau, n)$, $k=0, \pm 1, \pm 2, \ldots$, are mapped into one point of the plane (x, y).

e) The mapping (6) is now one-to-one. However, for any fixed s_0, it is one-to-one on the "half-open" rectangle in the (s, n) plane, defined by the inequalities

$$s_0\leqslant s<s_0+\tau,\quad -\delta<n<\delta,\qquad\qquad (W)$$

and it maps each of these rectangles onto Ω.

It follows from property e that (6) is a locally one-to-one mapping, i.e., it is one-to-one in any sufficiently small region of the strip (3). Since moreover $\Delta(s, n)\neq 0$ and the mapping (6) is continuous, it is r e g u l a r in every small region of this kind. In other words, the mapping (6) is l o c a l l y r e g u l a r. s and n can be considered as curvilinear coordinates in Ω. To every point in Ω corresponds one value of the coordinate n and an

infinity of values of s, differing by multiples of the period τ (the situation is analogous in this respect to the polar coordinates on a plane).

2. Transformation to the variables s, n in a dynamic system

Take the system (A)

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y) \tag{A}$$

and change over — quite formally at this stage — to the new coordinates s and n. Differentiating

$$x=\bar{\varphi}(s, n), \quad y=\bar{\psi}(s, n)$$

with respect to t and using (A), we find

$$\begin{aligned}\frac{dx}{dt}&=\bar{\varphi}_s'\frac{ds}{dt}+\bar{\varphi}_n'\frac{dn}{dt}=P(\bar{\varphi}(s, n), \bar{\psi}(s, n)),\\ \frac{dy}{dt}&=\bar{\psi}_s'\frac{ds}{dt}+\bar{\psi}'\frac{dn}{dt}=Q(\bar{\varphi}(s, n), \bar{\psi}(s, n)).\end{aligned} \tag{7}$$

By condition 4, the determinant

$$\Delta(s, n)=\begin{vmatrix}\bar{\varphi}_s' & \bar{\varphi}_n'\\ \bar{\psi}_s' & \bar{\psi}_n'\end{vmatrix}$$

does not vanish in the strip (3). Equations (7) are therefore solvable for $\frac{ds}{dt}, \frac{dn}{dt}$:

$$\frac{ds}{dt}=\frac{P(\bar{\varphi}, \bar{\psi})\bar{\psi}_n'-Q(\bar{\varphi}, \bar{\psi})\bar{\varphi}_n'}{\Delta(s, n)}, \quad \frac{dn}{dt}=\frac{Q(\bar{\varphi}, \bar{\psi})\bar{\varphi}_s'-P(\bar{\varphi}, \bar{\psi})\bar{\psi}_s'}{\Delta(s, n)}. \tag{8}$$

Since (see (2))

$$\bar{\varphi}(s, 0)=\varphi(s), \quad \bar{\psi}(s, 0)=\psi(s)$$

and

$$x=\varphi(t), \quad y=\psi(t), \quad 0\leqslant t\leqslant\tau,$$

is a solution of system (A), we have

$$\frac{\partial\bar{\varphi}(s, 0)}{\partial s}=P(\bar{\varphi}(s, 0), \bar{\psi}(s, 0)), \quad \frac{\partial\bar{\psi}(s, 0)}{\partial s}=Q(\bar{\varphi}(s, 0)\ \bar{\psi}(s, 0)) \tag{9}$$

for all s, $0\leqslant s\leqslant\tau$. Hence it follows that

$$\begin{aligned}P(\bar{\varphi}(s, 0), \bar{\psi}(s, 0))\bar{\psi}_n'(s, 0)-Q(\bar{\varphi}(s, 0), \bar{\psi}(s, 0))\bar{\varphi}_n'(s, 0)=\\ =\bar{\varphi}_s'(s, 0)\bar{\psi}_n'(s, 0)-\bar{\psi}_s'(s, 0)\bar{\varphi}_n'(s, 0)=\Delta(s, 0).\end{aligned} \tag{10}$$

By condition (5), $\Delta(s, 0)\neq 0$ for all s, $0\leqslant s\leqslant\tau$. The left-hand side of the last equality therefore does not vanish for all s, $0\leqslant s\leqslant\tau$. But then, from the

continuity of all the relevant functions and the compactness of the segment $0 \leqslant s \leqslant \tau$, we have for all sufficiently small n and for all s, $0 \leqslant s \leqslant \tau$,

$$P(\bar{\varphi}(s, n), \bar{\psi}(s, n))\bar{\psi}'_n(s, n) - Q(\bar{\varphi}(s, n), \bar{\psi}(s, n))\bar{\varphi}_n(s, n) \neq 0. \qquad (11)$$

Since the functions $\bar{\varphi}$ and $\bar{\psi}$ and their derivatives are periodic in s, the last relation is satisfied for all s and for sufficiently small n. We will assume that it is satisfied everywhere in the strip (3); this obviously can be accomplished by choosing $\delta > 0$ sufficiently small.

By (11), the right-hand side of the first equation in (8) does not vanish in the strip (3). Therefore, equations (8) now can be replaced by a single differential equation

$$\frac{dn}{ds} = \frac{Q(\bar{\varphi}, \bar{\psi})\bar{\varphi}'_s - P(\bar{\varphi}, \bar{\psi})\bar{\psi}'_s}{P(\bar{\varphi}, \bar{\psi})\bar{\psi}'_n - Q(\bar{\varphi}, \bar{\psi})\bar{\varphi}'_n} = R(s, n), \qquad (12)$$

which is obtained when the second equation in (8) is divided through by the first.

Let us consider the principal properties of the solutions of equation (12) and the relation of these solutions to the paths of (A).* The function $R(s, n)$ is defined and continuous in the strip (3), and it is readily seen that it is continuously differentiable with respect to n in (3). Therefore, both the theorem of existence and uniqueness and the theorem of continuous dependence on the initial conditions are applicable to equation (12). From the theorem of existence it follows that for any s_0 and n_0, $|n_0| < \delta$, there exists a unique solution of equation (12),

$$n = f(s;\ s_0, n_0), \qquad (13)$$

which is defined on some (maximal) interval (s_1, s_2) containing the point s_0 and satisfies the initial condition

$$f(s_0;\ s_0, n_0) = n_0. \qquad (14)$$

By (6) and (9),

$$R(s, 0) \equiv 0. \qquad (15)$$

Therefore $n = 0$ is the solution, and the axis s in the plane (s, n) is an integral curve of equation (12).

All the integral curves of equation (12) lying in the strip (3) evidently coincide with the paths of (8). If

$$s = s(t), \qquad n = n(t) \qquad (16)$$

is a path $\hat{L}$ of (8), and s_0, n_0 is a point on this path, the solution $n = f(s; s_0, n_0)$ of equation (8) is an equation of $\hat{L}$ in the coordinates s, n. The mapping (6) moves the path $\hat{L}$ into the line

$$x = \bar{\varphi}(s(t), n(t)), \qquad y = \bar{\psi}(s(t), n(t)), \qquad (17)$$

* The treatment of these subjects is analogous to the treatment of paths near a focus. See §10.1 and also QT, §8.3.

which, in virtue of local regularity of (6), is a path of system (A) lying in the ring Ω. Let this path be L. It is readily seen that every path L of (A) in Ω is an image (under (6)) of at least one path $\hat{L}$ of (8) in the strip (3), i.e., it is an image of at least one integral curve of equation (12).*

The equation

$$n = f(s;\ s_0,\ n_0) \tag{13}$$

of the path $\hat{L}$ in the plane (s, n) may be treated as an equation in curvilinear coordinates s, n of the path L on the plane (x, y). We will make use of this fact in our analysis of the succession function. Let l be the segment of the normal to L_0 which lies in $\overline{\Omega}$ and passes through the point M_0 of the path L_0 corresponding to $s = 0$. As we know, l is an arc without contact for (A).

From the theorems of existence and continuous dependence of the solution on the initial conditions, and also from the fact that $n = 0$ solves equation (12), we directly have the following propositions:

I. Any solution $n = f(s;\ s_0,\ n_0)$ of equation (12) for all possible s_0, $0 \leqslant s_0 \leqslant \tau$ and sufficiently small n_0 is defined for all s, $0 \leqslant s \leqslant \tau$ and can be written in the form

$$n = f(s;\ 0,\ n_0^*).$$

For system (A) this means, in geometrical terms, that every path of the system passing through a point in a sufficiently small neighborhood of L_0 crosses the normal l (for $s = 0$) and also all the other normals to L_0 in the ring Ω, and then crosses the normal l again (for $s = \tau$) (Figure 38). Note that for $n = 0$, $\frac{ds}{dt} = 1$ (see (10) and (8)), and since the numerator and the denominator of the expression for $\frac{ds}{dt}$ in (8) do not reverse their sign in the strip (3), we have $\frac{ds}{dt} > 0$ in this strip, i.e., an increase in s corresponds to an increase in t, and vice versa.

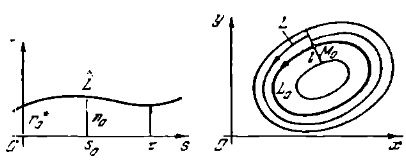

FIGURE 38

II. For any $\varepsilon > 0$, there exists $\eta > 0$, $\eta = \eta(\varepsilon)$, such that if $|n_0| < \eta$, then

$$|f(s,\ 0,\ n_0)| < \varepsilon$$

for all s, $0 \leqslant s \leqslant \tau$.

This implies that the part of a path of system (A) lying between two successive intersection points of the path with the arc without contact l is

* The path L of (A) is an image of one (if it is closed) or infinitely many (if it is open) paths of system (8). We will not go into this problem. See QT, §8.3.

completely contained in $U_\varepsilon(L_0)$ if the first intersection point corresponds to n_0 which is less than η in absolute value.

3. Succession function on a normal to a closed path

We will now consider the succession function on the arc without contact l defined in the previous section, i.e., on a normal through the point M_0. Since s increases with increasing t (see proposition I, §13.2), the succession function on the arc l is evidently $f(\tau; 0, n_0)$. For brevity, we will write simply $f(n_0)$. Thus,

$$f(n_0) = f(\tau;\ 0,\ n_0). \tag{18}$$

Since $n=0$ is a solution of equation (12), we have

$$f(s;\ 0,\ 0) \equiv 0, \tag{19}$$

and therefore

$$f(0) = 0. \tag{20}$$

In §12.3 and §12.4 we have shown that the derivatives of the function $d(n_0) = f(n_0) - n_0$ play an important role in investigating the topological structure of a dynamic system in the neighborhood of a closed path. We will now calculate the first derivative of this function at the point $n_0 = 0$. The calculation method is completely analogous to the calculation of the first vocal value of an equilibrium state with complex characteristic numbers (see §10.2).

By definition, $f(s; 0, n_0)$ is a solution of the differential equation (12), so that

$$\frac{df(s;\ 0,\ n_0)}{ds} = R(s,\ f(s;\ 0,\ n_0)). \tag{21}$$

Differentiating with respect to n_0, we obtain

$$\frac{\partial}{\partial n_0}\left(\frac{df(s;\ 0,\ n_0)}{ds}\right) = \frac{\partial R(s,\ f(s;\ 0,\ n_0))}{\partial n}\,\frac{\partial f(s;\ 0,\ n_0)}{\partial n_0}. \tag{22}$$

Since $R(s, n)$ is a continuous function with a continuous partial derivative with respect to n, the mixed derivative in the left-hand side of (22) is continuous and independent of the order of differentiation (see /13/, §24, Theorem 16). Therefore,

$$\frac{d}{ds}\left(\frac{\partial f(s;\ 0,\ n_0)}{\partial n_0}\right) = \frac{\partial R(s;\ f(s;\ 0,\ n_0))}{\partial n}\,\frac{\partial f(s;\ 0,\ n_0)}{\partial n_0} \tag{23}$$

(a variational equation with respect to n_0).

By definition, $f(0;\ 0,\ n_0) \equiv n_0$. Therefore

$$\left.\frac{\partial f(s;\ 0,\ n_0)}{\partial n_0}\right|_{s=0} = 1. \tag{24}$$

Setting $n_0 = 0$ in equation (23) and integrating with the initial condition (24), we find

$$\left.\frac{\partial f(s;\,0,\,n_0)}{\partial n_0}\right|_{n_0=0} = e^{\int\limits_0^s \frac{\partial R(s;\, f(s;\,0,\,0))}{\partial n}\, ds} \tag{25}$$

Since $f(s;0,0)=0$ (see (19)), the integrand in (25) is equal to $\frac{\partial R(s,\,0)}{\partial n}$. Let us calculate it explicitly.

Differentiation of (12) with respect to n gives

$$\frac{\partial R(s,\,n)}{\partial n} = \frac{-[P'_x\bar{\varphi}'_n + P'_y\bar{\psi}'_n]\,\bar{\psi}'_s + [Q'_x\bar{\varphi}'_n + Q'_y\bar{\psi}'_n]\,\bar{\varphi}'_s - P\bar{\psi}''_{sn} + Q\bar{\varphi}''_{sn}}{P\bar{\psi}'_n - Q\bar{\varphi}'_n} +$$
$$+ \frac{P\bar{\psi}'_s - Q\bar{\varphi}'_s}{(P\bar{\psi}'_n - Q\bar{\varphi}'_n)^2}\,\frac{\partial}{\partial n}(P\bar{\psi}'_n - Q\bar{\varphi}'_n). \tag{26}$$

where $P = P(\bar{\varphi},\,\bar{\psi}), P'_x = P'_x(\bar{\varphi},\,\bar{\psi})$, etc.

From (9) we see that for $n=0$, $P(\bar{\varphi},\,\bar{\psi})\,\bar{\psi}'_s - Q(\bar{\varphi},\,\bar{\psi})\,\bar{\varphi}'_s \equiv 0$. Therefore, for $n=0$, only the first fraction remains in the right-hand side of (26).

We have

$$\frac{\partial \varphi}{\partial s} = P(\varphi(s),\ \psi(s)), \qquad \frac{\partial \psi}{\partial s} = Q(\varphi(s),\ \psi(s)).$$

Differentiating with respect to s, we find

$$\varphi''(s) = P'_x(\varphi(s),\ \psi(s))\,\varphi'(s) + P'_y(\varphi(s),\ \psi(s))\,\psi'(s),$$
$$\psi''(s) = Q'_x\varphi'(s) + Q'_y\psi'(s).$$

Hence

$$P'_y(\varphi(s),\ \psi(s))\,\psi'(s) = \varphi''(s) - P'_x(\varphi(s),\ \psi(s))\,\varphi'(s),$$
$$Q_x(\varphi(s),\ \psi(s))\,\varphi'(s) = \psi''(s) - Q'_y(\varphi(s),\ \psi(s))\,\psi'(s). \tag{27}$$

Taking $n=0$ in (26) and using (6), (9), and (27), we find after simple manipulations

$$\frac{\partial R(s,\,0)}{\partial n} = P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s)) - \frac{2(\varphi'(s)\,\varphi''(s) + \psi'(s)\,\psi''(s))}{[\varphi'(s)]^2 + [\psi'(s)]^2}$$

or

$$\frac{\partial R(s,\,0)}{\partial n} = P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s)) - [\ln([\varphi'(s)]^2 + [\psi'(s)]^2)]'. \tag{28}$$

The left-hand side of the last relation is the integrand in (25). Therefore,

$$\left.\frac{\partial f(s;\,0,\,n_0)}{\partial n_0}\right|_{n_0=0} = e^{\int\limits_0^s \{[P'_x + Q'_y] - [\ln([\varphi'(s)]^2 + [\psi'(s)]^2)]'\}\, ds} =$$
$$= \frac{[\varphi'(0)]^2 + [\psi'(0)]^2}{[\varphi'(s)]^2 + [\psi'(s)]^2}\, e^{\int\limits_0^s [P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s))]\, ds} \tag{29}$$

From (18) and (29), setting $s=\tau$ and remembering that the functions φ and ψ and their derivatives are periodic, we finally obtain an expression

for $f'(0)$:

$$f'(0) = \frac{\partial f(\tau;\ 0,\ n_0)}{\partial n_0}\Big|_{n_0=0} = e^{\int\limits_0^\tau [P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s))]\, ds} \tag{30}$$

Thus,

$$d'(0) = f'(0) - 1 = e^{\int\limits_0^\tau [P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s))]\, ds} - 1. \tag{31}$$

Definition 17. The number

$$\chi = \frac{1}{\tau}\int\limits_0^\tau [P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s))]\, ds \tag{32}$$

is called the characteristic index of a closed path L_0.

Direct calculations show that $\int\limits_0^\tau [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\, ds$, and therefore the characteristic index χ, are invariant under a transformation of coordinates, i.e., they do not depend on the particular system of coordinates in which the dynamic system is described. This integral is therefore completely determined by the closed path L_0. If the characteristic index $\chi \neq 0$, then $d'(0) \neq 0$, and the closed path L_0 is a limit cycle (§12.3 and §12.4).

Definition 18. A closed path L_0 is called a simple limit cycle if

$$\int\limits_0^\tau [P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s))]\, ds \neq 0 \tag{33}$$

or, equivalently, if the characteristic index χ does not vanish. If L_0 is a limit cycle and $\chi = 0$, i.e., condition (33) *is not met, the path L_0 is said to be a multiple limit cycle.*

Note that if $\chi = 0$, the closed path L_0 is not necessarily a limit cycle (see §12.4).

From §12.3, §12.4, and equation (31), we have the following theorem.

Theorem 17. A simple limit cycle L_0 is stable if

$$\int\limits_0^\tau [P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s))]\, ds < 0,$$

and unstable if this integral is positive.

§14. PROOF OF STRUCTURAL STABILITY OF A SIMPLE LIMIT CYCLE

Let

$$\frac{dx}{dt} = P(x,\ y), \quad \frac{dy}{dt} = Q(x,\ y) \tag{A}$$

be a dynamic system of class $N \geqslant 1$ or an analytical system, defined in $\bar{G}$. Let L_0 be a closed path of (A) in G, which is a simple limit cycle. Let also

$$x = \varphi(t), \qquad y = \psi(t), \tag{1}$$

where φ and ψ are periodic functions of period τ, describe the motion corresponding to the cycle L_0.

The present section is concerned with the proof of the following theorem.

Theorem 18. Any simple limit cycle L_0 of a dynamic system (A) *is a structurally stable path.*

Proof. We draw a normal through some point M_0 of the path L_0 and define a parameter n on this normal, as in the previous section.* The point M_0 corresponds to $n = 0$ on the normal. To fix our ideas, let the negative values of n correspond to the points of the normal which lie inside L_0, and positive n to the points of the normal which lie outside L_0 (Figure 39). Consider the part of the normal corresponding to n, $0 \leqslant |n| \leqslant n^*$. If $n^* > 0$ is sufficiently small, the corresponding part of the normal is an arc without contact, which we denote l.

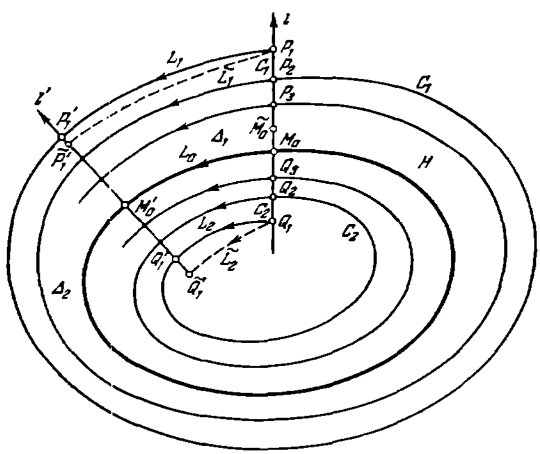

FIGURE 39

Let $\bar{n} = f(n)$ be a succession function constructed in the direction of increasing t. We may take it to have been defined everywhere along the arc l (this can be accomplished by using a sufficiently small n^*). Consider the function

$$d(n) = f(n) - n.$$

Clearly $d(0) = 0$, and from the definition of a simple limit cycle, $d'(0) \neq 0$. Let $d'(0) < 0$, i.e., the limit cycle L_0 is stable. If L_0 is an unstable limit cycle, the proof is completely analogous.

* If the point M_0 corresponds to $t = 0$ on the path L_0, the equations of the normal are (see §13.1, (6))

$$x = \varphi(0) + n\psi'(0), \qquad y = \psi(0) - n\varphi'(0).$$

Since 0 is a simple root of the function $d(n)$, this function has no other roots in a sufficiently small neighborhood of 0. We assume that $d(n)$ has no roots for $0 < |n| \leqslant n^*$. Under this assumption, all the paths crossing the arc l are not closed, going to the limit cycle L_0 for $t \to +\infty$.

Let L_1 be a path through some point P_1 of the arc l corresponding to $n = n_1$, $0 < n_1 < n^*$ (i.e., a path through some point outside L_0), and L_2 a path through some point Q_1 corresponding to $n = n_2$, $n_2 < 0$, $|n_2| < n^*$ (i.e., through a point inside L_0). We have thus generated a sequence of points

$$P_1, P_2, \ldots, P_n, \ldots$$

of the path L_1 lying on the part of the arc l outside L_0 which evidently go to the point M_0, and another sequence of points

$$Q_1, Q_2, \ldots, Q_n, \ldots$$

of the path L_2 lying on the part of the arc l inside L_0, which also go to the point M_0 (see QT, §3.7, Corollary 2 of Lemma 11). Evidently, any path crossing the segment P_1P_2 (Q_1Q_2) of the arc l at any of the points other than the ends successively crosses every other segment P_kP_{k+1} (Q_kQ_{k+1}) at one and only one point which is not an end point.

Together with l, let us consider another arc without contact l' passing through the point M_0' of the path L_0, which has no common points with l. A section of the normal through the point M_0' can be chosen as such an arc. We take $n^* > 0$ to be so small that each path crossing the arc l for $t = t_0$ crosses the arc l' for $t > t_0$, and has no other common points with either l or l'. Let the paths passing through the points P_1 and Q_1 (i.e., the paths L_1 and L_2) cross the arc l' at the points P_1' and Q_1', respectively (Figure 39).

Let C_1 be a simple closed curve formed by the arc P_1P_2 of the path L_1 and the segment P_2P_1 of the arc l, and C_2 a simple closed curve formed by the arc Q_1Q_2 of the path L_2 and the segment Q_2Q_1 of the arc l. Let further H be the region limited by the curves C_1 and C_2. All the paths of system (A) passing through the points of H evidently cross the arc l at points which fall between Q_1 and P_1, and L_0 is the only closed path between these paths. The segments P_1Q_1 and $P_1'Q_1'$ of the arcs without contact l and l' divide the region $\bar{H}$ into two elementary quadrangles $P_1P_1'Q_1'Q_1$ and $P_1'P_2Q_2Q_1'$. We denote the first quadrangle by Δ_1, and the second by Δ_2.

Let us now consider modified systems $(\tilde{A})$ sufficiently close to (A). By Lemmas 1, 2, and 11 of §4 and the remark to Theorem 5 (§1.3), there exists a number $\delta_1 > 0$ such that for any $(\tilde{A})$ which is δ_1-close to (A)

1) the arcs l and l' are arcs without contact;

2) on the arc l for all n, $|n| \leqslant n^*$, a succession function $\tilde{n} = \tilde{f}(n)$ is defined;

3) the equation $\tilde{d}(n) = \tilde{f}(n) - n = 0$ has a single root $\tilde{n}$ such that $|\tilde{n}| < n^*$, and this root satisfies the conditions $n_2 < \tilde{n} < n_1$ and $\tilde{d}'(\tilde{n}) < 0$;

4) all the paths of system $(\tilde{A})$ which for $t = t_0$ cross the segment P_1Q_1 of the arc l, cross for $t > t_0$ the arc l', without meeting again the arc l before this time. Let $\tilde{L}_1'$ and $\tilde{L}_2'$ be the paths of system $(\tilde{A})$ δ_1-close to (A) which pass through the points P_1 and Q_1 of the arc l, respectively, and $\tilde{P}_1'$ and $\tilde{Q}_1'$ the intersection points of these paths with the arc l'.

Condition 3 signifies that among the paths of system $(\tilde{A})$ δ_1-close to (A) which cross the arc l, there is only one closed path $\tilde{L}_0$, and this path is a stable limit cycle crossing the arc l at a point $\tilde{M}_0$ between the points P_1 and Q_1 on this arc. Evidently, the path $\tilde{L}_1$ lies outside the closed path $\tilde{L}_0$, and

the path $\tilde{L}_2$ inside $\tilde{L}_0$. The segment $P_1\tilde{M}_0$ of the arc l contains a sequence of points

$$\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \ldots$$

of the path $\tilde{L}_1'$ converging to the point $\tilde{M}_0$ (here $\tilde{P}_1$ coincides with P_1), and the segment $Q_1\tilde{M}_0$ of the arc l contains a sequence of points

$$\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \ldots$$

of the path $\tilde{L}_2$ converging to the point $\tilde{M}_0$ (here $\tilde{Q}_1$ coincides with Q_1). Consider simple closed curves analogous to the curves C_1 and C_2, namely the curve $\tilde{C}_1$ consisting of the arc $\tilde{P}_1\tilde{P}_2$ of the path $\tilde{L}_1$ and the segment $\tilde{P}_2\tilde{P}_1$ of the arc l, and the curve $\tilde{C}_2$ consisting of the arc $\tilde{Q}_1\tilde{Q}_2$ of the path $\tilde{L}_2$ and the segment $\tilde{Q}_2\tilde{Q}_1$ of the arc l. Let us further consider the region $\tilde{H}$ enclosed by the curves $\tilde{C}_1$ and $\tilde{C}_2$. All the paths of system $(\tilde{A})$ through the points of region $\tilde{H}$ evidently cross the arc l at points lying between Q_1 and P_1. Therefore, not a single closed path of system $(\tilde{A})$, except the path $\tilde{L}_0$, passes through any of the points in $\tilde{H}$. The segments $\tilde{P}_1\tilde{Q}_1$ and $\tilde{P}_1'\tilde{Q}_1'$ of the arcs without contact l and l' partition the region $\tilde{H}$ into two elementary quadrangles $\tilde{P}_1\tilde{P}_1'\tilde{Q}_1'\tilde{Q}_1$ and $\tilde{P}_1'\tilde{P}_2\tilde{Q}_2\tilde{Q}_1'$, which are designated $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$, respectively (these quadrangles are analogous to the quadrangles Δ_1 and Δ_2 of system (A)).

We will show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if system $(\tilde{A})$ is δ-close to system (A), then

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}).$$

In this way we shall have proved that L_0 is a structurally stable path (see §7.1 and Definition 10, §6.1).

Let σ be any positive number. We shall first show that if δ_2, $0 < \delta_2 < \delta_1$ is sufficiently small, and system $(\tilde{A})$ is δ_2-close to system (A), we can construct a topological mapping φ of the segment P_1Q_1 of the arc l onto itself which has the following properties:

(a) The points P_1 and Q_1 are mapped into themselves, i.e., $\varphi(P_1) = P_1$, $\varphi(Q_1) = Q_1$. Points lying on one path of system (A) are mapped into points of one path of system $(\tilde{A})$ (and vice versa).

(b) The mapping φ is a σ-translation (i.e., for any point M of the segment P_1Q_1, $\rho(M, \varphi(M)) < \sigma$).

We will first construct a topological mapping of the arc P_1Q_1 onto itself which satisfies property (a) above, without bothering with property (b) for the time being. This mapping is constructed separately on the segment P_1M_0 and then on the segment Q_1M_0. The segment P_1M_0 is mapped onto the segment $\tilde{P}_1\tilde{M}_0$ in the following manner:

1) Take any topological mapping φ_1 of the segment P_1P_2 onto the segment $\tilde{P}_1\tilde{P}_2$ such that

$$\varphi_1(P_1) = \tilde{P}_1, \quad \varphi_1(P_2) = \tilde{P}_2.$$

2) Assuming that the mapping φ_{k-1} of the segment $P_{k-1}P_k$ onto $\tilde{P}_{k-1}\tilde{P}_k$ ($k = 2, 3, 4, \ldots$) has been constructed, we define φ_k as the mapping of P_kP_{k+1} onto $\tilde{P}_k\tilde{P}_{k+1}$ *induced* by the mapping φ_{k-1} in the following sense: if M_{k-1} and M_k are two successive intersection points at which some path L of system (A) meets the path l as t increases, which lie in the segments $P_{k-1}P_k$ and P_kP_{k+1},

respectively, and $\tilde{M}_{k-1}$ and $\tilde{M}_k$ are two successive intersection points at which path $\tilde{L}$ of system $(\tilde{A})$ meets the arc l as t increases, which lie in the segments $\tilde{P}_{k-1}\tilde{P}_k$ and $\tilde{P}_k\tilde{P}_{k+1}$, respectively, and if $\varphi_{k-1}(M_{k-1}) = \tilde{M}_{k-1}$, then

$$\varphi_k(M_k) = \tilde{M}_k$$

(Figure 40). In this figure, the intersections of the arc l with the path L and with the path $\tilde{L}$ are shown separately for clarity. Since every arc $L(\tilde{L})$ of system (A) $((\tilde{A}))$ passing through points of the segment P_1P_2 $(\tilde{P}_1\tilde{P}_2)$ meets each of the segments P_kP_{k+1} $(\tilde{P}_k\tilde{P}_{k+1})$, $k = 2, 3, 4, \ldots$, precisely in one point, the mapping φ_k is single-valued. Clearly,

$$\varphi_k(P_k) = \tilde{P}_k, \quad \varphi_k(P_{k+1}) = \tilde{P}_{k+1}.$$

3) Let $\varphi(M_0') = M_0$ and suppose that φ coincides with φ_k on the segment P_kP_{k+1} $(k = 1, 2, \ldots)$.

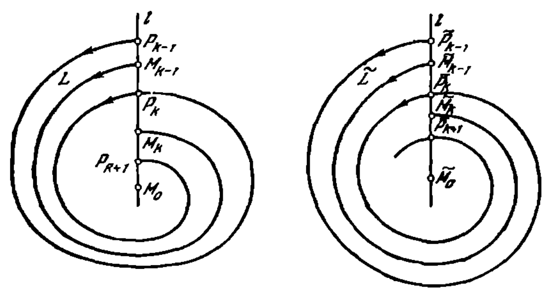

FIGURE 40

We obtain a mapping φ defined on the segment P_1M_0. Exactly the same mapping φ is constructed on the segment Q_1M_0 (i.e., we first take an arbitrary mapping φ_1^* of Q_1Q_2 on $\tilde{Q}_1\tilde{Q}_2$, and then the induced mappings φ_k^* of Q_kQ_{k+1} on $\tilde{Q}_k\tilde{Q}_{k+1}$). As a result, we obtain a mapping φ of P_1Q_1 onto $\tilde{P}_1\tilde{Q}_1$. The mapping φ constructed in this way satisfies condition (a) above and is a topological mapping.*

* *We can readily write an analytical expression defining the mapping φ. Let $f(n)$ be the succession function on the arc l defined by system (A), and $\tilde{f}(n)$ an analogous function for system $(\tilde{A})$. Let $g(n)$ be a function which defines a topological mapping of the segment P_1P_2 onto $\tilde{P}_1\tilde{P}_2$. Clearly, the iteration $[f(n)]^{k-1} = f^{(k-1)}(n)$ defines a mapping of P_1P_2 on P_kP_{k+1}. This is a topological mapping, and the inverse mapping therefore exists. The corresponding function for the inverse mapping will be designated $f^{-(k-1)}(n)$. The functions $\tilde{f}^{(k-1)}$ and $f^{-(k-1)}$ are treated similarly. Then if M is a point of the segment P_kP_{k+1}, n the value of the parameter corresponding to this point M, and $\tilde{n}$ the value of the parameter corresponding to $\varphi(M)$, we have*

$$\tilde{n} = \tilde{f}^{(k-1)}\, g f^{-(k-1)}(n).$$

An analogous expression is obtained for the mapping φ on the segment Q_kQ_{k+1}. $f^{(k)}(n) = [f(n)]^k$ is not a symbol of a derivative here: this is the k-th iteration of the function f, i.e., $f(f(f(\ldots f(n)\ldots)))$.

We will now show that for a sufficiently small δ_2 and an appropriate choice of φ_1 and φ_1^*, the mapping φ constructed according to the above formula will satisfy condition (b), i.e., it will be a σ-translation. To prove this, we start with a fixed $\sigma > 0$ and choose a sufficiently large natural number I, so that the segment P_IQ_I of the arc l is entirely contained in $U_{\sigma/2}(M_0)$ (the point M_0 evidently lies on this segment).

All the paths passing for $t = t_0$ through points of the segment P_1P_2 (Q_1Q_2, respectively) cross all the segments P_iP_{i+1} (Q_iQ_{i+1}, respectively) for $i = 2, 3, \ldots, I-1$ in a finite time (see QT, §3.8, Lemma 13). It follows from this and from §4.2, Lemmas 7 through 11, that for sufficiently small δ_2 and an appropriate choice of the mapping φ_1 and φ_1^*

1) the section $\tilde{P}_I\tilde{Q}_I$ of the arc l (containing the point $\tilde{M}_0$) is entirely contained in $U_{\sigma/2}(M_0)$;

2) the mapping φ_k of the segment P_kP_{k+1} onto the segment $\tilde{P}_k\tilde{P}_{k+1}$ ($k = 1, 2, \ldots, I-1$) is a σ-translation;

3) the mapping φ_k^* of the segment Q_kQ_{k+1} onto the segment $\tilde{Q}_k\tilde{Q}_{k+1}$ ($k = 1, 2, \ldots, I-1$) is a σ-translation.

Since the two segments P_IQ_I and $\tilde{P}_I\tilde{Q}_I$ lie in $U_{\sigma/2}$ and the mapping φ moves the former into the latter, the mapping φ is a σ-translation on the segment P_IQ_I. Hence and from conditions 2 and 3 it follows that under these conditions the mapping φ of P_1Q_1 is a σ-translation.

We have thus shown that if δ_2 is sufficiently small, and system $(\tilde{A})$ is δ_2-close to system (A), we can construct (for any $\sigma > 0$) a mapping φ of the segment P_1Q_1 of the arc l onto itself which satisfies conditions (a) and (b).

We will now show that for any $\varepsilon > 0$ and for system $(\tilde{A})$ sufficiently close to (A), we have

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}).$$

Let $\varepsilon > 0$ be fixed. By Lemma 8, §4.2, there exist $\sigma' > 0$ and $\delta_3 > 0$, $\delta_3 < \delta_2$ which satisfy the following condition: if φ' is a mapping of the segment $P_1'Q_1'$ of the arc l' onto the segment $\tilde{P}_1'\tilde{Q}_1'$ of the same arc, which is a σ'-translation, and systems (A) and $(\tilde{A})$ are δ_3-close, there exists a mapping T_2 of the elementary quadrangle Δ_2 onto the quadrangle $\tilde{\Delta}_2$ which coincides with φ' on $P_1'Q_1'$, moves paths into paths, and is an ε-translation. Let $\delta_4 > 0$, $\delta_4 < \delta_2$ be so small that if system $(\tilde{A})$ is δ_4-close to system (A), there exists a mapping T of the elementary quadrangle Δ_1 on the quadrangle $\tilde{\Delta}_1$ satisfying the following conditions:

1) T_1 coincides on P_1Q_1 with the previously described mapping φ of this segment onto itself;

2) T_1 moves paths into paths;

3) T_1 is an ε-translation;

4) on $P_1'Q_1'$, the mapping T is a σ'-translation (moving this segment into $\tilde{P}_1'\tilde{Q}_1'$).

The existence of a number $\delta_4 > 0$ with the above properties follows from Lemma 8, §4.2, and from the fact that for close systems the mapping φ is an arbitrarily small translation (property b). As the number δ we choose an arbitrary positive number, smaller than either δ_4 or δ_3. Let system $(\tilde{A})$ be δ-close to system (A). We construct a mapping φ of the segment P_1Q_1, then a mapping T_1 of the quadrangle Δ_1 onto $\tilde{\Delta}_1$ with properties 1 through 4. The mapping φ' is identified with the mapping T_1 on the segment $P_1'Q_1'$ (which

moves this segment into $\tilde{P}_1'\tilde{Q}_1'$), and as the last step we construct a mapping T_2 of the elementary quadrangle Δ_2 onto $\tilde{\Delta}_2$, which coincides with φ' on $P_1'Q_1'$, moves paths into paths, and is an ε-translation.

It is readily seen that the mapping T which coincides with T_1 on Δ_1 and with T_2 on Δ_2 moves the region H into $\tilde{H}$, conserves paths, and is an ε-translation. This indicates that

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}).$$

The proof of the theorem is thus complete.

Remark. It follows from our proof that if L_0 is a simple limit cycle of a dynamic system (A), there exist $\varepsilon^* > 0$ and $\delta^* > 0$ such that any system $(\tilde{A})$ δ^*-close to (A) has a single limit cycle $\tilde{L}_0$ in the ε^*-neighborhood of the paths L_0, and the characteristic indices of the cycles L_0 and $\tilde{L}_0$ have the same sign (i.e., the cycles L_0 and $\tilde{L}_0$ are either both stable or both unstable).

§15. STRUCTURALLY UNSTABLE CLOSED PATHS

1. The fundamental lemma

Let, as before,

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system of class $N \geqslant 1$ or an analytical system in region G, L_0 a closed path of this system ($L_0 \subset G$),

$$x = \varphi(t), \quad y = \psi(t) \tag{1}$$

the motion corresponding to this path, $\varphi(t)$, $\psi(t)$ are periodic functions of period $\tau > 0$. φ, ψ, being a solution of system (A) of a class $\geqslant 1$, are a priori functions of class 2.

Lemma 1. There exists a function of class 2, $z = F(x, y)$, defined in $\bar{G}$ such that for all s, $-\infty < s < +\infty$,

(a) $$F(\varphi(s), \psi(s)) \equiv 0;$$

(b) $$(F_x'(\varphi(s), \psi(s)))^2 + (F_y'(\varphi(s), \psi(s)))^2 \neq 0.$$

Proof of (a). We will first construct a function $F(x, y)$ of class 2 satisfying conditions a and b not in the entire region $\bar{G}$ but only in some neighborhood of the path L_0.

To this end, consider the set of equations

$$x = \varphi(s) + na(s), \quad y = \psi(s) + nb(s), \tag{2}$$

where $a(s)$ and $b(s)$ are periodic functions of period τ which belong to class 2, such that the functional determinant

$$\left.\frac{D(x, y)}{D(s, n)}\right|_{n=0} = \Delta(s) = \begin{vmatrix} \varphi'(s) & a(s) \\ \psi'(s) & b(s) \end{vmatrix} \tag{3}$$

does not vanish for any s:

$$\Delta(s) \neq 0, \quad -\infty < s < +\infty. \tag{4}$$

For $a(s)$ and $b(s)$ we may take the functions $\psi'(s)$ and $-\varphi'(s)$, respectively, provided they are functions of class 2 (this will be so if (A) is a system of class 2, say). If, however, $\psi'(s)$ and $-\varphi'(s)$ are functions of class 1, but not of class 2, $a(s)$ and $b(s)$ can be identified with trigonometric polynomials of period τ which ensure an adequate approximation to $\psi'(s)$ and $-\varphi'(s)$. These polynomials exist in virtue of Weierstrass's theorem (see /11/, Vol. 3. Sec. 734, p. 580) evidently meet condition (4).

Let δ be some sufficiently small positive number. We will consider (2) as equations which define the mapping of the strip

$$-\infty < s < +\infty, \quad -\delta \leqslant n \leqslant \delta \tag{5}$$

in the plane (s, n) into the plane (x, y).

Under this mapping, the axis s is clearly moved into the path L_0, and every vertical segment

$$s = \text{const}, \quad -\delta \leqslant n \leqslant \delta$$

of the strip (5) is moved to a straight segment l_s through the point $M(s)$ of the path L_0 corresponding to the value s of the parameter. In virtue of condition (4), the segment l_s does not touch the path L_0 at the point $M(s)$. Moreover, from the theorem of implicit functions and from condition (4) it follows that in a sufficiently small neighborhood of any point $(s, 0)$ of the axis s, the mapping (2) is one-to-one. But then it can be shown, precisely as in Lemma 1, §13.1, that if δ is sufficiently small, no two segments l_s corresponding to different points of the path L_0 (e.g., for $0 \leqslant s < \tau$) intersect. We will assume that this condition is indeed satisfied. The strip (5) is then moved by mapping (2) into some closed ring $\bar{\Omega}$ in the plane (x, y) enclosed by simple closed curves Γ_1 and Γ_2 (Figure 41). If δ is sufficiently small, then at any point of the strip (5)

$$\frac{D(x, y)}{D(s, n)} \neq 0. \tag{6}$$

We will assume that this condition is also satisfied.

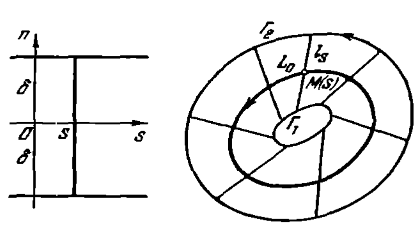

FIGURE 41

Mapping (2) of strip (5) onto the region $\bar{\Omega}$ is entirely analogous to the mapping (6) considered in §13.1 and it has similar properties. In

particular, the mapping (2) is locally one-to-one in strip (5), and by virtue of condition (6) it is also regular, i.e., this mapping is locally regular.

Let us consider (2) as a set of equations in s and n, assuming that $(x, y) \in \Omega$. In virtue of the above, to every point $(x, y) \in \Omega$ there corresponds precisely one value of n and infinitely many values of s, such that the numbers s, n satisfy equations (2) (the various s differ by a multiple of τ). The particular n is therefore a single-valued function of x, y. We will designate it by $F(x, y)$:

$$n = F(x, y). \tag{7}$$

If we consider only the local situation, s may also be regarded as a single-valued function of x and y. Moreover, n and s are functions of class 2 (in virtue of the theorem of implicit functions).

We will show that the function of class 2 defined by relation (7), $F(x, y)$, satisfies conditions (a) and (b) of the lemma.

Condition (a) follows directly from the fact that at every point (x, y), i.e., $(\varphi(s), \psi(s))$, the value of n corresponding to the path L_0 is zero in virtue of equations (2).

The proof of (b) will be based on local considerations, and both n and s will be regarded as single-valued functions of x and y.

Proof of (b). Differentiation of (2) with respect to x and y respectively gives

$$\begin{aligned} 1 &= [\varphi'(s) + na'(s)]\frac{\partial s}{\partial t} + a(s)\frac{\partial n}{\partial x}, \\ 0 &= [\psi'(s) + nb'(s)]\frac{\partial s}{\partial x} + b(s)\frac{\partial n}{\partial x} \end{aligned} \tag{8}$$

and

$$\begin{aligned} 0 &= [\varphi'(s) + na'(s)]\frac{\partial s}{\partial y} + a(s)\frac{\partial n}{\partial y}, \\ 1 &= [\psi'(s) + nb'(s)]\frac{\partial s}{\partial y} + b(s)\frac{\partial n}{\partial y}. \end{aligned} \tag{9}$$

For $n = 0$, each of the systems (8) and (9), considered as a linear system of equations in the partial derivatives, has a determinant $\Delta(s)$ which does not vanish in virtue of (4). Solving these systems and remembering that $n = F(x, y)$ and that for $n = 0$, $x = \varphi(s)$, $y = \psi(s)$, we find

$$\begin{aligned} \left(\frac{\partial n}{\partial x}\right)_{\substack{x=\varphi(s)\\ y=\psi(s)}} &= F'_x(\varphi(s), \ \psi(s)) = -\frac{\psi'(s)}{\Delta(s)}, \\ \left(\frac{\partial n}{\partial y}\right)_{\substack{x=\varphi(s)\\ y=\psi(s)}} &= F'_y(\varphi(s), \ \psi(s)) = \frac{\varphi'(s)}{\Delta(s)}. \end{aligned} \tag{10}$$

Hence,

$$(F'_x(\varphi(s), \ \psi(s)))^2 + (F'_y(\varphi(s), \ \psi(s)))^2 = \frac{[\varphi'(s)]^2 + [\psi'(s)]^2}{\Delta}. \tag{11}$$

The last expression does not vanish, since if $\varphi'(s) = \psi'(s) = 0$ for some s, $(\varphi(s), \psi(s))$ is an equilibrium state of system (A), and this contradicts the fact that the point $(\varphi(s), \psi(s))$ belongs to the closed path L_0. Condition (b) is thus proved.

We have thus shown that there exists a function $F(x, y)$ of class 2 in the ring $\overline{\Omega}$ which satisfies conditions (a) and (b) of the lemma.

The boundary of the ring $\bar{\Omega}$ comprises the two simple curves Γ_1 and Γ_2 with the parametric equations

$$x = \varphi(s) + \delta a(s), \quad y = \psi(s) + \delta b(s)$$

and

$$x = \varphi(s) - \delta a(s), \quad y = \psi(s) - \delta b(s).$$

The right-hand sides of these equations are functions of class 2. Therefore, by Whitney's theorem (see /11/, Vol. 1, para. 260, p. 594), the function $F(x, y)$ can be continued to the entire region $\bar{G}$ without changing its class. The continuation of the function evidently retains the same properties (a) and (b). This completes the proof.

Remark 1. If (A) is a system of class $N > 1$, there exists a function $F(x, y)$ of class $N + 1$ which satisfies conditions (a) and (b) of the lemma. The proof is the same as before, but $a(s)$ and $b(s)$ are functions of class $N + 1$.

Remark 2. Let the plane (x, y) in which system (A) is defined be a coordinate plane of the three-dimensional space (x, y, z). The equation $z = F(x, y)$, where F is the function discussed in Lemma 1, is the equation of a surface through the path L_0. It follows from condition (b) that at the points of this path the surface does not touch the plane (x, y).

2. The theorem of the creation of a closed path from a multiple limit cycle

We recall that a closed path L_0 is said to be a limit cycle if it is isolated, i.e., if any neighborhood of the path does not contain any closed path except L_0 itself. In this case, as we have seen in §12.2, all the paths which pass through points sufficiently close to the path L_0 either wind onto L_0 (a stable limit axis) or unwind from L_0 (unstable limit cycle), or else paths on one side of L_0 unwind from it and paths on the other side wind onto it (a semistable cycle). A limit cycle is called multiple if its characterisitic index is not zero (Definition 18, §13.3).

We will now prove a theorem which is the analog of the theorem of the creation of a closed path from a multiple focus (Theorem 14).

Theorem 19 (theorem of the creation of a closed path from a multiple limit cycle). Let (A) *be a dynamic system of class $N \geq 1$ (or analytical), L_0 a multiple limit cycle of the system. For any $\varepsilon > 0$ and $\delta > 0$, there exists a system* $(\tilde{A})$ *of the same class which is δ-close to* (A) *to rank $r \leq N$ $(r < +\infty)$ and which has at least two closed paths in the ε-neighborhood of L_0.*

Proof. The proof of Theorem 19 is based on Lemma 1, and in all other respects follows the proof of Theorem 14. We will consider, as in the previous sections, a succession function $f(n)$ and a function $d(n) = f(n) - n$ on some normal to the path L_0. Let n be the parameter along the normal, defined by equations (2) in §12.4. Since by assumption L_0 is a limit cycle, there exists $n^* > 0$ such that for all n, $|n| < n^*$, $n \neq 0$, a succession function $f(n)$ and $f(n) - n = d(n) \neq 0$ $(d(0) = 0)$ are defined. Let l

be the segment of the normal corresponding to the values of n such that $|n| \leqslant n^*$. To fix ideas, let L_0 be a stable limit cycle (the proof for an unstable or a semistable cycle proceeds along the same lines). Then for all n, $|n| < n^*$, $n \neq 0$,

$$\begin{aligned} d(n) &= f(n) - n < 0, \quad \text{if} \quad n > 0; \\ d(n) &= f(n) - n > 0, \quad \text{if} \quad n < 0. \end{aligned} \tag{12}$$

We will first prove the theorem for system (A) of class 1.

Let $\varepsilon > 0$ and $\delta > 0$ be given. Let $n^* > 0$ be so small that each arc of a path of system (A) whose ends lie on our normal and correspond to the values n and $f(n)$ of the parameter (i.e., an arc between two successive intersection points with the normal), $|n| \leqslant n^*$, is contained in $U_\varepsilon(L_0)$.

Take any n_1, $0 < n_1 < n^*$. By (12),

$$d(n_1) < 0. \tag{13}$$

Let further δ_1, $0 < \delta_1 < \delta$, be so small that if system $(\tilde{A})$ is δ_1-close to system (A) then (a) the normal l remains an arc without contact for the paths of system $(\tilde{A})$; (b) for all n, $|n| \leqslant n^*$, the functions $\tilde{f}(n)$ and $\tilde{d}(n)$ are defined on the normal; (c) the arcs of the paths of system $(\tilde{A})$ between two successive intersection points n and $\tilde{f}(n)$ with the normal ($|n| \leqslant n^*$) are contained in $U_\varepsilon(L_0)$, and (d) the following inequality is satisfied:

$$\tilde{d}(n_1) < 0. \tag{14}$$

The existence of a number δ satisfying the above conditions follows from Lemmas 1, 2, and 11 of §4.

We will consider modified systems of a special form, as follows.

Let $F(x, y)$ be a function of class 2 satisfying the conditions of Lemma 1, and μ any real number. We take a modified system $(\tilde{A})$ of the form

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) + \mu F(x, y) F'_x(x, y) = \tilde{P}(x, y), \\ \frac{dy}{dt} &= Q(x, y) + \mu F(x, y) F'_y(x, y) = \tilde{Q}(x, y). \end{aligned} \tag{$\tilde{A}$}$$

Clearly, $(\tilde{A})$ is a system of class 1, and if μ is sufficiently small in absolute value, $(\tilde{A})$ is arbitrarily close to (A).

Since $x = \varphi(t)$, $y = \psi(t)$ is a solution of system (A) corresponding to the path L_0, and the function $F(x, y)$ has been defined so that it satisfies the equality

$$F(\varphi(t), \psi(t)) \equiv 0, \tag{15}$$

the functions $\varphi(t)$, $\psi(t)$ are also a solution of system $(\tilde{A})$, i.e., the path L_0 of system (A) is also a path of system $(\tilde{A})$.

Let μ be so small that $(\tilde{A})$ is δ_1-close to (A). Inequality (14) is then satisfied, i.e., $\tilde{d}(n_1) < 0$.

Let us compute $\tilde{d}'(0)$. By definition $d'(0) = 0$, i.e.,

$$\int_0^\tau [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\, ds = 0. \tag{16}$$

Differentiating $\tilde{P}$ and $\tilde{Q}$ with respect to x and y, respectively, and using (15) and (16), we find

$$\int_0^\tau [\tilde{P}'_x(\varphi(s), \psi(s)) + \tilde{Q}'_y(\varphi(s), \psi(s))]\, ds =$$
$$= \mu \int_0^\tau [(F'_x(\varphi(s), \psi(s)))^2 + (F'_y(\varphi(s), \psi(s)))^2]\, ds = \mu J. \qquad (17)$$

The letter J identifies the integral in the right-hand side of the equality. From property (b) of function F it follows that $J > 0$. Finally, from (17) we see that

$$\tilde{d}'(0) = e^{\int_0^\tau (\tilde{P}'_x + \tilde{Q}'_y)\, ds} - 1 = e^{\mu J} - 1. \qquad (18)$$

Let $\mu > 0$. Then $\tilde{d}'(0) > 0$, i.e., the path L_0 is an unstable limit cycle of system $(\tilde{\mathrm{A}})$. Hence, for all sufficiently small $n > 0$, $\tilde{d}(n) > 0$.

In particular for some n_2, $0 < n_2 < n_1$,

$$\tilde{d}(n_2) > 0. \qquad (19)$$

The continuity of the function $\tilde{d}$ and inequalities (14) and (19) show that between n_1 and n_2 there is at least one value $n^* > 0$ such that $\tilde{d}(n^*) = 0$. This number corresponds to a closed path L^* of $(\tilde{\mathrm{A}})$ which does not coincide with L_0. By condition (c) imposed on δ_1, $L^* \subset U_\varepsilon(L_0)$. The second closed path of $(\tilde{\mathrm{A}})$ lying in $U_\varepsilon(L_0)$ is L_0 itself. The theorem is thus proved for the case when (A) is a system of class 1.

If (A) is a system of class $N > 1$, $r \leqslant N$, and the closeness is considered to rank r, the proof proceeds along the same lines, but $F(x, y)$ is a function of class $N+1$ (see Remark 1 to Lemma 1) and μ is sufficiently small for $(\tilde{\mathrm{A}})$ to be δ-close to (A) to rank r.

Let now (A) be an analytical system. In this case, Lemma 1 does not apply in its original form. Indeed, we can construct in the neighborhood of L_0 an analytical function $F(x, y)$ satisfying conditions (a) and (b); this may be accomplished as for systems of class N. This function, however, cannot be continued to the entire region $\bar{G}$ in general. We will therefore adopt a slightly different course.

To fix ideas, let us suppose, as before, that L_0 is a stable (multiple) limit cycle. We choose some n_1, $0 < n_1 < n^*$. Then $d(n_1) < 0$. Taking a fixed $\delta > 0$, we construct a function $F(x, y)$ of class $(r+1)$ which satisfies the conditions of Lemma 1 and consider the system $(\tilde{\mathrm{A}})$. μ is chosen as a positive number, sufficiently small for $(\tilde{\mathrm{A}})$ to be $\frac{\delta}{2}$-close to (A) to rank r and for the following inequality to be satisfied:

$$\tilde{d}(n_1) < 0.$$

Since for $\mu > 0$, $\tilde{d}'(0) > 0$ and also $\tilde{d}(0) = 0$, we have for sufficiently small $n_2 > 0$ and $n_3 < 0$

$$\tilde{d}(n_2) > 0 \qquad (19)$$

and

$$\tilde{d}(n_3) < 0 \tag{20}$$

(Figure 42a).

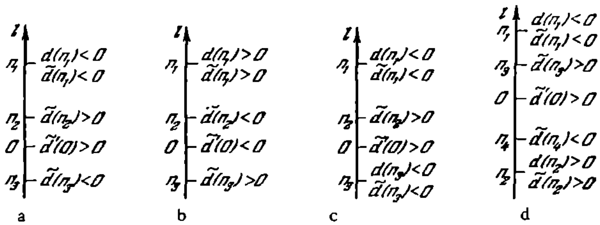

FIGURE 42

Let us now consider an analytical system (A^*) η-close to $(\tilde{A})$ to rank r, where $\eta < \frac{\delta}{2}$. η is taken to be sufficiently small so that a succession function $f^*(n)$ is defined on the arc l for system (A^*) and the following inequalities are satisfied:

$$d^*(n_1) < 0, \quad d^*(n_2) > 0, \quad d^*(n_3) < 0. \tag{21}$$

Clearly, (A^*) can be chosen as any system whose right-hand sides are polynomials providing a sufficiently close fit to the right-hand sides of $(\tilde{A})$. Since $\eta < \frac{\delta}{2}$, system $(\tilde{A})$ is δ-close to (A) to rank r. It follows from (21) that there exist at least two values of n (which lie between n_1 and n_2 and between n_2 and n_3, respectively) for which the function d^* vanishes. These values of the parameter correspond to two closed paths of system (A^*). For sufficiently small δ, these paths clearly lie in $U_\varepsilon(L_0)$).

The case of unstable or semistable L_0 is treated along the same lines. These instances are illustrated in Figure 42, b and c. The proof of the theorem is thus complete.

Remark 1. It is readily seen that if L_0 is a stable or an unstable multiple limit cycle, there exist systems $(\tilde{A})$ arbitrarily close to (A) which have at least three closed paths in any arbitrarily small neighborhood of L_0. The validity of this proposition can be established reasoning along the same lines as in the proof of the theorem (see Figure 42a illustrating the stable case). For a semistable cycle L_0, however, the last proposition (regarding the existence of at least three closed paths) is in general inapplicable.

Remark 2. It can be readily seen that if L_0 is a multiple limit cycle of system (A), there exists a system (A^*) arbitrarily close to (A) which has at least two simple, i.e., structurally stable, limit cycles in any arbitrarily small neighborhood of the cycle L_0. We will prove this

proposition for the case of analytical systems. The first step is to construct, as before, a system (A_1) of class 1 of the type $(\bar{A})$ for which L_0 is a simple limit cycle and which has another closed path L_1 close to L_0. If L_1 is a simple limit cycle, we take a polynomial (and therefore analytical) system (A^*) sufficiently close to (A_1). Because of the structural stability of the cycles L_0 and L_1 of system (A_1), system (A^*) will have structurally stable cycles L_0^* and L_1^* in a close neighborhood of L_0 and L_1. If L_1 is not a simple limit cycle of (A_1), we construct a system $(\tilde{A}_1)$ related to the cycle L_1 in the same manner as system $(\tilde{A})$ is related to cycle L_0 in the proof of Theorem 19. For an appropriate choice of the number μ, $(\tilde{A}_1)$ will be so close to (A_1) that in any arbitrarily small neighborhood of L_0 the system $(\tilde{A}_1)$ will have a simple limit cycle $\tilde{L}_0$. The path L_1 will be a simple limit cycle of $(\tilde{A}_1)$ (see proof of Theorem 19). It now remains to approximate to $(\tilde{A}_1)$ with a sufficiently close analytical system.

Let us prove another lemma which will be needed in the following. Its proof is based directly on Lemma 1.

Lemma 2. Let

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system of class $N \geq 1$ defined in region $\bar{G}$, and $x = \varphi(t)$, $y = \psi(t)$ a closed path L of the system ($L \subset G$) which is not a structurally stable (i.e., simple) limit cycle. For any $\delta > 0$, there exists an arbitrarily small neighborhood U of the path L and a dynamic system (B) of class N with the following properties:

(a) System (B) is δ-close to rank N to system (A) in $\bar{G}$.

(b) System (B) coincides with system (A) outside the neighborhood U.

(c) The path L of system (A) is a structurally stable limit cycle of system (B).

Proof. Since (A) is a system of class N, φ and ψ are functions of class $N+1$. In the neighborhood of the path L, we introduce curvilinear coordinates s and n defined by the relations

$$x = \varphi(s) + n\psi'(s), \qquad y = \psi(s) - n\varphi'(s). \tag{22}$$

In Lemma 1 we have seen that to every point (x, y) of a sufficiently small neighborhood of the path L corresponds precisely one value of the parameter n,

$$n = F(x, y),$$

and infinitely many values of s, such that the numbers s and n satisfy equations (22). By the theorem of implicit functions, $F(x, y)$ is a function of class N.

Let n_1 be a sufficiently small positive number, and n_2 and n_3 some positive numbers such that $n_3 < n_2 < n_1$. Let W, U, and V be the neighborhoods of L defined by the respective inequalities

$$|n| < n_3, \qquad |n| < n_2, \qquad |n| < n_1.$$

Clearly,

$$W \subset U \subset V$$

(see Figure 43a, where the neighborhood W is cross-hatched).

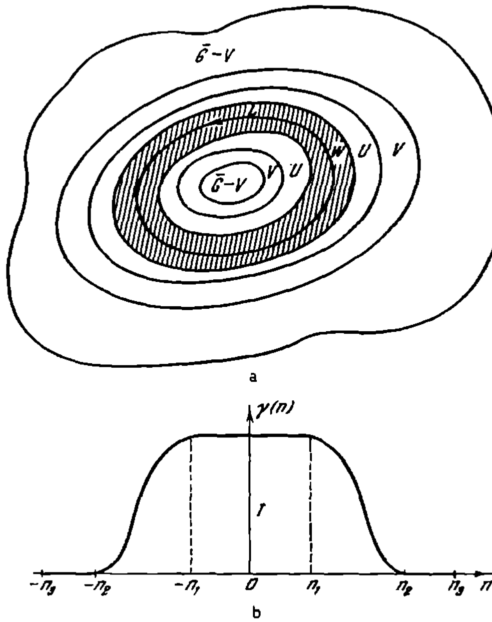

FIGURE 43

Let $\gamma(n)$ be a function satisfying the following conditions:

(a) $\gamma(n)$ is a function of class N defined for all n, $|n| < n_1$;

(b)
$$\gamma(n) = 1 \quad \text{for} \quad |n| \leqslant n_3,$$
$$\gamma(n) = 0 \quad \text{for} \quad n_2 \leqslant |n| \leqslant n_1,$$
$$0 \leqslant \gamma(n) \leqslant 1 \quad \text{for} \quad n_3 \leqslant |n| < n_2$$

(Figure 43b). There evidently exist functions $\gamma(n)$ of any class satisfying condition (b).

In our proof of Theorem 19 we established that if L is not a structurally stable limit cycle of (A), there exists a system

$$\frac{dx}{dt} = \tilde{P}(x, y), \quad \frac{dy}{dt} = \tilde{Q}(x, y) \qquad (\tilde{A})$$

of class N arbitrarily close to (A) to rank N for which L is a structurally stable cycle.

Consider a system (B) which coincides with system (A) in $\bar{G}-V$ and is expressed by the equations

$$\begin{aligned}\frac{dx}{dt} &= P(x, y) + [\tilde{P}(x, y) - P(x, y)]\,\gamma(F(x, y)),\\ \frac{dy}{dt} &= Q(x, y) + [\tilde{Q}(x, y) - Q(x, y)]\,\gamma(F(x, y))\end{aligned} \tag{B}$$

in the neighborhood V of the closed path. It is readily seen that in the neighborhood W, system (B) coincides with system ($\tilde{\mathrm{A}}$), i.e., L is a structurally stable limit cycle for (B). Moreover, in $V-U$ system (B) coincides with system (A). Hence, (B) coincides with (A) in $\bar{G}-U$ too. Finally, (B) is evidently a system of class N and is arbitrarily close to rank N to system (A), provided that ($\tilde{\mathrm{A}}$) is sufficiently close to (A). System (B) thus satisfies all the propositions of the lemma. This completes our proof.

3. Structural instability of a closed path with a zero characteristic index

Theorem 20. A closed path L_0 of system (A) *for which*

$$\int_0^{\tau} [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\, ds = 0$$

is structurally unstable (in relation to any of the spaces $R_N^{(r)}$, $r \leqslant N$, if (A) *is a system of class N, and $R_a^{(r)}$, if* (A) *is an analytical system).*

Proof. We shall first prove Theorem 20 assuming that (A) is a system of class N and structural instability is treated in relation to the space $R_N^{(r)}$, where $1 \leqslant r \leqslant N$.

Suppose, contrary to the proposition of the theorem, that the path L_0 is structurally stable in some neighborhood H of L_0. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if system (A^*) of class N is δ-close to rank r to system (A), we have

$$(H, A) \overset{\varepsilon}{\equiv} (H^*, A^*), \tag{23}$$

where H^* is some region. We will now show that L_0 in this case is necessarily a limit cycle of (A).

Let l be the arc without contact for the paths of system (A) considered above (which is a normal to L_0), Ω a neighborhood of the path L_0 with the property that each path of system (A) passing through a point of Ω meets the arc without contact l both for increasing and decreasing t. Any sufficiently small canonical neighborhood of the path L_0 may be taken as Ω (see QT, §24.3). Regarding the neighborhood H, we assume that it lies inside Ω at a positive distance from the boundaries of this region. Let $\varepsilon > 0$ be so small that a region H^* generated by an ε-translation of H is contained in Ω, i.e., $H^* \subset \Omega$. Let $\delta > 0$ be a number corresponding to this ε in a sense that if (A) and (A^*) are δ-close, relation (23) is satisfied. Let δ_1, $0 < \delta_1 < \delta$, be so small that if system (A^*) is δ_1-close to (A), every path of (A^*)

passing through the points of Ω crosses the arc without contact l both for increasing and decreasing t, and a succession function f^*, and hence also d^*, can be defined on this arc for system (A^*).

Let us first consider system $(\tilde{A})$ of class N of the form

$$\frac{dx}{dt} = P(x, y) + \mu F(x, y) F'_x(x, y), \quad \frac{dy}{dt} = Q(x, y) + \mu F(x, y) F'_y(x, y) \qquad (\tilde{A})$$

(see §15.2), choosing $\mu > 0$ so that $(\tilde{A})$ is $\frac{\delta_1}{2}$-close to rank r to (A). By (18), $\tilde{d}'(0) > 0$, i.e., $\tilde{d}(n) \not\equiv 0$. Now let (A^*) be an analytical system δ_2-close to rank r to $(\tilde{A})$. Also $0 < \delta_2 < \frac{\delta_1}{2}$. If δ_2 is sufficiently small, $d^*(n)$ is not greatly different from $\tilde{d}(n)$ and therefore $d^*(n) \not\equiv 0$. We will assume that this condition is satisfied. System (A^*) is δ_1-close to (A) to rank r and is analytical; the function $d^*(n)$ is therefore also analytical. Since $d^*(n) \not\equiv 0$, this function may have only a finite number of roots in a finite interval of n values. Hence it evidently follows that system (A^*) may only have a finite number of closed paths in Ω. Then each of these paths is isolated, and is therefore a limit cycle.

In virtue of the assumption of structural stability of system (A) in H and in virtue of the conditions imposed on δ_1, δ, and ε, relation (23) is satisfied and $H^* \subset \Omega$. The mapping of H onto H^* ensuring the ε-identity (23) moves the closed path L_0 of (A) into some closed path L_0^* of (A^*) which lies in H^* and is therefore isolated. Then L_0 is evidently also isolated, i.e., it is a limit cycle of (A).

We have thus shown so far that if L_0 is a structurally stable path, then L_0 is a limit cycle. Let L_0 be structurally stable in neighborhood H. We may assume that H contains a single closed path L_0. Let U be a neighborhood of path L_0 such that $U \subset H$ and U is at a positive distance from the boundary of H.

We take $\varepsilon > 0$ to be so small that an ε-translation of H leaves U inside the "translated" neighborhood.* Let δ be a number corresponding to ε in the sense of the definition of the structural stability of system (A) in H. By Theorem 19 (the theorem of the creation of a closed path from a multiple limit cycle), there exists a system $(\tilde{A})$ of class N, δ-close to rank r to system (A) which has at least two closed paths in U.

In virtue of the particular choice of the numbers δ and ε and the neighborhood U,

$$(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A}),$$

where $\tilde{H}$ is some region and $U \subset H$. It follows from these relations that $\tilde{H}$ contains at least two closed paths of $(\tilde{A})$, so that H contains at least two closed paths of (A). This contradicts the original assumption that $\tilde{H}$ contains a single closed path of (A), namely L_0. This contradiction establishes the structural instability of the path L_0 in relation to the space $R_N^{(r)}$.

If (A) is an analytical system, the structural instability of L_0 in relation to $R_a^{(r)}$ is proved along the same lines. The proof of the theorem is complete.

* See footnote on p. 67.

Corollary. A closed path L_0 is structurally stable if and only if it is a simple limit cycle, i.e., the characteristic index

$$\chi = \frac{1}{\tau} \int_0^\tau [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\, ds$$

does not vanish.

The validity of this proposition follows directly from Definition 18 and Theorems 18 and 20.

Chapter VI

NECESSARY AND SUFFICIENT CONDITIONS OF STRUCTURAL STABILITY OF SYSTEMS

INTRODUCTION

In the previous chapters we derived a number of necessary conditions of structural stability of a dynamic system. We established, in particular, that if a dynamic system (A) is structurally stable in a closed bounded region $\bar{G}^*$, then:

I. It may have only a finite number of equilibrium states in $\bar{G}^*$, which are of necessity simple nodes, saddle points, or foci ($\Delta < 0$ or $\Delta > 0$ and $\sigma \neq 0$; see §7, Theorems 10 and 11, and §10, Theorem 15).

II. All the closed paths of system (A) are simple limit cycles (i.e., paths $x = \varphi(s)$, $y = \psi(s)$ for which $J = \int\limits_0^\tau (P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s)))\, ds \neq 0$, where τ is the period of the functions φ and ψ; see §15, Theorem 20).

III. System (A) does not have saddle-to-saddle separatrices in $\bar{G}^*$ (§11, Theorem 16).

In the present chapter, we will prove that conditions I through III are both necessary and sufficient for structural stability of a system in $\bar{G}^*$. A rigorous proof of this proposition, although essentially simple, requires a fairly lengthy and tedious analysis. It is analogous to the proof of Theorem 76 in QT, §29.4.

Chapter VI consists of three sections (§16, §17, §18).

In §16 we prove that if system (A) is structurally stable in $\bar{G}^*$, it may have only a finite number of closed paths in $\bar{G}^*$ (Theorem 21), and hence only a finite number of orbitally unstable paths and semipaths (Theorem 22). The concept of a region with a normal boundary is also introduced in §16. A normal boundary is made up of a finite number of simple closed curves, each of which is either a cycle without contact or consists of an even number of alternating arcs without contact and arcs of paths. The fundamental theorem of structural stability is proved for regions with a normal boundary, since this assumption greatly simplifies the proof without imposing a significant restriction.

§17 is completely devoted to supplementary material. It is proved in this section that a region $\bar{G}^*$ with a normal boundary can be partitioned into canonical neighborhoods of equilibrium states and limit cycles and into elementary quadrangles. This partition is actively used in the proof of the fundamental theorem.

In §18, we give a complete proof of the fact that conditions I through III are the necessary and sufficient conditions of structural stability for a

system (A) in $\bar{G}^*$ (Theorem 23). In §18.3 we prove that these are also necessary and sufficient conditions for structural stability on a sphere (Theorem 24). At the end of the section, §18.4 offers a number of significant remarks and supplements. These include a theorem which states that the structurally stable systems form an open set in the space of all dynamic systems in a plane region (Theorem 25) and on a sphere (Theorem 25') and a theorem according to which the structurally stable systems are everywhere dense in the space of the dynamic systems (Theorem 26). These theorems indicate that "almost all" the dynamic systems are structurally stable, and structurally unstable systems are an exception.

A reader wishing to get his teeth into the meat of the theory without further delay may skip the proof of the fundamental theorems 23 (§18.2) and 24 (§18.3) and only peruse §16.1, the statement of Theorem 23 in §18.2, and also §18.3 and §18.4.

§16. SINGULAR PATHS AND SEMIPATHS OF DYNAMIC SYSTEMS

1. Finite number of closed paths for structurally stable systems

Before proceeding with the proof of the sufficiency of conditions I through III (see Introduction to this chapter) for structural stability of a system, we will show that these conditions allow only a finite number of closed paths in a structurally stable system. Note that in virtue of condition II, each closed path in a structurally stable system is isolated. This conclusion in itself, however, does not establish that the number of closed paths is finite, since a condensation point for the points of closed paths evidently need not belong to a closed path.

Theorem 21. If system (A) is structurally stable in $\bar{G}^*$, it may have only a finite number of closed paths in $\bar{G}^*$.

Proof. Suppose that this proposition is not true, i.e., a system (A) has an infinite number of closed paths in $\bar{G}^*$. Take a sequence $L_1, L_2, L_3, \ldots$ of these paths and choose an arbitrary point on each path. Let $M_1, M_2, M_3, \ldots$ be the sequence of these points, $M_i \in L_i$. Since $\bar{G}^*$ is compact, the sequence $\{M_i\}$ has at least one condensation point, and without loss of generality we may assume that $\{M_i\}$ is a convergent sequence (if this is not so, we can always choose a convergent subsequence). Let the sequence converge to a point M^*. Thus closed paths completely contained in $\bar{G}^*$ pass through any neighborhood of M^*. We will now show that if (A) is a structurally stable system, no such point M^* may exist in $\bar{G}^*$. If M^* is not a state of equilibrium, we will designate by L^* the path through M^*. If the α- and ω-limit sets of the path L^* lie in $\bar{G}^*$, we will denote them by K_α and K_ω, respectively.

We have to consider the following alternatives:

1) M^* is a simple focus or a simple node.
2) M^* is a simple saddle point.
3) The path L^* leaves $\bar{G}^*$ as t increases or decreases.
4) K_α or K_ω is a node or a focus.
5) K_α or K_ω is a closed path.

6) K_α or K_ω is a saddle point.

7) K_α or K_ω is a limit continuum, comprising a saddle point and saddle-to-saddle separatrices which are continuations of one another.

Let us consider each of the seven cases separately.

Case 1 is inapplicable, since all the paths passing sufficiently close to a node or a focus go to that singular point, and are therefore not closed.

Case 3 is inapplicable, since if L^* leaves $\bar{G}^*$, any closed path L_k passing sufficiently close to M^* also leaves $\bar{G}^*$. This contradicts our assumption that all the closed paths are completely contained in $\bar{G}^*$.

Case 7 is inapplicable, since a structurally stable system has no saddle-to-saddle separatrices.

We can now concentrate on the remaining cases.

Case 4. Let K_ω be a node or a focus O. Let $U_\varepsilon(O)$ be a sufficiently small neighborhood of the point O, so that all the paths through this neighborhood go to the point O for $t \to +\infty$ and are thus not closed. Since the path L^* has points in $U_\varepsilon(O)$, the theorem of the continuous dependence on the initial values indicates that any path L_k passing sufficiently close to M^* also passes inside $U_\varepsilon(O)$, and is thus not closed, contrary to the definition of L_k.

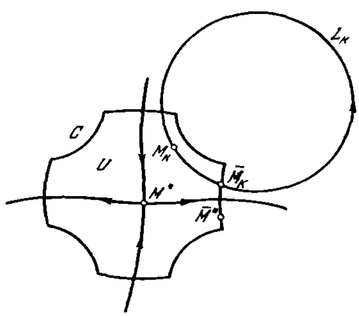

FIGURE 44

Case 5. Let K_ω be a closed path. We choose an arbitrary point S on this path. Any neighborhood $U_\varepsilon(S)$ contains points o the path L^*, and by the theorem of continuous dependence on the initial values, it also contains points of any closed path L_k passing sufficiently close to M^*. But then K_ω is not an isolated closed path, which clashes with the structural stability of (A).

Case 6. Let K_α be a saddle point, so that L^* is a separatrix of one of the saddle points of the system. Since a structurally stable system has no saddle-to-saddle separatrices, there are two alternatives: L^* leaves $\bar{G}^*$ as t increases (case 3), or K_ω is a node, a focus, or a closed path in $\bar{G}^*$ (cases 4 and 5). None of these cases is applicable, as we have shown above.

Case 2. Let M^* be a saddle point. Consider a sufficiently small canonical neighborhood U of this point, limited by a simple closed curve C, which is made up of four arcs without contact and four arcs of paths (Figure 44). We take U to be so small that the saddle point M^* is the only equilibrium state of (A) contained in U. By assumption, U contains an infinite number of points M_k which belong to the paths L_k. However, none of the closed paths of system (A) may be completely contained in U, since such a closed path, if it existed, would not enclose any equilibrium states or would enclose at most one equilibrium state, which is a saddle point. This is forbidden by QT, §11.2, Theorem 30 and Corollary 1 from Theorem 29. Therefore each closed path L_k has at least one point $\bar{M}_k$ lying on the curve C. Since this curve is compact, we can select a sequence of points $\{\bar{M}_k\}$ converging to some point $\bar{M}^* \in C$. $\bar{M}^*$ is not a state of equilibrium, and it can be taken as the initial point M^*. We have thus reduced our problem to one of the cases 3—7, which are inapplicable.

None of the cases 1 through 7 is thus applicable. This establishes the validity of the theorem.

In QT, §15, we introduced the concept of orbital stability and distinguished between orbitally stable and orbitally unstable paths. We will not repeat the previous results here. It suffices to say that if system (A) is structurally stable in region G^*, its orbitally unstable paths in this region are all the equilibrium states, limit cycles, and saddle-point separatrices.*

Theorem 22. If system (A) *is structurally stable in* $\bar{G}^*$, *it may have only a finite number of orbitally unstable paths and semipaths in* $\bar{G}^*$.

The validity of this theorem follows directly from Theorem 10, Theorem 11 (§7), and Theorem 21.

Remark. Theorems 21 and 22 clearly remain valid when the requirement of structural stability of system (A) is replaced by a requirement that system (A) satisfies conditions I through III (these requirements are actually equivalent, but this still remains to be proved).

2. Regions with normal boundary

The proof of sufficiency of conditions I–III (see Introduction to this chapter) for structural stability of a system in $\bar{G}^*$ will be carried out for a particular case, assuming that $\bar{G}^*$ has a so-called normal boundary. Regions with a normal boundary were defined in QT, §16.2, and we will repeat here the corresponding definition in full. Note that the requirement of a normal boundary does not impose a significant restriction, but it helps us to avoid various complications in the proof of sufficiency.**

Definition 19. The boundary of a compact connected region is called normal for a given dynamic system (A) *if the following conditions are satisfied:*

1) The boundary is made up of a finite number of simple closed curves.

2) Each of these closed curves is either a cycle without contact or consists of a finite number of alternating arcs without contact and arcs of paths.† *The common point of an arc of a path and an arc without contact making the boundary will be called a corner point; a semipath lying in* G^* *and terminating at a corner point will be called a corner semipath; an arc of a path which is completely contained in* $\bar{G}^*$, *except for its end points which lie on boundary arcs without contact so that at least one of them is a corner point, will be called a corner arc; the corresponding arc of a path for which neither end point is a corner point will be called a whole non-singular arc.*

3) For any corner arc, only one end point is a corner point.

4) None of the corner semipaths is a saddle-point separatrix.

5) None of the boundary arcs of paths belong to a closed path completely contained in $\bar{G}^*$.

Conditions 4 and 5 clearly indicate that the boundary arcs of paths do not belong to orbitally unstable paths or semipaths lying completely in $\bar{G}^*$.

* Saddle-point separatrices may be whole paths or semipaths, depending on whether they are completely contained in $\bar{G}^*$ or not.

** If the boundary of a region $\bar{G}^*$ consists of a finite number of simple closed non-intersecting piecewise-smooth curves, it can be fitted by an arbitrarily close normal boundary.

† Boundary arcs of paths will sometimes be called for brevity boundary arcs.

A boundary arc and a corner arc of a path with a common end point, which thus form a single arc of a path of system (A), are said to be a c o n t i n u a t i o n o f e a c h o t h e r (in the direction of increasing or decreasing t). Similarly, a boundary arc and a corner path with a common end point are regarded as a c o n t i n u a t i o n o f e a c h o t h e r.

Figure 45 shows a triply connected region $\bar{G}^*$ with a normal boundary. The boundary consists of three simple closed curves, one of which is a cycle without contact Λ, and each of the other two is made up of an even number of arcs of paths b_i and arcs without contact λ_i ($1 \leqslant i \leqslant 5$ for the exterior boundary curve and $6 \leqslant i \leqslant 7$ for the interior boundary curve). The arcs without contact λ_i are marked in the figure by straight segments.

The corner points are A_i ($1 \leqslant i \leqslant 10$) and B_j ($1 \leqslant j \leqslant 4$). The figure also shows the corner arcs C_3A_4, C_5A_6, A_7C_7, B_1C_8; a corner semipath L_1^+ with an end point A_1 going to a stable focus O_1; a corner semipath L_2^+ going to a stable limit cycle Z, which encloses an unstable node or focus O_2; a saddle point O_3 with four separatrices.

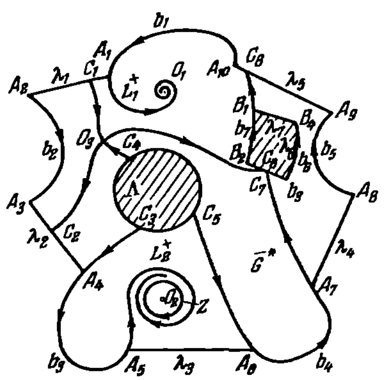

FIGURE 45

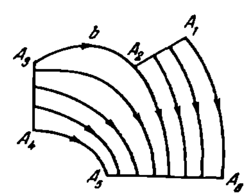

FIGURE 46

The boundary arcs b_2 and b_5 have no continuation in $\bar{G}^*$. Each of the other boundary arcs has continuations in two directions. The continuation of the boundary arc b_3 in the direction of decreasing t is the corner arc A_4C_3, and the continuation in the direction of increasing t is the corner semipath L_2^+. Figure 46 shows a boundary arc b with a continuation in one direction only (in the direction of increasing t).

D e f i n i t i o n 20. Orbitally unstable paths, saddle-point separatrices and corner semipaths, boundary and corner arcs of paths, boundary arcs without contact and boundary cycles without contact in $\bar{G}^*$ will be respectively called singular paths, semipaths, arcs (of paths), arcs without contact, and cycles without contact. All the other paths, semipaths, and arcs of paths will be called non-singular. Singular paths, semipaths, etc., will also be referred to as singular elements.

We will now give a complete list of the various paths, semipaths, and arcs of a structurally stable system (A) in a region $\bar{G}^*$ with a normal boundary.

A) Singular orbitally unstable paths and semipaths:

1) An equilibrium state (a stable node or focus, an unstable node or focus, a saddle point).

2) A limit cycle (stable or unstable).

3) A separatrix which goes to a saddle point for $t \to +\infty$ $(-\infty)$ and to an unstable (stable) focus, node, or limit cycle for $t \to -\infty$ $(+\infty)$, or which leaves $\bar{G}^*$ through a boundary arc or through a cycle without contact as t decreases (increases).

B) Singular orbitally stable semipaths.

4) A corner semipath going for $t \to -\infty$ $(+\infty)$ to an unstable (stable) node, focus, or limit cycle.

C) Singular arcs and cycles without contact:

5) A corner arc.

6) A boundary arc of a path.

7) A boundary arc without contact.

8) A boundary cycle without contact.

D) Non-singular whole paths and semipaths:

9) Paths going for $t \to -\infty$ to an unstable and for $t \to +\infty$ to a stable node, focus, or limit cycle (9 different possibilities).

10) Semipaths going for $t \to -\infty$ $(+\infty)$ to an unstable (stable) node, focus, or limit cycle and emerging from $\bar{G}^*$ through a boundary arc or cycle without contact as t increases (decreases).

E) Non-singular whole arcs:

11) An arc of a path which is neither a corner arc nor a boundary arc, whose end points lie on boundary arcs or cycles without contact, and all the other points are in G^*.

In what follows, stable nodes, foci, and limit cycles of system (A) lying in $\bar{G}^*$ (or in fact only in G^*) will be called *attraction elements* or *sinks*, and unstable nodes, foci, and limit cycles will be called *repulsion elements* or *sources*. Boundary arcs or cycles without contact through which paths emerge from $\bar{G}^*$ as t increases can also be interpreted in a sense as attraction elements (sinks), whereas boundary arcs and cycles without contact through which paths enter $\bar{G}^*$ may be regarded as repulsion elements (sources).

Since a structurally stable system in $\bar{G}^*$ has only a finite number of singular elements, all the propositions regarding the partition of $\bar{G}^*$ into cells, formulated in QT, §16, remain valid. In particular, the set E of all the points of $\bar{G}^*$ which belong to singular elements is a closed set. Its complement, i.e., the open set $\bar{G}^*-E$, consists of a finite number of components, called *cells*. Each cell is filled with non-singular paths, semipaths, or arcs of paths which show "identical" behavior in a certain sense (see QT, §16, Theorems 46–48, 53, 57). The cells are either singly connected or doubly connected. In the next chapter we will consider the various types of cells of structurally stable systems.

§17. A REGULAR SYSTEM OF NEIGHBORHOODS AND THE PARTITION OF $\bar{G}^*$ INTO CANONICAL NEIGHBORHOODS AND ELEMENTARY QUADRANGLES

1. A regular system of canonical neighborhoods for structurally stable systems

As we know, a cycle without contact can be drawn around any node or focus, so that it encloses no equilibrium states other than that node or focus and no closed paths. The cycle can be drawn in any arbitrarily small neighborhood of the node or the focus.* We will say that this cycle without contact belongs to the given node or focus, and conversely, that the node or the focus belongs to the particular cycle without contact.

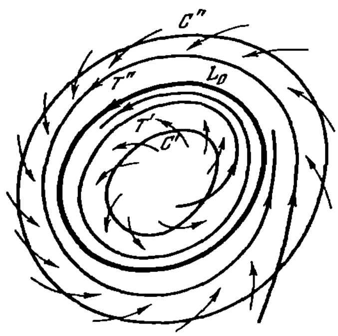

FIGURE 47

The closed region comprising the points inside this cycle without contact and the points of the cycle itself will be called a closed canonical neighborhood of the node or the focus. All the paths passing through the points of a canonical neighborhood of a node or a focus O, which do not coincide with the point O, evidently do not leave that neighborhood, going to O for $t \to +\infty$ (if O is a stable node or focus) or for $t \to -\infty$ (in the unstable case).

Let us consider a limit cycle L_0 of system (A). In QT, §24.3, it is shown that in any neighborhood of the cycle L_0 we can pass two cycles without contact C' and C'', one lying inside L_0, and the other enclosing L_0, so that the annular region T between the cycles C' and C'' contains no equilibrium states and no closed paths other than L_0 (Figure 47). The closed annular region T is a union of two closed unilateral canonical neighborhoods T' and T'', limited by the closed curves L_0 and C' and L_0 and C'', respectively. We will say that the cycles without contact C' and C'' belong to the limit cycle L_0. T will be called a bilateral closed canonical neighborhood, or simply a canonical neighborhood of the limit cycle L_0.

Any path, other than L_0, passing through a point of the canonical neighborhood T goes, without leaving T, to the limit cycle L_0 for $t \to +\infty$ if L_0 is stable and for $t \to -\infty$ if it is unstable. As t decreases or increases, respectively, this path leaves T through one of the cycles without contact C' or C''. Clearly, for any $\varepsilon > 0$, the cycles C' and C'' can be always drawn so that T is entirely contained in $U_\varepsilon(L_0)$.

In what follows, we will not deal separately with the canonical neighborhoods of nodes, foci, or limit cycles; we will speak invariably of the canonical neighborhoods of sinks or sources (attraction or repulsion elements). We will also say that a cycle without contact belongs to a given sink or source, and conversely, that a sink or a source belongs to a given cycle without contact.

* See QT, §18.1, Lemma 3 and the remark to this lemma.

Apart from canonical neighborhoods of nodes, foci, and limit cycles, we will also consider canonical neighborhoods of a saddle point. This neighborhood is delimited by four arcs without contact, each crossing one of the saddle-point separatrices, and by four arcs of paths (Figure 44; the canonical neighborhoods of a saddle point are defined in QT, §19.2). For any $\varepsilon > 0$, we can choose a canonical neighborhood of the saddle point O which is entirely contained in $U_\varepsilon(O)$.

In what follows, we invariably assume that all the dynamic systems satisfy conditions I through III in the Introduction to Chapter VI and consider regions with a normal boundary.

Lemma 1. If the arcs without contact forming the boundary of a canonical neighborhood of a saddle point O are sufficiently small, each of these arcs has only one common point with the corresponding separatrix of the saddle point O, and has no common points with any of the other singular paths or semipaths (i.e., limit cycles, separatrices, and corner semipaths) or with any of the corner arcs.

Proof. The validity of the lemma follows directly from the finite number of singular paths and semipaths in the relevant dynamic system (§16, Theorems 21 and 22; remark to Theorem 22 and Definitions 19 and 20) and from the fact that, in virtue of condition III, the saddle-point separatrices of these systems cannot be limit paths (i.e., they do not enter the limit α- or ω-continuum of any path).

Lemma 2. There exists $\varepsilon_0 > 0$ such that if the canonical neighborhood of each saddle point O_i lies inside $U_{\varepsilon_0}(O_i)$, none of the paths through the points of the canonical neighborhood of one saddle point has points in the canonical neighborhood of any other saddle point.

Proof. We first choose the canonical neighborhoods of sinks and sources (nodes, foci, and limit cycles) in such a way that they have no common points. All the saddle points of system (A) evidently lie outside these neighborhoods. Let us consider the set Ω comprising all the cycles without contact which enter the boundaries of the canonical neighborhoods of the sinks and sources, all the boundary cycles without contact, and all the open boundary arcs without contact (i.e., the boundary arcs without their end points). By condition III, every α-separatrix (ω-separatrix) of each saddle point intersects with increasing (decreasing) t one and only one element of the set Ω at a single point.

Let O_i ($i = 1, 2, \ldots, m$) be the saddle points of system (A) lying in $\bar{G}^*$, and $L_i^{(k)}$ ($k = 1, 2, 3, 4$) the separatrices of the saddle point O_i. Consider a semipath which is part of the separatrix $L_i^{(k)}$, ending at the point of its intersection with the corresponding cycle or arc without contact of the set Ω. To avoid introducing new symbols, $L_i^{(k)}$ will denote this semipath. The set comprising the saddle point O_i and all the points of the semipaths $L_i^{(k)}$ ($k = 1, 2, 3, 4$) will be denoted F_i ($i = 1, 2, \ldots, m$). Each of these F_i is evidently a closed set, and they have no common points as there are no saddle-to-saddle separatrices. The distances between any two sets F_i are therefore positive, and there exist nonintersecting closed neighborhoods $\bar{W}_i$ of these sets. By $\bar{W}_i$ ($i = 1, 2, \ldots, k$) we mean a closed neighborhood of the set F_i in the set comprising the points of $\bar{G}^*$ which lie outside or on the boundary of the canonical neighborhoods of the sinks or sources. The neighborhoods $\bar{W}_i$ are typically "cross-shaped" (Figure 48).

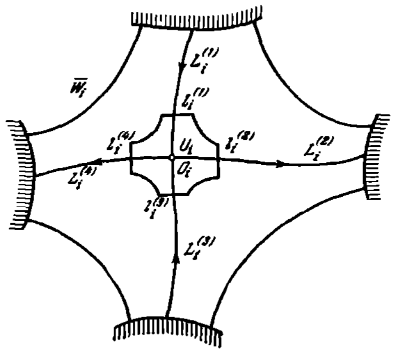

FIGURE 48

Let us consider the canonical neighborhoods $U_i(O_i)$ completely contained in W_i. Let the boundary of U_i comprise the arcs without contact $l_i^{(k)}$ ($k = 1, 2, 3, 4$). If these arcs are sufficiently small, then for both decreasing and increasing t, every path passing through the points of these arcs will either leave the region $\overline{G}^*$ or enter one of the canonical neighborhoods of the sources or sinks, remaining until that time in the set $\overline{W}_i$. But then the same property is characteristic of every path through any point of the neighborhood U_i.

Let ε_0 be a sufficiently small positive number, so that for any $i = 1, 2, \ldots, m$, $U_{\varepsilon_0}(O_i) \subset U_i$. This number clearly satisfies the proposition of the lemma, which completes our proof.

Definition 21. Let (A) *be a dynamic system satisfying conditions I through III in the Introduction of Chapter VI. Consider a region with a normal boundary. A system of canonical neighborhoods of the system* (A) *is said to be regular* if the following conditions are satisfied:*

1) The canonical neighborhoods of various states of equilibrium are all disjoint and do not intersect with any of the canonical neighborhoods of the limit cycles; the canonical neighborhoods of different limit cycles are also nonintersecting.

2) None of the paths of system (A) *passes through the canonical neighborhoods of two different saddle points.*

3) Every arc without contact contained in the boundary of the canonical neighborhood of a saddle point satisfies the conditions of Lemma 1, i.e., it has precisely one common point with the corresponding separatrix and has no common points with any of the other singular paths and semipaths or corner arcs.

Lemmas 1 and 2 establish the existence of regular systems of canonical neighborhoods. In what follows, we will only deal with regular systems of canonical neighborhoods.

* In QT, §27, the concept of a regular system of canonical neighborhoods was defined for dynamic systems of a more general type. The dynamic systems considered in this chapter satisfy conditions I–III, and therefore their regular system of canonical neighborhoods meets the condition formulated below (these conditions are stronger than in the general case).

The following obvious propositions derive from the above list of paths, semipaths, arcs of paths, etc., and from the properties of canonical neighborhoods:

A) Any non-singular path completely contained in $\bar{G}^*$ crosses one cycle without contact which belongs to a source and one cycle without contact which belongs to a sink.

B) Any non-singular semipath crosses one cycle without contact which belongs to a source or a sink, and one boundary arc or boundary cycle without contact.

C) Any α-separatrix (ω-separatrix) either crosses one cycle without contact which belongs to a sink (source) or leaves $\bar{G}^*$ with increasing (decreasing) t through a boundary cycle or a boundary arc without contact.

D) Any positive (negative) corner semipath crosses one cycle without contact which belongs to a sink (source).

We will now present the terminology and some facts pertaining to cycles and arcs without contact comprising the set Ω and the division of these arcs and cycles into parts by singular paths and semipaths (the set Ω is defined in the proof of Lemma 2).

Let C be a cycle without contact contained in the boundary of some canonical neighborhood or in the boundary of the region $\bar{G}^*$. If this cycle has no common points with singular paths, semipaths, and corner arcs of the system, it is said to be *free*. We say that C is a free ω-cycle (α-cycle), if C belongs to a sink (source) or if C is a boundary cycle through which the paths of the system leave $\bar{G}^*$ (enter $\bar{G}^*$).

A cycle without contact C is said to be *non-free* if it has at least one common point with singular paths, semipaths, or corner arcs (a corner arc may have a common point with C only if C is a boundary cycle without contact). If a non-free cycle without contact C has more than one point in common with singular semipaths, paths, or corner arcs, it is divided by these points into a finite number of simple arcs, which have no common points with the singular elements, except the end points. These arcs are called *simple elementary arcs*. If a non-free cycle C has only one point M in common with singular paths, semipaths, and corner arcs, C is called a *cyclic elementary arc*, and M is the *end point of a cyclic arc*. An elementary arc — whether simple or cyclic — is called an *elementary ω-arc* (*α-arc*) or simply a *ω-arc* (*α-arc*) if the cycle without contact C containing this arc belongs to a sink (a source) or if C is a limit cycle without contact through which the paths of the system leave $\bar{G}^*$ (enter $\bar{G}^*$).

Boundary arcs without contact are also divided by points which belong to singular elements — specifically to separatrices or corner arcs — into *simple elementary arcs*, which have no common points with singular elements, except their end points. In particular, a simple elementary arc may coincide with a boundary arc without contact. These elementary arcs are also ω- or α-arcs according as the paths of the system leave or enter the region $\bar{G}^*$ through these arcs.

We will now formulate without proof some auxiliary proposition which will be helpful in the following. The proof can be found in QT, §27.4 for a more general class of dynamic systems (namely, systems which do not necessarily satisfy conditions I — III). From the list of paths, semipaths, etc., in §16.2 and from the definition of free cycles without contact and

elementary arcs, it follows directly that every non-singular path of a dynamic system (A) lying in $\bar{G}^*$, whether a whole path, a semipath, or a whole arc, crosses one α-arc or free α-cycle and one ω-arc or free ω-cycle.

Lemma 3. All the paths passing through the points of one free α-cycle (ω-cycle) cross the same free ω-cycle (α-cycle), one of the two cycles lying inside the other.

Remark. The paths of this lemma are non-singular (from the definition of a free cycle). They are either all whole paths, or all semipaths, or all whole arcs. Free α- and ω-cycles through which pass the same paths are called conjugate.

Lemma 4. All the paths passing through the (inner) points of one α-arc (ω-arc) cross the same ω-arc (α-arc).

Remark. Here again we are dealing with non-singular paths. Two elementary arcs without contact through which pass the same paths are called conjugate elementary arcs. Two conjugate elementary arcs of a structurally stable system are both simple elementary arcs; alternatively one of them is simple and the other is cyclic.

If (A) is a structurally unstable system, it may have two conjugate cyclic arcs. However, structurally unstable systems have no conjugate cyclic arcs. This will be proved in Chapter VII, §19.3, Lemma 3.

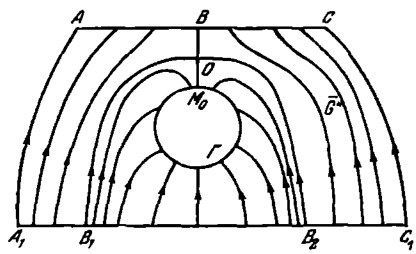

FIGURE 49

In Figure 49, the simple α-arcs A_1B_1 and B_2C_1 are the conjugates of the simple ω-arcs AB and BC, and the simple α-arc B_1B_2 is the conjugate of the cyclic ω-arc Γ with M_0 as its end point. All these arcs are contained in the boundary of the doubly connected region $\bar{G}^*$.

2. The partition of the region $\bar{G}^*$ into canonical neighborhoods and elementary quadrangles

We again regard (A) as a dynamic system which satisfies conditions I–III of the Introduction in region $\bar{G}^*$ with a normal boundary. Let some regular system of canonical neighborhoods be defined in $\bar{G}^*$. Let K be the set of all the interior points of all the canonical neighborhoods. The points of $\bar{G}^*$ which do not belong to K constitute the closed set $\bar{G}^* - K$.

L e m m a 5. *The closed set $\bar{G}^* - K$ can be partitioned into a finite number of closed elementary quadrangles, so that every arc without contact contained in the boundary of any of these quadrangles is either a part of a boundary cycle without contact, or a part of a cycle without contact which belongs to some sink or source, or a part of a boundary arc without contact, or an arc without contact contained in the boundary of the canonical neighborhood of a saddle point.*

Moreover,

a) any two quadrangles of the partition either have no common points or their common points form an arc of a path which is part of the boundary of each quadrangle;

b) an arc without contact forming part of the boundary of a quadrangle of the partition either has one point, other than the end point, in common with the saddle-point separatrix and no other common points with singular elements; or one of its end points belongs to a corner semipath or a corner arc or is a corner point of the boundary, and all the other points belong to non-singular paths; or it entirely consists of points which belong to non-singular paths.

R e m a r k. A partition of $\bar{G}^*$ into canonical neighborhoods forming a regular system and into elementary quadrangles satisfying the conditions of Lemma 5 will be called a r e g u l a r p a r t i t i o n o f $\bar{G}^*$.

P r o o f. First let us consider the canonical neighborhoods of the saddle points of system (A). Let O be some saddle point, H its canonical neighborhood, γ an arc without contact entering the boundary of H, A and B the end points of this arc, L a separatrix of the saddle point O crossing the arc γ, D the intersection point of L and γ (Figure 50). To fix our ideas, let L be an α-separatrix. Then, for increasing t, L leaves the neighborhood H through the point D and crosses either a cycle without contact C (this is a boundary cycle or a cycle belonging to a sink) or a boundary arc without contact. Let the separatrix L cross the cycle without contact C at point D' (if L crosses a boundary arc without contact, the argument remains the same). From the definition of a regular system of canonical neighborhoods we see that all the paths through points of the arc γ will cross with increasing t an arc without contact γ' which is part of the cycle C. The end points of γ' (A' and B') lie respectively on the paths L_A and L_B through the end points A and B of the arc γ, and γ' has no points which belong to singular paths, except the point D'.

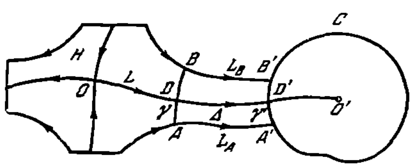

FIGURE 50

The arcs γ and γ' will be called c o n j u g a t e a r c s. The quadrangle Δ delimited by γ and γ' will be called an elementary quadrangle.

Thus, to each arc without contact γ entering the boundary of a canonical neighborhood of a saddle point corresponds an elementary quadrangle. Let Δ_i ($i = 1, 2, \ldots, N$) be all such quadrangles, γ_i and γ_i' the corresponding arcs without contact. The quadrangles Δ_i are evidently all disjoint (see Definition 21, 2).

Let us now consider a corner path $\hat{L}$ which ends at a corner point R of the boundary. Let $\hat{L}$ be a negative semipath. Then $\hat{L}$ has a common point $\hat{R}$

with a cycle without contact $\hat{C}$ which belongs to a source (Figure 51). On the cycle $\hat{C}$ we choose two points $\hat{A}$ and $\hat{B}$ on the two sides of $\hat{R}$ and sufficiently close to it so that the arc $\hat{A}\hat{B}$ of the cycle $\hat{C}$ contains no points of the previously defined arcs γ_i' and no other points of any corner paths, except the point $\hat{R}$. Let us consider the arcs $\hat{R}\hat{A}$ and $\hat{R}\hat{B}$ separately, denoting them $\hat{\chi}_1$ and $\hat{\chi}_2$.

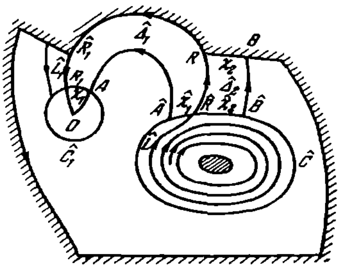

FIGURE 51

As t increases, the paths through the points of one of these arcs (in Figure 51, through the points of $\hat{\chi}_2$) will cross the arc χ_2 which is part of a boundary arc without contact and has R as one of its end points (all the other points of χ_2 belong to non-singular paths). The paths through the points of the second arc, $\hat{\chi}_1$, will cross with increasing t the arc χ_1, which is also part of a boundary arc or cycle without contact, or (as in Figure 51) is part of a cycle without contact which belongs to a sink. One of the end points of χ_1 — we denote it by R_1 — belongs to a corner arc or semipath or is a corner point of the boundary, and all the other points of the arc χ_1 belong to non-singular paths. The quadrangle delimited by the arcs without contact χ_i and $\hat{\chi}_i$ $(i = 1, 2)$ and the arcs of the paths through their end points is the elementary quadrangle $\hat{\Delta}_i$. The arcs χ_i and $\hat{\chi}_i$ will be called conjugate, as before.

We take all the corner semipaths $\hat{L}_i$, whether positive or negative, and construct the corresponding elementary quadrangles $\hat{\Delta}_i$ in the same way. Note that if there are two different semipaths, a positive and a negative one, which are a continuation of the same boundary arc of a path, the two corresponding arcs χ and $\hat{\chi}$ will be chosen "compatibly." This is best illustrated with an example: in Figure 51, the semipaths $\hat{L}$ and $\hat{L}_1$ with the end points R and R_1, respectively, are continuations of the same boundary arc $R_1\hat{R}_1$. We may therefore choose arbitrarily one of the two arcs χ_1 and $\hat{\chi}_1$, and the other arc is automatically determined by this choice. Both these arcs define the same elementary quadrangle $\hat{\Delta}_1$.

Let $\hat{\Delta}_j$ $(j = 1, 2, \ldots, \hat{N})$ be all the different elementary quadrangles constructed in this way. All the arcs χ_i (and $\hat{\chi}_i$) are taken sufficiently small, so that they have no common inner points between themselves and no common points (whether inner points or end points) with the previously defined arcs γ_i. Under these conditions no two elementary quadrangles Δ_i $(i = 1, 2, \ldots)$ and $\hat{\Delta}_j$ $(j = 1, 2, \ldots)$ have any inner points in common (but $\hat{\Delta}_j$ and $\hat{\Delta}_k$ may have common arcs of paths in their boundaries or parts of boundary arcs. The second case is observed, e.g., for the quadrangles $\hat{\Delta}_1$ and $\hat{\Delta}_2$ in Figure 51).

The elementary quadrangles $\hat{\Delta}_j$ adjoin corner paths (and possibly their continuations also; for example, the quadrangle $\hat{\Delta}_2$ in Figure 51 adjoins the semipath $\hat{L}$, and $\hat{\Delta}_1$ adjoins the semipath $\hat{L}$, its continuation — the boundary arc $R\hat{R}_1$ — and the continuation of this arc, the semipath $\hat{L}_1$).

Following the same procedure as for the quadrangles Δ_j, we define the elementary quadrangles Δ_k^* $(k = 1, 2, \ldots, N^*)$ adjoining corner arcs. The arcs without contact forming the boundaries of the quadrangle Δ_k^* will be designated λ_k and λ_k^*; these arcs will also be called conjugate. Note that a

quadrangle adjoining a corner semipath may at the same time adjoin a corner arc also (e.g., $\hat{\Delta}_2$ in Figure 52). Therefore, Δ_k^* $(k = 1, 2, \ldots, N^*)$ are defined specifically as elementary quadrangles adjoining corner arcs, but not adjoining corner semipaths. We moreover take the arcs λ_k and λ_k^* to be sufficiently small. Under these conditions, the quadrangles Δ_i, $\hat{\Delta}_j$, and Δ_k^* are all different and no two of these quadrangles have interior points in common (but $\hat{\Delta}_j$ and Δ_k^* may have common arcs of paths in their boundaries or parts of boundary arcs; such are, e.g., the pairs of quadrangles Δ_1^* and $\hat{\Delta}_2$, $\hat{\Delta}_1$ and $\hat{\Delta}_2$, Δ_1^* and Δ_2^* in Figure 52).

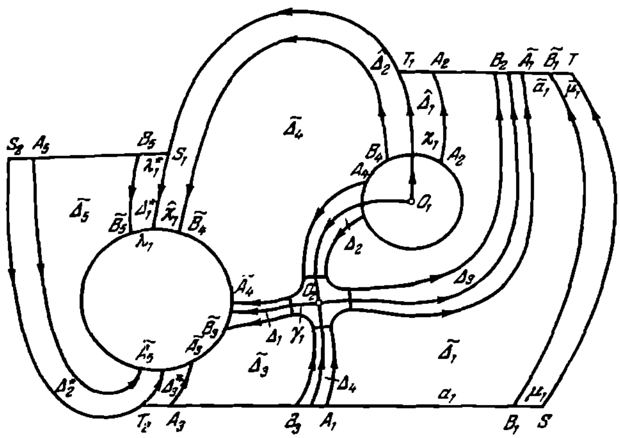

FIGURE 52

Let us now consider boundary arcs of paths. Those which are the continuations of corner arcs or corner semipaths form the boundary of the quadrangles Δ_k^* and $\hat{\Delta}_j$ (such are, e.g., the arcs S_1T_1 and S_2T_2 in Figure 52, which form the boundaries of $\hat{\Delta}_2$ and Δ_2^*, respectively). Now suppose that a boundary arc of a path ST has no continuation in G^*. It is readily seen that the corner points of this arc belong to two different boundary arcs without contact. Take a sufficiently small section μ of one of these arcs, adjoining a corner point (e.g., the section SB_1 in Figure 52) and draw through the points of μ paths of the system until they emerge through a section $\bar{\mu}$ of the second boundary arc without contact. We have thus formed an elementary quadrangle adjoining a boundary arc of a path ST (in Figure 52, this is the quadrangle $\bar{\Delta}_1$). Let $\bar{\Delta}_l$ $(l = 1, 2, \ldots, \bar{N})$ be all such quadrangles. Their boundaries contain the conjugate arcs without contact μ_m and $\bar{\mu}_m$.

Let us now consider all the elementary quadrangles Δ_i, $\hat{\Delta}_j$, Δ_k^*, and $\bar{\Delta}_l$ and all the arcs without contact γ_i', χ_j and $\bar{\chi}_j$, λ_k and λ_k^*, μ_l and $\bar{\mu}_l$ entering their boundaries (except the arcs γ_i, entering the boundaries of the canonical neighborhoods of saddle points). For simplicity, we will designate all these quadrangles by Δ_i $(i = 1, 2, \ldots, s$, where $s = N + N + N^* + \bar{N})$ and the relevant arcs without contact by $\gamma_m'^{(\alpha)}$ $(\gamma_m^{(\omega)})$ $(m = 1, 2, \ldots, k)$ if they are parts of cycles without contact which belong to sources (sinks) or if these arcs belong to

the boundary of $\bar{G}^*$, the paths of system (A) entering $\bar{G}^*$ (leaving $\bar{G}^*$) through them.*

Let us further consider all the non-free cycles without contact $C^{(\alpha)}$ which belong to sources, and all the boundary arcs without contact $c^{(\alpha)}$ and the non-free cycles without contact $Z^{(\alpha)}$ through which paths of system (A) enter $\bar{G}^*$. All the arcs $\gamma_m^{(\alpha)}$ ($m = 1, 2, \ldots, r$)belong to these cycles without contact and boundary arcs $C^{(\alpha)}$, $c^{(\alpha)}$, and $Z^{(\alpha)}$. Removing from each cycle $C^{(\alpha)}$ and $Z^{(\alpha)}$ and from each boundary arc $c^{(\alpha)}$ the points of the arcs $\gamma_m^{(\alpha)}$ which belong to the corresponding elements, we obtain a finite number of open arcs without contact which have no common points. The closures of these arcs are designated $a_1^{(\alpha)}, a_2^{(\alpha)}, \ldots, a_p^{(\alpha)}$. Note that the end points of each of these arcs belong to non-singular paths.

Similarly treating all the non-free cycles without contact $C^{(\omega)}$ which belong to sinks and the boundary arcs without contact $c^{(\omega)}$ and the cycles $Z^{(\omega)}$, we remove from them the points of the arcs $\gamma_m^{(\omega)}$ to obtain arcs without contact whose closures are designated $a_1^{(\omega)}, a_2^{(\omega)}, \ldots$.

Let us now consider all the paths which at some t pass through points of the arc $a_i^{(\alpha)}$ ($i = 1, 2, \ldots, p$). It is readily seen that as t increases, they all cross one of the arcs $a^{(\omega)}$ (by an appropriate choice of our notation, we can ensure that this is the arc $a_i^{(\omega)}$). Hence it follows directly that the number of arcs $a_i^{(\omega)}$ is exactly equal to the number of arcs $a_i^{(\alpha)}$, i.e., p in both cases. The arcs of paths extending between the arcs without contact $a_i^{(\alpha)}$ and $a_i^{(\omega)}$ form an elementary quadrangle, which we designate Δ_i' ($i = 1, 2, \ldots, p$). All the quadrangles Δ_i', like the quadrangles Δ_i, are elements of the set $\bar{G}^* - K$.

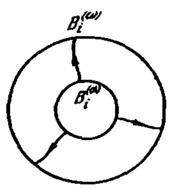

FIGURE 53

Finally, let us consider the free α-cycles of system (A) in $\bar{G}^*$ (if any). They are designated $B_i^{(\alpha)}$ ($i = 1, 2, \ldots, q$). A cycle $B_i^{(\alpha)}$ is conjugate, as we know (see §18.1, Lemma 3), with the free ω-cycle $B_i^{(\omega)}$, and both these cycles delimit an annular region filled with sections of non-singular paths (Figure 53). Drawing three path sections in each of these annuli, we partition them into elementary quadrangles Δ_i'' ($i = 1, 2, \ldots, 3q$). They are all evidently elements of the set $\bar{G}^* - K$.

It is readily seen that each point of the set $\bar{G}^* - K$ belongs at least to one of the quadrangles Δ_j' ($j = 1, 2, \ldots, p$), Δ_i ($i = 1, 2, \ldots, s$), and Δ_k'' ($k = 1, 2, \ldots, 3q$), and that all these quadrangles satisfy the conditions of Lemma 5. This completes the proof of the lemma.

Figure 52 shows a doubly connected region $\bar{G}^*$ with a normal boundary. It comprises two canonical neighborhoods, that of an unstable node O_1 and that of a saddle point O_2. The complement $\bar{G}^* - K$ is partitioned into 15 elementary quadrangles. This partition meets all the requirements of the lemma.

§18. THE FUNDAMENTAL THEOREM OF STRUCTURAL STABILITY OF A DYNAMIC SYSTEM

1. Lemmas

We will give here a number of lemmas that will be needed in connection with the proof of the fundamental theorem of structural

* It is readily seen that the number of the arcs $\gamma_m^{(\alpha)}$ is equal to the number of $\gamma_m^{(\omega)}$ (we designate this common number by r). If there are no saddle points in $\bar{G}^*$, then $s = r$. If $\bar{G}^*$ has k saddle points, then $r = s - 2k$.

stability of a dynamic system. Some of these lemmas are presented without proof.

L e m m a 1. *Let* (A) *be a dynamic system defined in* $\bar{G}$, *and let* $\bar{G}^*$ ($\bar{G}^* \subset G$) *be a region with a normal boundary. For any* $\varepsilon > 0$, *there exists a region with a normal boundary* $\bar{G}^{**}$ *such that* $\bar{G}^* \subset G^{**} \subset \bar{G}^{**} \subset G$, *and all the points of* $\bar{G}^{**}$ *which are not points of* $\bar{G}^*$ *are contained in an* ε*-neighborhood of the boundary of* $\bar{G}^*$; *moreover, the boundaries of* $\bar{G}^*$ *and* $\bar{G}^{**}$ *have identical schemes.**

Lemma 1 is geometrically self-evident, and its proof is omitted. Region $\bar{G}^{**}$ which satisfies the proposition of the lemma will be called an e x t e n s i o n or an ε-e x t e n s i o n of $\bar{G}^*$. Figure 54 shows a doubly connected region $\bar{G}^*$ and its extension. The region $\bar{G}^{**}-\bar{G}^*$ is cross-hatched.

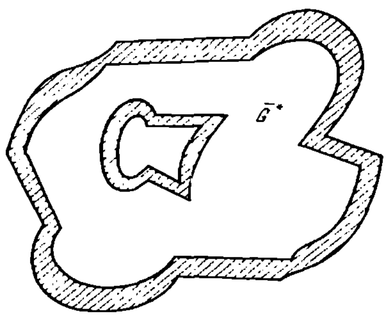

FIGURE 54

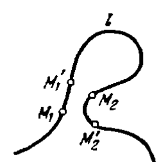

FIGURE 55

R e m a r k. If conditions I — III formulated in the Introduction to this chapter are satisfied in $\bar{G}^*$ and if $\varepsilon > 0$ is sufficiently small and $\bar{G}^{**}$ is an ε-extension of $\bar{G}^*$, system (A) has no equilibrium states and no closed paths in $\bar{G}^{**}$ other than those that it has in $\bar{G}^*$, i.e., conditions I — III are also satisfied in $\bar{G}^{**}$.

The truth of the remark regarding the equilibrium states follows from the fact that the equilibrium states of a system cannot lie arbitrarily close to the boundary of $\bar{G}^*$, since there are no equilibrium states on the boundary. The truth of the remark regarding closed paths is proved by the same argument that we have used in the proof of Theorem 21.**

L e m m a 2. *Let* l *be a simple arc,* M_1 *and* M_2 *two points on this arc. If* $\eta > 0$ *is sufficiently small, and* M_1' *and* M_2' *are two points of the arc* l

* Regarding the scheme of a boundary of a region, see QT, §26.

** Let $\varepsilon_n \to 0$, $\varepsilon_n > 0$. H_n is an ε_n-extension of $\bar{G}^*$, L_n is a closed path in H_n which is not contained completely in $\bar{G}^*$. M_n is a point of L_n lying outside H_n. M_n can be taken to converge to M_0. M_0 lies on the boundary of $\bar{G}^*$. Let L_0 be a path through M_0. If L_0 has a point S outside $\bar{G}^*$, at a distance $\sigma > 0$ from the boundary of $\bar{G}^*$, then for large n, L_n passes arbitrarily close to S and yet lies inside an arbitrarily small neighborhood of $\bar{G}^*$, which is impossible. If, on the other hand, L_0 is completely contained in $\bar{G}^*$, M_0 belongs to a boundary arc of a path, whose continuations in the two directions are corner semipaths, one going to a source in $\bar{G}^*$ and the other to a sink. But then a path L_n passing sufficiently close to M_0 goes to the same source for $t \to -\infty$ and to the same sink for $t \to +\infty$, i.e., it cannot be closed. This proves the proposition.

lying respectively in the η*-neighborhoods of* M_1 *and* M_2, *the direction of the arc* l *defined by the motion from* M_1 *to* M_2 *coincides with the direction defined by the motion from* M_1' *to* M_2' *(Figure 55).*

Lemma 3. *Let* C *be a simple closed curve,* M_i $(i = 1, 2, \ldots, n;\ n \geqslant 3)$ *points lying on this curve. Let these points be ordered* $M_1, M_2, \ldots, M_n$ *for the motion in a certain direction (one of the two possible directions) along the curve* C. *Then if* $\eta > 0$ *is sufficiently small, and* M_i' $(i = 1, 2, \ldots, n)$ *are points of the curve* C *so that* $M_i' \in U_\eta(M_i)$, *the motion in the same direction along the curve* C *will find the points* M_i' *in the order* $M_1', M_2', \ldots, M_n'$ *(Figure 56).*

The proof of Lemmas 2 and 3 is omitted.

Together with a given system (A), we will consider modified systems $(\tilde{A})$ sufficiently close to (A).

FIGURE 55

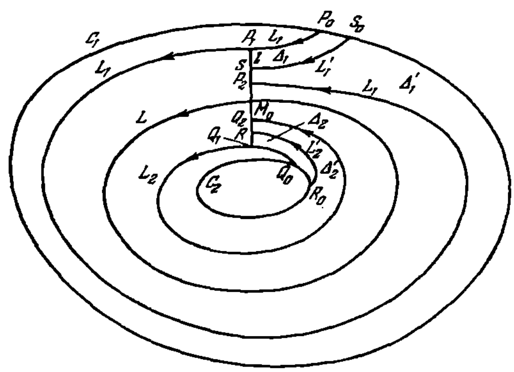

FIGURE 57

Lemma 4. *Let* L *be a structurally stable limit cycle of system* (A), *i.e., a limit cycle with a non-zero characteristic index), and* γ *a canonical neighborhood (ring) of the limit cycle bounded by cycles without contact* C_1 *and* C_2. *Then for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if system* $(\tilde{A})$ *is* δ*-close to* (A), *then*

(a) C_1 *and* C_2 *are cycles without contact for the paths of system* $(\tilde{A})$, *and these paths cross each of the cycles* C_1 *and* C_2 *in the same direction as the paths of the original system* (A);

(b) the ring γ *contains a single closed path* $\tilde{L}$ *of system* $(\tilde{A})$, *and this* $\tilde{L}$ *is a stable structurally stable limit cycle if* L *is stable, and an unstable structurally stable limit cycle if* L *is unstable;*

(c)
$$(\gamma, A) \overset{\varepsilon}{\equiv} (\gamma, \tilde{A}). \tag{1}$$

Proof. To fix ideas, let us consider a stable limit cycle L. We choose some point M_0 on this cycle and draw a normal to the path L through this point. Let l be a segment of this normal, P_1, Q_1 its end points, L_1 and L_2 the paths of system (A) passing through P_1 and Q_1, respectively. We assume that M_0 is an interior point of the segment l. Moreover, the segment l is taken to be so small that l is an arc without contact for the paths of system (A)

and the paths L_1 and L_2 with increasing t will again cross the segment l at points P_2 and Q_2, the arcs P_1P_2 and Q_1Q_2 of these paths lying completely inside γ (Figure 57; see §12.1).

Let Γ_1 be a simple closed curve made up of the turn P_1P_2 of the path L_1 and the segment P_1P_2 of the normal l; Γ_2 is the analogous curve consisting of the turn Q_1Q_2 of the path L_2 and the segment Q_1Q_2 of the normal l (Figure 57). We use W to designate the region between the curves Γ_1 and Γ_2. Clearly, $L \subset W \subset \gamma$.

From the definition of a canonical neighborhood, γ and therefore W do not contain any closed paths of (A), except L. Therefore, as t decreases, the paths L_1 and L_2 passing through the points P_1 and Q_1 will emerge from γ through the points P_0 and Q_0 lying on the cycles without contact C_1 and C_2, respectively (Figure 57).

Take some point S on the segment P_1P_2 of the normal l and some point R on the segment Q_1Q_2. The paths L_1' and L_2' of system (A) through these points emerge with decreasing t from the neighborhood γ through the respective points S_0 and R_0 of the cycles C_1 and C_2 (Figure 57). The arc P_0P_1 of path L_1 and the arc S_0S of path L_1' partition the region between the closed curves C_1 and Γ_1 into two elementary quadrangles, which we denote Δ_1 and Δ_1' (Figures 57, 58). Also the arc Q_0Q_1 of path L_2 and the arc R_0R of path L_2' partition the region between the curves C_2 and Γ_2 into two elementary quadrangles Δ_2 and Δ_2'.

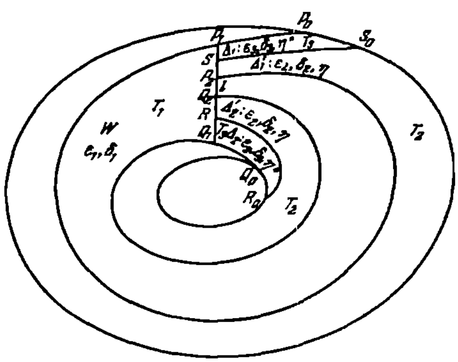

FIGURE 58

Let $(\tilde{A})$ be a dynamic system sufficiently close to (A), and $\tilde{P}_2$, $\tilde{Q}_2$, $\tilde{S}$, $\tilde{R}$, $\tilde{L}_1$, $\tilde{L}_1'$, $\tilde{\Delta}_1$, etc., the elements of $(\tilde{A})$ corresponding to the elements P_2, Q_2, S, R, L_1, L_1', Δ_1, etc., of (A) (it is assumed that the points P_1, Q_1, the normal l, and the closed curves C_1 and C_2 do not change on passing to system $(\tilde{A})$).

In Chapter V, §14, in our proof of Theorem 18 (on the structural stability of a simple limit cycle) we established that for any $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ with the following property: if system $(\tilde{A})$ is δ_1-close to system (A), then

$$(\overline{W}, A) \overset{\varepsilon_1}{\equiv} (\overline{W}, \tilde{A}), \tag{2}$$

and the mapping T_1 which realizes this relation is defined in $\overline{W}$ and can be chosen so that

$$T_1(P_1) = P_1, \quad T_1(Q_1) = Q_1 \text{ and } T_1(l) = l. \tag{3}$$

We will now use Lemma 9 of §4 (Chapter II, §4.2). By this lemma, for any $\varepsilon_2 > 0$, we can find two numbers $\delta_2 > 0$, $\eta > 0$ with the following property: if system $(\tilde{A})$ is δ_2-close to system (A) and a topological mapping φ is given,

transforming the arcs $P_0P_1P_2$, SS_0, $Q_0Q_1Q_2$ and RR_0 of the paths of system (A) into the arcs $\tilde{P}_0P_1\tilde{P}_2$, $\tilde{S}\tilde{S}_0$, $\tilde{Q}_0Q_1\tilde{Q}_2$ and $\tilde{R}\tilde{R}_0$ of the paths of system $(\tilde{A})$, respectively, and the arcs without contact SP_2 and RQ_2 into the arcs $\tilde{S}\tilde{P}_2$ and $\tilde{R}\tilde{Q}_2$, and φ is an η-translation, then the mapping φ can be continued to a mapping T_2 which moves the quadrangles Δ_1' and Δ_2' into $\tilde{\Delta}_1'$ and $\tilde{\Delta}_2'$, respectively, conserves paths, is an ε_2-translation, and coincides with the mapping φ wherever the latter is defined.

By Lemma 8, §4 (Chapter II, §4.2), for any $\varepsilon_3>0$, we can find such $\delta_3>0$ and $\eta^*>0$ that if system $(\tilde{A})$ is δ_3-close to (A) and a topological mapping φ^* is given moving the arcs without contact SP_1 and RQ_1 of system (A) into the arcs without contact $\tilde{S}P_1$ and $\tilde{R}Q_1$ of system $(\tilde{A})$, respectively, and this φ^* is an η^*-translation, the mapping φ^* can be continued to a mapping T_3 which moves the elementary quadrangles Δ_1 and Δ_2 into $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$, respectively, conserves paths, is an ε_3-translation, and coincides with the mapping φ^* on the segments SP_1 and RQ_1.

Finally, let $\delta_4>0$ be so small that if system $(\tilde{A})$ is δ_4-close to system (A), the cycles without contact C_1 and C_2 and the arc without contact l of system (A) are respectively cycles without contact and an arc without contact of system $(\tilde{A})$, the paths of $(\tilde{A})$ crossing each of the cycles C_1 and C_2 and the arc l in the same directions as the paths of (A).

Let $\varepsilon>0$. We take $\varepsilon_2=\varepsilon$ and for this ε_2 find the numbers δ_2 and η. We take $\varepsilon_3<\varepsilon$, $\varepsilon_3<\eta$, and for this ε_3 we find the numbers η^* and δ_3.

We take $\varepsilon_1<\varepsilon$, $\varepsilon_1<\eta$, $\varepsilon_1<\eta^*$, and for this ε_1 find δ_1.

Finally, we take

$$\delta<\min\{\delta_1, \delta_2, \delta_3, \delta_4\}.$$

The number $\delta>0$ obtained in this way meets all the requirements of the lemma.

Indeed, let system $(\tilde{A})$ be δ-close to system (A). Then, according to the choice of δ_1, there exists a mapping T_1 of $\overline{W}$ onto $\overline{\tilde{W}}$ which satisfies conditions (2) and (3).

Further, since $\delta<\delta_3$ and $\varepsilon_1<\eta^*$, $\varepsilon_1<\varepsilon$, we can construct a mapping T_3 which moves the elementary quadrangles Δ_1 and Δ_2 into $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$, respectively, and has the required property (φ^* is identified with the mapping T_1 previously defined on the segments P_1S and Q_1R).

Finally, since $\varepsilon_2=\varepsilon$, $\varepsilon_1<\eta$, $\varepsilon_3<\eta$, and $\delta_2<\delta$, we can construct a mapping T_2 of the quadrangles Δ_1' and Δ_2' onto $\tilde{\Delta}_1'$ and $\tilde{\Delta}_2'$, respectively, which has the required properties (φ is identified with the mappings T_3 and T_1 previously defined on the corresponding segments of the boundaries of Δ_1' and Δ_2').

It is readily seen that the mappings T_1, T_2, T_3 jointly define a mapping T of the canonical neighborhood γ onto itself which is path-conserving and is an ε-translation. Therefore, if $(\tilde{A})$ is δ-close to (A), we have

$$(\gamma, A) \overset{\varepsilon}{\equiv} (\gamma, \tilde{A}),$$

i.e., proposition (c) of the lemma is satisfied. Proposition (b) follows directly from (c). Proposition (a) is satisfied in virtue of the peculiar choice of the number δ_4 and the relation $\delta<\delta_4$. This completes the proof of the lemma.*

* Figure 58 shows the numbers ε, δ, η corresponding to the regions W, Δ_1, Δ_1', Δ_2, Δ_2'. The first step in the construction of the mapping is the construction of T_1 (in W), then the construction of T_3 (in the quadrangles Δ_1 and Δ_2), and finally the construction of T_2 (in the quadrangles Δ_1' and Δ_2').

The following lemma is a stronger version of Lemma 4.

Lemma 5. Let L be a structurally stable limit cycle of system $(\tilde{A})$, γ its canonical neighborhood limited by cycles without contact C_1 and C_2. Then, for any $\varepsilon > 0$, there exist $\eta > 0$ and $\delta > 0$ with the following property; if system $(\tilde{A})$ is δ-close to system (A), and φ is a topological mapping moving each of the cycles C_1 and C_2 into itself which is an η-translation, then C_1 and C_2 are cycles without contact for system $(\tilde{A})$ and there exists a mapping T of the neighborhood γ into itself which is an ε-translation, which conserves paths, and which coincides with the mapping φ on the boundary of the neighborhood γ (i.e., on the cycles C_1 and C_2).

Proof of Lemma 5 is analogous to the proof of Lemma 4, differing only in an obvious modification of the argument. Specifically, we first construct the mapping T of the elementary quadrangles $\Delta_1, \Delta_1', \Delta_2, \Delta_2'$ onto $\tilde{\Delta}_1, \tilde{\Delta}_1', \tilde{\Delta}_2, \tilde{\Delta}_2'$, respectively, so that it coincides with the mapping φ on the boundary of γ. This mapping induces a mapping of the segments P_1P_2 and Q_1Q_2 onto the segments $\tilde{P}_1\tilde{P}_2$ and $\tilde{Q}_1\tilde{Q}_2$, respectively (the points P_1 and $\tilde{P}_1$ need not coincide in this case, whereas in Lemma 4 this was the same point. This also applies to the points Q_1 and $\tilde{Q}_1$). This induced mapping is continued first to a mapping of P_1Q_1 onto $\tilde{P}_1\tilde{Q}_1$, and then to a mapping of W onto $\tilde{W}$ by the technique developed in the proof of Theorem 18 (see §14).

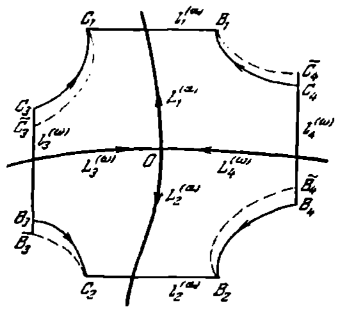

FIGURE 59

We will require two further lemmas. The first deals with the neighborhood of a saddle point.

Let O be a structurally stable saddle point of system (A), γ its canonical neighborhood limited by arcs without contact $l_1^{(\alpha)}, l_2^{(\alpha)}, l_3^{(\omega)}, l_4^{(\omega)}$* and arcs of paths C_1C_3, C_2B_3, B_1C_4, and B_2B_4 (Figure 59). The separatrices of saddle point O crossing the arcs without contact will be designated $L_1^{(\alpha)}, L_2^{(\alpha)}, L_3^{(\omega)}, L_4^{(\omega)}$, respectively.

Let $\lambda_i^{(\omega)}$ $(i = 3, 4)$ be the "elongation" of the arc without contact $l_i^{(\omega)}$, i.e., an arc without contact incorporating $l_i^{(\omega)}$ whose end points are not the end points of $l_i^{(\omega)}$ (in this way, all the points of $l_i^{(\omega)}$ are the inner points of $\lambda_i^{(\omega)}$). Let $(\tilde{A})$ be a dynamic system which is sufficiently close to (A). As t decreases, the paths of $(\tilde{A})$ passing through the end points C_i and B_i of the arcs without contact $l_i^{(\alpha)}$ $(i = 1, 2)$ will cross the arcs $\lambda_i^{(\omega)}$ $(i = 3, 4)$ at four points designated $\tilde{C}_3$, $\tilde{B}_3$, $\tilde{C}_4$, $\tilde{B}_4$, respectively. The sections $\tilde{C}_3\tilde{B}_3$ and $\tilde{C}_4\tilde{B}_4$ of the arcs $\lambda_3^{(\omega)}$ and $\lambda_4^{(\omega)}$ will be designated $\tilde{l}_3^{(\omega)}$ and $\tilde{l}_4^{(\omega)}$, respectively. Let $\tilde{\gamma}$ denote the region limited by the arcs l_1, l_2, $\tilde{l}_3$, and $\tilde{l}_4$ and the sections $C_1\tilde{C}_3$, $C_2\tilde{B}_3$, $B_1\tilde{C}_4$, and $B_2\tilde{B}_4$ of the paths of system $(\tilde{A})$. Finally, φ is a topological mapping moving each of the arcs l_1 and l_2 onto itself, so that the end points of l_1 and l_2 remain fixed, and the intersection points of these arcs with the separatrices $L_1^{(\alpha)}$ and $L_2^{(\alpha)}$, respectively, move into intersection points with the separatrices $\tilde{L}_1^{(\alpha)}$ and $\tilde{L}_2^{(\alpha)}$.

Lemma 6. For any $\varepsilon > 0$, there exist $\delta > 0$ and $\eta > 0$ with the following property: if system $(\tilde{A})$ is δ-close to system (A) and the mapping φ is an η-translation, the arcs l_1, l_2, λ_3, and λ_4 are arcs without contact for the

* We will sometimes use the shorter notation l_1, l_2, l_3, l_4.

paths of $(\tilde{A})$ *and there exists a mapping* T *of region* γ *onto* $\tilde{\gamma}$ *which coincides with* φ *on the arcs* $l_1^{(\alpha)}$ *and* $l_2^{(\alpha)}$, *conserves paths, and is an* ε*-translation.*

Remark. It evidently follows from Lemma 6 that $\tilde{\gamma}$ contains a single equilibrium state $\tilde{O}$ of $(\tilde{A})$, $\tilde{O}$ being a saddle point and $\tilde{\gamma}$ its canonical neighborhood. Also

$$(\gamma, A) \stackrel{\varepsilon}{\equiv} (\tilde{\gamma}, \tilde{A}).$$

Proof of Lemma 6 is analogous to the proof of Lemma 4, §9.2, from which it is obtained by obvious modifications.

Lemma 7. Let O *be a structurally stable state of equilibrium of system* (A), *which is either a node or a focus,* γ *is its canonical neighborhood limited by a cycle without contact* C. *For any* $\varepsilon > 0$, *there exist* $\eta > 0$ *and* $\delta > 0$ *with the following property: if system* $(\tilde{A})$ *is* δ*-close to system* (A) *and* φ *is a topological mapping of cycle* C *into itself, which is an* η*-translation, then* C *is a cycle without contact for system* $(\tilde{A})$ *and there exists a mapping* T *of* γ *into itself which is a path-conserving* ε*-translation that coincides with the mapping* φ *on the boundary of* γ *(i.e., on* C*).*

Proof of Lemma 7 is analogous in all respects to the proof given in the remark to Theorem 12 (§8.2).

2. The fundamental theorem for a plane region

Theorem 23. For a dynamic system (A) *defined in a plane region* $\bar{G}$ *to be structurally stable in a region* $\bar{G}^*$ *with a normal boundary* $(\bar{G}^* \subset G)$, *it is necessary and sufficient that conditions I through III in the Introduction to Chapter VI be satisfied:*

I. System (A) *has in* $\bar{G}^*$ *only a finite number of equilibrium states, which are simple nodes, saddle points, or foci.*

II. The closed paths of system (A) *in* $\bar{G}^*$ *are simple limit cycles.*

III. System (A) *has no saddle-to-saddle separatrices in* $\bar{G}^*$.

Proof. The necessity of conditions I–III for structural stability of a system (A) in $\bar{G}^*$ has been proved in previous chapters (§7, Theorems 10 and 11; §10, Theorem 15; §15, Theorem 20; §11, Theorem 16). We thus have to prove the sufficiency of conditions I–III for the structural stability of a system.

Suppose conditions I–III are satisfied. By Lemma 1, §18.1, and the remark to that lemma, for a sufficiently small $\sigma > 0$, any σ-extension of $\bar{G}^*$ contains no equilibrium states and no closed paths of system (A) other than those contained in $\bar{G}^*$. Let $\bar{H}$ be such an extension of G^*. We will show that $\bar{H}$ has the following property: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if system $(\tilde{A})$ is δ-close to system (A), then

$$(\bar{H}, A) \stackrel{\varepsilon}{\equiv} (\bar{\tilde{H}}, \tilde{A}), \tag{4}$$

where $\tilde{H}$ is a region. Since $\bar{G}^* \subset H$, this implies, by definition, that system (A) is structurally stable in $\bar{G}^*$.

Clearly, $\bar{H}$ has a normal boundary and system (A) satisfies conditions I–III in this region. Thus there exist regular partitions of $\bar{H}$ into canonical

neighborhoods and elementary quadrangles (§17.2, Lemma 5). We choose and fix one of these partitions, which we denote Π.

We introduce the following notation:

U_i ($i = 1, 2, \ldots, p$) are the canonical neighborhoods of the sources and the sinks of the partition Π;

V_j ($j = 1, 2, \ldots, q$) are the canonical neighborhoods of the saddle points O_j;

$l_{1j}^{(\alpha)}, l_{2j}^{(\alpha)}, l_{3j}^{(\omega)}, l_{4j}^{(\omega)}$ ($j = 1, 2, \ldots, q$) are arcs without contact making up the boundary of the canonical neighborhood of the saddle point O_i;

$F_{1j}^{(\alpha)}, F_{2j}^{(\alpha)}, F_{3j}^{(\omega)}, F_{4j}^{(\omega)}$ ($j = 1, 2, \ldots, q$) are the elementary quadrangles of the partition Π whose boundaries incorporate the arcs $l_{1j}^{(\alpha)}, l_{2j}^{(\alpha)}, l_{3j}^{(\omega)}, l_{4j}^{(\omega)}$, respectively. We will use the following abbreviated notation for these quadrangles and arcs: $F_1, F_2, \ldots, F_{4q}$; $l_1, l_2, \ldots, l_{4q}$;

R_k ($k = 1, 2, \ldots, r$) are all the other elementary quadrangles of the partition Π;

a_k (b_k) ($k = 1, 2, \ldots, r$) are the arcs without contact making up the boundary of quadrangle R_k through which the paths of system (A) enter R_k (leave R_k);

A_i ($i = 1, 2, \ldots, s$) are the "vertices" of the elementary quadrangles which belong to the boundaries of the canonical neighborhoods of the *sources* or to the boundary arcs without contact through which the paths of (A) *enter* the region H;

B_i ($i = 1, 2, \ldots, s^*$) are the vertices of the elementary quadrangles which belong to the boundaries of the canonical neighborhoods of the *sinks* or to the boundary arcs without contact through which the paths of (A) *leave* the region H (all the corner points of the boundary of H are evidently included among the A_i and B_i).

Let $(\tilde{A})$ be a dynamic system sufficiently close to (A). We will first define the region $\overline{\tilde{H}}$ in which this system is considered. Let C be some closed boundary curve of $\overline{H}$. If C is a cycle without contact, it will be used as the boundary curve of $\tilde{H}$. Suppose C is made up of arcs without contact $l_1, l_2, \ldots, l_n$ and arcs of paths $z_1, z_2, \ldots, z_n$ of system (A), so that when the curve C is traversed in the positive direction the various arcs are encountered in the order $l_1, z_1, l_2, z_2, \ldots, l_n, z_n$ (Figure 60), their end points (corner points) being respectively $X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be some fixed elongations of the arcs $l_1, l_2, \ldots, l_n$.* Through the points $Y_1, Y_2, \ldots, Y_n$ we pass the arcs $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$ of paths of system $(\tilde{A})$ to their intersection with the arcs $\lambda_2, \lambda_3, \ldots, \lambda_n, \lambda_1$ at the points $\tilde{X}_2, \tilde{X}_3, \ldots, \tilde{X}_n, \tilde{X}_1$, respectively (in Figure 60 these arcs are marked by dashed curves). The arcs without contact $\tilde{X}_1 Y_1, \tilde{X}_2 Y_2, \ldots, \tilde{X}_n Y_n$ will be designated $\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_n$, respectively. If $(\tilde{A})$ is sufficiently close to (A), the curve $\tilde{C}$ made up of the arcs $\tilde{l}_1, \tilde{z}_1, \tilde{l}_2, \tilde{z}_2, \ldots, \tilde{l}_n, \tilde{z}_n$ is a simple closed curve. For each boundary curve C_i of H we substitute a curve $\tilde{C}_i$ constructed in this way. The region limited by all the curves $\tilde{C}_i$ is $\tilde{H}$.

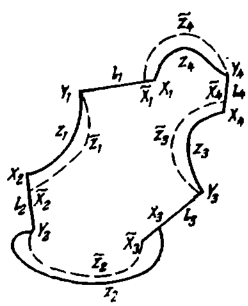

FIGURE 60

Let us now construct a partition $\tilde{\Pi}$ of $\overline{\tilde{H}}$, analogous to the regular partition Π of $\overline{H}$, making each element U_i, V_j, F_m, R_k of the partition Π to

* If λ is an arc without contact and l is the part of this arc consisting entirely of interior points of λ, then λ is called an *elongation* of the arc without contact l.

correspond to an element $\tilde{U}_i$, $\tilde{V}_j$, $\tilde{F}_m$, $\tilde{R}_k$. In particular, we assume that $\tilde{U}_i$ coincides with U_i $(i = 1, 2, \ldots, p)$.

Each canonical neighborhood V_j of a saddle point O_j $(j = 1, 2, \ldots, q)$ is made to correspond to a region $\tilde{V}_j$, whose construction is described in the statement of Lemma 6 (see Figure 59; the regions V_j and $\tilde{V}_j$ in Lemma 6 are designated γ and $\tilde{\gamma}$, respectively). Furthermore, we will identify $\tilde{l}_{1j}^{(\alpha)}$, $\tilde{l}_{2j}^{(\alpha)}$, $j = 1, 2, \ldots, q$, with $l_{1j}^{(\alpha)}$, $l_{2j}^{(\alpha)}$, respectively. $\tilde{l}_{3j}^{(\omega)}$ $(\tilde{l}_{4j}^{(\omega)})$ is an arc without contact entering the boundary of $\tilde{V}_j$. It is part of the arc $l_{3j}^{(\omega)}$ $(l_{4j}^{(\omega)})$ or of its elongation (we assume that all the elongations have been chosen and fixed beforehand).

Every elementary quadrangle F_m $(m = 1, 2, \ldots, 4q)$ of the partition Π is made up of arcs of paths of system (A) and is limited on one side by the arc l_m, and on the other by an arc without contact which belongs either to a limit cycle Γ of the neighborhood of a source or a sink, or to a limit cycle (or an arc without contact) of the region $\bar{H}$. The quadrangle F_m is made to correspond to the quadrangle $\tilde{F}_m$ made up from arcs of paths of system $(\tilde{A})$ and limited on one side by the arc $\tilde{l}_m$, and on the other by an arc without contact belonging to the same limit cycle Γ (or the same boundary cycle without contact, or the same boundary arc without contact).

The corner points of the boundary of $\tilde{H}$ are the corner points Y_i of the boundary of H and the points $\tilde{X}_i$ described in the construction of $\tilde{H}$ (Figure 60). Corner semipaths and corner arcs of paths in $\tilde{H}$ are determined by the region itself and the system $(\tilde{A})$.

Consider an elementary quadrangle R_k and the arc a_k $(k = 1, 2, \ldots, r$; see above) entering its boundary. Let $A_k^{(1)}$ and $A_k^{(2)}$ be the end points of the arc a_k. Let $\Gamma^{(\alpha)}$ be the cycle without contact (a boundary cycle or a cycle belonging to a source) or the boundary arc without contact incorporating a_k.

The point $A_k^{(1)}$ (or $A_k^{(2)}$) may be a corner point of the boundary of H, or a point of a corner arc or of a corner semipath of system (A), or it may be a vertex of one of the quadrangles F_m. In each of these cases, the corresponding point $\tilde{A}_k^{(1)}$ (or $\tilde{A}_k^{(2)}$) is naturally determined by the preceding construction. If $A_k^{(1)}$ (or $A_k^{(2)}$) is not a point of one of the above types, we may take $\tilde{A}_k^{(1)}$ to coincide with $A_k^{(1)}$ ($\tilde{A}_k^{(2)}$ to coincide with $A_k^{(2)}$, respectively). The points $\tilde{A}_k^{(1)}$ and $\tilde{A}_k^{(2)}$ are thus well defined. The arc a_k is made to correspond to an arc without contact $\tilde{a}_k$ which is part of the cycle (or the arc without contact) Γ between the end points $\tilde{A}_k^{(1)}$ and $\tilde{A}_k^{(2)}$. Finally, the elementary quadrangle R_k consisting of the arcs of paths of system (A) and delimited (on one side) by the arc a_k is made to correspond to the elementary quadrangle $\tilde{R}_k$ consisting of the arcs of paths of system $(\tilde{A})$ and delimited by the arc without contact $\tilde{a}_k$.

The points $\tilde{B}_i$ $(i = 2, 3, \ldots, s^*)$ corresponding to the points B_i, and the arcs without contact $\tilde{b}_k$ corresponding to the arcs b_k $(k = 1, 2, \ldots, r)$ are determined in a natural way using the previously constructed arcs $\tilde{a}_k$ and points $\tilde{A}_i$.

Let system $(\tilde{A})$ be sufficiently close to system (A). Moreover, when going around the cycle without contact $\Gamma^{(\alpha)}$ which belongs to a source or to a boundary of H (or when traveling along a boundary arc without contact) in a certain direction, let the points A_i be encountered in the order $A_{i_1}, A_{i_2}, \ldots, A_{i_s}$. From Lemmas 2 and 3 it follows that when going around the cycle $\tilde{\Gamma}^{(\alpha)}$ in the same direction, the points $\tilde{A}_i$ lie on $\tilde{\Gamma}^{(\alpha)}$ in the order $\tilde{A}_{i_1}, \tilde{A}_{i_2}, \ldots, \tilde{A}_{i_s}$. A similar proposition is valid for the cycle $\Gamma^{(\omega)}$ (or the boundary arc without contact) and for the points B_i and $\tilde{B}_i$.

It is readily seen that if system $(\tilde{A})$ is sufficiently close to (A), then:

a) $\tilde{U}_i$ (they coincide with U_i) are canonical neighborhoods of sources or sinks of system $(\tilde{A})$; if U_i is a canonical neighborhood of an equilibrium state (a limit cycle) of system (A), $\tilde{U}_i$ is a canonical neighborhood of an equilibrium state (respectively, a limit cycle) of the same stability of system $(\tilde{A})$.

b) To every saddle point O_j of system (A) corresponds a saddle point $\tilde{O}_j$ of system $(\tilde{A})$ and $\tilde{V}_j$ $(j=1, 2, \ldots, q)$ is a canonical neighborhood of $\tilde{O}_j$.

c) The previously constructed regions $\tilde{F}_m$ $(m=1, 2, \ldots, 4q)$ and $\tilde{R}_k$ $(k = 1, 2, \ldots, r)$ are elementary quadrangles of system $(\tilde{A})$.

d) The set of all points of the quadrangles $\tilde{F}_m$ and $\tilde{R}_k$ and the neighborhoods $\tilde{U}_i$ and $\tilde{V}_j$ coincides with the entire region $\bar{H}$.

e) System $(\tilde{A})$ has no other equilibrium states and closed paths in $\bar{\bar{H}}$, except those which lie in the canonical neighborhoods $\tilde{U}_i$ and $\tilde{V}_j$.

This proposition follows directly from the preceding statement and from the fact that each elementary quadrangle $\tilde{F}_m$ and $\tilde{R}_k$ adjoins either the boundary of $\bar{H}$ or the boundary of one of the canonical neighborhoods $\tilde{U}_i$ or $\tilde{V}_j$.

f) The canonical neighborhoods $\tilde{U}_i$ and $\tilde{V}_j$ and the elementary quadrangles $\tilde{F}_m$ and $\tilde{R}_k$ form a regular partition of $\bar{\bar{H}}$, which will be designated $\tilde{\Pi}$.

g) The canonical partitions Π and $\tilde{\Pi}$ of the regions $\bar{H}$ and $\bar{\bar{H}}$, respectively, are isomorphic in the obvious sense.*

We will now show that for any $\varepsilon>0$ and a sufficiently small $\delta>0$, we have

$$(\bar{H}, A) \overset{\varepsilon}{\equiv} (\bar{\bar{H}}, \tilde{A})$$

for a system $(\tilde{A})$ which is δ-close to (A).

Choose a fixed $\varepsilon>0$. The corresponding $\delta>0$ will be selected in several steps.

1) We choose a number $\delta_\Pi>0$ such that if system $(\tilde{A})$ is δ_Π-close to system (A), $\bar{H}$ satisfies the above conditions (a) through (g).

2) For every canonical neighborhood U_i we select two numbers $\eta_i>0$ and $\delta_i>0$ $(i = 1, 2, \ldots, p)$ in accordance with Lemmas 5 and 7 (i.e., such that if system $(\tilde{A})$ is δ_i-close to system (A) and φ_i is a topological mapping of the boundary of U_i into itself, which is an η_i-translation, there exists a mapping T_i of U_i into itself which is an ε-translation, conserves paths, and coincides with φ_i on the boundary of U_i).

Let

$$\delta_U=\min\{\delta_1, \delta_2, \ldots, \delta_p\}, \quad \eta_U=\min\{\eta_1, \eta_2, \ldots, \eta_p\}.$$

3) Consider an elementary quadrangle F_j and an arc without contact l_j $(j = 1, 2, \ldots, 4q)$ entering its boundary (note that l_j enters the boundary of a canonical neighborhood of a saddle point). Let c_j and d_j be the arcs of paths of system (A) entering the boundary of the quadrangle F_j. Let $\varepsilon_F = \min\{\varepsilon, \eta_U\}$.

By Lemma 9, §4.2, there exist $\delta_j>0$ and $\eta_j>0$ such that if system $(\tilde{A})$ is δ_j-close to system (A), and φ_j is a mapping of the arcs l_j, c_j, d_j onto the

* To be precise, if any two elements E_1 and E_2 of the partition are incidental, i.e., one of these elements enters the boundary of its counterpart, the corresponding elements $\tilde{E}_1$ and $\tilde{E}_2$ of partition $\tilde{\Pi}$ are also incidental. By *elements of the partition* Π we mean the neighborhoods U_i and V_j, the elementary quadrangles F_m and R_k, the arcs without contact and the arcs of paths entering their boundaries, and the end points of these arcs. The previous construction established a natural one-to-one correspondence between the elements of partitions Π and $\tilde{\Pi}$.

corresponding boundary arcs $\tilde{l}_j$, $\tilde{c}_j$, $\tilde{d}_j$ which is also an η_j-translation, there exists an ε_F-translation T_j of the elementary quadrangle F_j onto $\tilde{F}_j$ which coincides with φ_j on l_j, c_j, d_j and conserves paths.

Let

$$\delta_F = \min\{\delta_1, \delta_2, \ldots, \delta_{4q}\}, \qquad \eta_F = \min\{\eta_1, \eta_2, \ldots, \eta_{4q}\}.$$

4) Consider an elementary quadrangle R_k and the arc without contact a_k and arcs of paths c_j and d_j ($j = 1, 2, \ldots, r$) entering its boundary. As in case 3, we take $\varepsilon_R = \varepsilon_F = \min\{\varepsilon, \eta_U\}$ and select for each of the quadrangles R_k the two numbers δ_j and η_j (along the same lines as before). Let

$$\delta_R = \min\{\delta_1, \delta_2, \ldots, \delta_r\}, \quad \eta_R = \min\{\eta_1, \eta_2, \ldots, \eta_r\}.$$

5) Consider the arc without contact a_k entering the boundary of the quadrangle R_k. Let A_{k1} and A_{k2} be the end points of a_k. Let c_{k1} and c_{k2} denote the arcs of paths of system (A) passing through A_{k1} and A_{k2}, respectively, and entering the boundary of the quadrangle R_k ($k = 1, 2, \ldots, r$). The arc a_k lies either on a boundary cycle or a boundary arc without contact, or on a cycle without contact which belongs to a source.

If $(\tilde{A})$ is sufficiently close to (A), the elements of the regular partition $\tilde{\Pi}$ corresponding to the quadrangle R_k, its sides a_k, c_{k1}, c_{k2}, and the vertices A_{k1} and A_{k2} are the quadrangle $\tilde{R}_k$, its sides $\tilde{a}_k$, $\tilde{c}_{k1}$, $\tilde{c}_{k2}$, and the vertices $\tilde{A}_{k1}$ and $\tilde{A}_{k2}$. By Lemmas 7 and 8, §4.2, we can select $\zeta_k > 0$ and $\delta_k > 0$ such that if system $(\tilde{A})$ is δ_k-close to system (A), and $\rho(A_{k1}, \tilde{A}_{k1}) < \zeta_k$, $\rho(A_{k2}, \tilde{A}_{k2}) < \zeta_k$, there exists a mapping φ_k defined on the arcs a_k, c_{k1} and c_{k2} which maps these arcs into $\tilde{a}_k$, $\tilde{c}_{k1}$, $\tilde{c}_{k2}$, respectively, and the points A_{k1} and A_{k2} into $\tilde{A}_{k1}$ and $\tilde{A}_{k2}$, respectively, and which is also an η_R-translation (η_R is defined in step 4 of the procedure; see Figure 61). Let

$$\zeta_A = \min\{\zeta_1, \zeta_2, \ldots, \zeta_r\},$$
$$\delta_A = \min\{\delta_1, \delta_2, \ldots, \delta_r\}.$$

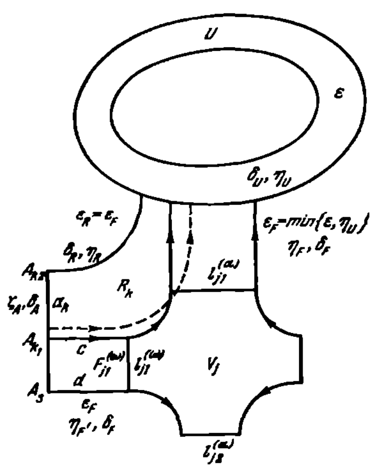

FIGURE 61

6) Consider a canonical neighborhood V_j of the saddle point O_j and the arcs without contact $l_{j1}^{(\alpha)}, l_{j2}^{(\alpha)}, l_{j1}^{(\omega)}, l_{j2}^{(\omega)}$ entering its boundary. Let $\delta_V > 0$ be a number with the following property: if system $(\tilde{A})$ is δ_V-close to system (A), $\tilde{V}_j$ is a canonical neighborhood of the partition $\tilde{\Pi}$ corresponding to the neighborhood V_j, and $\varepsilon_V = \min\{\varepsilon, \eta_F, \eta_R\}$, then

a) there exists a mapping T_j of V_j onto $\tilde{V}_j$ which conserves paths, is an ε_V-translation, and coincides with the mapping φ described before the statement of Lemma 6 on each of the arcs $l_{j1}^{(\alpha)}$, $l_{j2}^{(\alpha)}$;*

* In other words, the mapping T_j topologically maps each of the arcs l_{j1}^{α}, l_{j2}^{α} onto itself, leaving the end points of these arcs fixed and moving the intersection points of the arcs with the separatrices of system (A) into intersection points with the corresponding separatrices of system $(\tilde{A})$.

b) every arc of a path c (or d) entering the boundary of the elementary quadrangles $F_{1j}^{(\alpha)}$ (or $F_{2j}^{(\alpha)}$, $F_{1j}^{(\omega)}$, $F_{2j}^{(\omega)}$) can be mapped onto the corresponding arc $\tilde{c}(\tilde{d})$ by an η^*-translation, where $\eta^* = \min\{\eta_F, \eta_R\}$;

c) to every vertex A_s of the quadrangle $F_{j1}^{(\omega)}$ or $F_{j2}^{(\omega)}$ $(j = 1, 2, \ldots, q)$ there corresponds a vertex $\tilde{A}_s$ of the partition $\tilde{\Pi}$ such that

$$\rho(A_s, \tilde{A}_s) < \zeta_A$$

(ζ_A is selected in step 5 of the procedure).

The existence of δ_V follows from Lemma 6 (§18.1), from Lemmas 7 and 8, §4.2, and from the remark to Lemma 3, §9.2.

7) Consider an elementary quadrangle R_k and the arc without contact a_k $(k = 1, 3, \ldots, r)$ entering its boundary. Let $E_1, E_2, \ldots, E_{r^*}$ be the end points of the arcs a_k, which are corner points or belong to corner arcs or corner semipaths.* Let $\delta_E > 0$ be so small that if system $(\tilde{A})$ is δ_E-close to system (A) and $\tilde{E}_s$ is a point corresponding to point E_s $(s = 1, 2, \ldots, r^*)$, then $\rho(E_s, \tilde{E}_s) < \zeta_A$.

The existence of δ_E follows from the method of construction of $\tilde{H}$ and from Lemma 5, §4.1.

Let

$$\delta = \min\{\delta_\Pi, \delta_U, \delta_F, \delta_R, \delta_A, \delta_V, \delta_E\}.$$

We will prove that if system $(\tilde{A})$ is δ-close to system (A), then

$$(\bar{H}, A) \overset{\varepsilon}{\equiv} (\bar{\tilde{H}}, \tilde{A}). \tag{4}$$

To prove this proposition, we assume system $(\tilde{A})$ to be δ-close to (A) and construct a mapping T which realizes the relation (4).

Construction of mapping T.

Step I. In each saddle-point neighborhood $\bar{\Gamma}_j$ $(j = 1, 2, \ldots, q)$ we construct a mapping T_j satisfying the conditions described in step 6 above, and assign the symbol T to T_j.

Step II. We now construct a mapping T of the arcs of paths c and d entering the boundaries of the quadrangles $F_j^{(\alpha)}$, $F_j^{(\omega)}$ onto the arcs $\tilde{c}$ and $\tilde{d}$ so that this mapping is an η^*-translation (see step 6 above) and coincides with the previous mapping T at the points of the neighborhood $\bar{\Gamma}_j$.

Step III. The mapping T completed on the boundary arcs l, c, and of the quadrangles F_j $(j = 1, 2, \ldots, 4q)$ is now continued to $\bar{F}_j$ so that it is an ε_F-translation and conserves paths. This is feasible because of condition 3.

Step IV. Every point E_s $(s = 1, 2, \ldots, r^*)$ is made to correspond to a point $\tilde{E}_s$, taking $T(E_s) = \tilde{E}_s$.

Step V. Steps III and IV have defined the mapping T at the end points of each arc without contact a_k entering the boundary of the elementary quadrangle R_k. We continue this mapping to a mapping T of the entire arc a_k onto $\tilde{a}_k$, so that T is an η_R-translation. This is feasible because of 5.

Step VI. Let E_s be the end point of the arc without contact a_k entering the boundary of the quadrangle R_k (see step 7), c_s the arc of a path of system (A) passing through the point E_s and entering the boundary of the

* The end points of the arcs a_k may also belong to the elementary quadrangles F_j.

quadrangle R_k, $\tilde{c}_s$ the corresponding arc of a path of system $(\tilde{\mathrm{A}})$ entering the boundary of the quadrangle $\tilde{R}_k$. The mapping T defined at the points E_s by step IV is now continued to a mapping T of c_s onto $\tilde{c}_s$ $(s = 1, 2, \ldots, r^*)$, so that T is an η_R-translation. This is feasible because of 5.

Step VII. The mapping T has been defined by steps I through VI on the arc without contact a_k and on the arcs of paths entering the boundary of each elementary quadrangle R_k $(k = 1, 2, \ldots, r)$; this mapping is an η_R-translation. We continue it to a mapping T of the quadrangle $\overline{R}_k$ onto $\overline{\tilde{R}}_k$, which conserves paths and is an ε_R-translation. This is feasible because of 4.

Step VIII. The mapping T is defined by the previous steps on the boundary of each canonical neighborhood U_i $(i = 1, 2, \ldots, p)$ and is an η_U-translation. We continue this mapping to a mapping T of the neighborhood $\overline{U}_i$ onto itself which conserves paths and is an ε-translation. This is feasible because of 2.

The mapping T defined by steps I through VIII is evidently an ε-translation which maps $\overline{H}$ into $\overline{\tilde{H}}$ and conserves the paths. We thus have the relation

$$(H, A) \overset{\varepsilon}{\equiv} (\overline{\tilde{H}}, \tilde{A}). \tag{4}$$

This completes the proof of the fundamental theorem.

3. The fundamental theorem for a sphere

The definition of a structurally stable dynamic system on a sphere was given in Chapter III (§6.2, Definition 12). It amounts to the following: a dynamic system (A) defined on a sphere S is said to be structurally stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any system $(\tilde{\mathrm{A}})$ δ-close to (A),

$$(S, \tilde{A}) \overset{\varepsilon}{\equiv} (S, A).$$

The necessary and sufficient conditions of structural stability of a system defined on a sphere precisely coincide with the corresponding conditions for a system on a plane. We will now prove the following theorem.

Theorem 24. A dynamic system (A) *defined on a sphere S is structurally stable if and only if*

I. Each of the equilibrium states of system (A) *is a simple node, saddle point, or focus.*

II. The closed paths of system (A) *are simple limit cycles.*

III. System (A) *has no saddle-to-saddle separatrices.*

Proof. To fix our ideas, let (A) be an analytical system on a sphere, and we will prove structural stability relative to the space $R_a^{(1)}$. Structural stability relative to the space $R_a^{(r)}$ $(r > 1)$ is proved in the same way, and the proof relative to the space $R_N^{(r)}$ is even simpler.

Sufficiency. The sufficiency of conditions I through III for the structural stability of system (A) on a sphere is proved along the same lines as Theorem 23, but on the whole the proof is somewhat simpler. The simplification naturally derives from the fact that there are no boundary arcs of paths and boundary arcs and cycles without contact on a sphere.

Necessity. We will only prove the necessity of condition II, namely that every closed path of a structurally dynamic system on a sphere is a

simple limit cycle. The absence of multiple equilibrium states and saddle-to-saddle separatrices for a structurally stable system is proved along the same lines, with some simplification.

Without loss of generality, we may consider the systems on a sphere S in a three-dimensional space R^3, defined by the equation

$$x^2+y^2+z^2=1. \qquad (S)$$

As the closed covering of the sphere (see §5.2) we choose the covering which can be regarded as the simplest in a certain sense: it comprises two regions $\bar{G}_1$ and $\bar{G}_2$, where $\bar{G}_1$ is the set of all the points of the sphere S for which $z_1 \leqslant z \leqslant 1$, and $\bar{G}_2$ is the set of all the points of the sphere for which $-1 \leqslant z \leqslant z_2$. We further assume that $-1 < z_1 < 0$ and $z_1 < z_2 < 1$ (Figure 62). Let U_i, V_i be the local coordinates in G_i $(i=1, 2)$; $\bar{G}_i$ corresponds to a region $\bar{H}_i$ in the plane (u_i, v_i), which may be regarded as a circle centered at the origin.

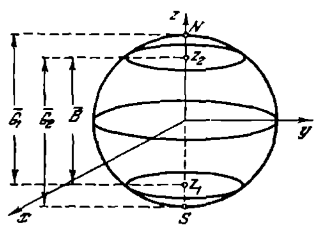

FIGURE 62

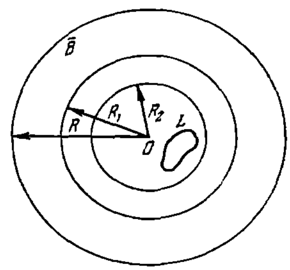

FIGURE 63

The intersection of $\bar{G}_1$ and $\bar{G}_2$ is the ring $\bar{B}$.

Consider a structurally stable analytical system (A) defined by the set of analytical equations

$$\frac{du_i}{dt}=P_i(u_i, v_i), \qquad \frac{dv_i}{dt}=Q_i(u_i, v_i), \qquad (A_i)$$

where $i=1, 2$, and P_i, Q_i are functions defined in G_i (or equivalently, in H_i); in the intersection $\bar{B}$ of these regions, system (A_1) is transformed into system (A_2) in virtue of the transformation equations (also analytical) between the coordinates u_1, v_1 and u_2, v_2 (§5.2).

The proof will be done by reductio ad absurdum. Indeed, suppose there is a closed path L of system (A) on the sphere S which is not a simple limit cycle (i.e., a path with a zero characteristic index). Again without loss of generality, we may take the path L to lie in $\bar{G}_1$, outside the ring $\bar{B}$, and we may then treat it as a closed path of the system

$$\frac{du_1}{dt}=P_1(u_1, v_1), \qquad \frac{dv_1}{dt}=Q_1(u_1, v_1), \qquad (A_1)$$

defined in the plane region $\bar{H}_1$. Since the characteristic index of L is zero, the results of Chapter V show that either

(1) the path L is a multiple limit cycle of system (A), or

(2) all the paths passing in a sufficiently small neighborhood of L are closed paths.

Let us first consider case (1). In virtue of Remark 2 to Theorem 19 (§15.2), for any $\delta_1 > 0$ in this case there exists a system (A_1^*) δ_1-close to (A_1) which has at least two structurally stable limit cycles in any arbitrarily small neighborhood of L. We will denote these limit cycles by L_1^* and L_2^*.

To simplify the presentation, we identify $\bar{G}_1$ with $\bar{H}_1$, i.e., consider as a circle of radius R with a center at the origin in the plane (u_1, v_1). Let the ring $\bar{B}$ be made up of points of the circle $\bar{G}_1$ for which the radius-vector ρ satisfies the inequality $R_1 \leqslant \rho \leqslant R$. Further let the cycle L lie inside the circle $0 \leqslant \rho \leqslant R_2$, where $R_2 < R_1$ (Figure 63), and let the system (A_1^*) have the form

$$\frac{du_1}{dt} = P_1^*(u_1, v_1), \quad \frac{dv_1}{dt} = Q_1^*(u_1, v_1). \tag{A_1^*}$$

Let us construct a system ($\hat{A}_1$) of class 1 which is sufficiently close to (A_1) and coincides with (A_1^*) in the circle $\rho \leqslant R_2$ and with the system (A_1) in the ring $\bar{B}$. We will use a function $\varphi(\rho)$ with the following properties:

a) $\varphi(\rho)$ is a function of class 1 defined for all ρ, $0 \leqslant \rho \leqslant R$;

b)
$$\varphi(\rho) = 1 \quad \text{for} \quad 0 \leqslant \rho \leqslant R_2,$$
$$\varphi(\rho) = 0 \quad \text{for} \quad R_1 \leqslant \rho \leqslant R,$$
$$0 \leqslant \varphi(\rho) \leqslant 1 \quad \text{for} \quad R_2 < \rho < R_1$$

(Figure 64; there evidently exist functions of any class $r > 1$ satisfying condition b).

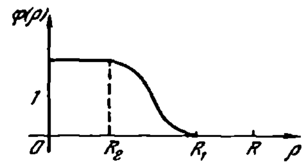

FIGURE 64

The right-hand sides of system ($\hat{A}_1$) — the functions $\hat{P}_1(u_1, v_1)$ and $\hat{Q}_1(u_1, v_1)$ — are defined by the equalities

$$\hat{P}_1 = P_1 + (P_1^* - P_1)\varphi(\rho), \quad \hat{Q}_1 = Q_1 + (Q_1^* - Q_1)\varphi(\rho), \tag{5}$$

where $\rho = \sqrt{u_1^2 + v_1^2}$.

System ($\hat{A}_1$) with right-hand sides defined by (5) is a system of class 1 which coincides with system (A_1) in the ring $\bar{B}$ and with the system (A_1^*) in the circle $0 \leqslant \rho \leqslant R_2$. Moreover, it is readily seen that if δ_1 is sufficiently small, ($\hat{A}_1$) can be made arbitrarily close to (A_1).

Since ($\hat{A}_1$) and (A_2) coincide in the ring $\bar{B}$ as before, taken together they constitute a certain dynamic system on the sphere S, which we denote by ($\hat{A}$).

In general, this is a system of class 1. We will show that it can be approximated with any degree of accuracy by an analytical system.

To this end, consider the vector field $\mathbf{w}(M)=\mathbf{w}(x, y, z)$ (where $M(x, y, z)$ is a point of the sphere) corresponding to the system $(\hat{A})$ in the space R^3. The vector $\mathbf{w}(M)$ lies in a plane tangent to the sphere at the point M.

Let $f_i(x, y, z)$, $i=1, 2, 3$, be the coordinates of the vector $\mathbf{w}(x, y, z)$. Consider a cube $\bar{E}$ with its center at the origin and with faces parallel to the coordinate planes, enclosing the sphere S. Consider a spherical layer $\bar{\Sigma}$, defined by the inequalities

$$1-\eta \leqslant r \leqslant 1+\eta,$$

where $r=\sqrt{x^2+y^2+z^2}$, and η is a positive number sufficiently small for the layer $\bar{\Sigma}$ to lie inside the cube $\bar{E}$. We define a function $F_i(x, y, z)$ at any point (x, y, z) of the layer Σ by the relation

$$F_i(x, y, z)=f_i\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$(i=1, 2, 3;\ r=\sqrt{x^2+y^2+z^2})$. Since $(\hat{A})$ is a dynamic system of class 1, it is readily seen that $F_i(x, y, z)$ are functions of class 1 in the layer $\bar{\Sigma}$. By Whitney's theorem (see /11/, Vol. I, Sec. 260), the functions $F_i(x, y, z)$ can be extended over the entire cube $\bar{E}$ without changing their class. Let the functions F_i be components of the vector $\mathbf{w}$; this approach yields a vector field of class 1 defined in the cube $\bar{E}$ which coincides with the field $\mathbf{w}$ on the sphere S. By the Weierstrass theorem, the field $\mathbf{w}$ can be approximated with any accuracy to rank 1 with an analytical field $\mathbf{w}_a$. On the sphere S, the vectors $\mathbf{w}_a$ are in general not tangent to the sphere. However, projecting these vectors on the corresponding tangent planes to the sphere, we obtain a field of vectors tangent to the sphere which define some dynamic system $(\tilde{A})$ on S. Clearly $(\tilde{A})$ is an analytical system which can be made as close as desired to $(\hat{A})$ and thus to the initial system. Now, since the cycles L_1^* and L_2^* are structurally stable, system $(\tilde{A})$ has structurally stable cycles $\tilde{L}_1$ and $\tilde{L}_2$ in the neighborhood of each of these cycles.

We have thus established that if an analytical system (A) has a multiple limit cycle L on the sphere S, there exists an arbitrarily close analytical system $(\tilde{A})$ which has at least two closed paths in any arbitrarily small neighborhood of the cycle L. This, however, implies that (A) is not structurally stable, as demonstrated in the proof of Theorem 20 (§15.3).

It now remains to consider case (2), when all the paths passing sufficiently close to L are closed. In this case, as in the proof of Theorem 20, we can construct an arbitrarily close system of class 1 with L as its simple limit cycle, and then proceed to approximate to it, as before, with an analytical system $(\tilde{A})$. System $(\tilde{A})$ will have a simple limit cycle $\tilde{L}$ arbitrarily close to the cycle L. If ε is sufficiently small, the mapping which realizes the ε-identity of the partition of the sphere by the paths of (A) and $(\tilde{A})$ moves an isolated closed path $\tilde{L}$ of $(\tilde{A})$ into a non-isolated closed path of (A) lying near L, which is impossible.

The problem is thus proved for the structural stability relative to the space $R_3^{(1)}$. In other cases, the proof is analogous with obvious modifications.

4. Remarks and supplements

a) Remark regarding structurally stable systems inside a cycle without contact.

As we have already noted (§6.1), structurally stable systems were originally considered in a region limited by a cycle without contact /4/, and not in any general region. The definition of a structurally stable system can be significantly simplified in this case (compared to Definition 10, §6.1).

Indeed, let (A) be a dynamic system defined in $\bar{G}$, and $\bar{G}^*$ a closed subregion of G, limited by a cycle without contact Γ.

First let us assume that (A) is structurally stable in G^*. Since $\bar{G}^*$ is evidently a region with a normal boundary, Theorem 23 applies and conditions I through III are thus satisfied in $\bar{G}^*$. Repeating the same arguments as in the proof of the sufficiency in Theorem 23, we readily see that the following lemma holds true.

Lemma 8. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if system $(\tilde{A})$ is δ-close to system (A), we have

$$(\bar{G}^*, \tilde{A}) \overset{\varepsilon}{\equiv} (\bar{G}^*, A). \tag{6}$$

Now suppose that Lemma 8 is satisfied for a system (A). Let $\bar{H}$ be a σ-extension of G^* (see Lemma 1, §18.1), where $\sigma > 0$ is sufficiently small. Clearly, $\bar{H}$ in this case is also a region limited by a cycle without contact, and $\bar{G}^* \subset H$. Then by the previous lemma and by Lemmas 12 and 8 of §4.2 we see (this conclusion is easy to demonstrate) that for any $\varepsilon > 0$

$$(\bar{H}, \tilde{A}) \overset{\varepsilon}{\equiv} (\bar{H}, A),$$

provided that systems $(\tilde{A})$ and (A) are sufficiently close, and this implies that (A) is structurally stable in $\bar{G}^*$.

We have thus shown that if (A) is structurally stable in $\bar{G}^*$, Lemma 8 holds true, and vice versa. Hence, the statement of Lemma 8 can be adopted as a definition of structural stability of a system (A) in a region limited by a cycle without contact.

A similar remark applies to the case when the system is considered in a region limited by several nonintersecting cycles without contact. The proof is entirely analogous.

In the general case, when $\bar{G}^*$ is a region with a normal boundary, the statement of Lemma 8 no longer provides a definition of structural stability. Indeed, if (6) is satisfied, the boundary arcs of paths of $\bar{G}^*$ should be arcs of paths of both (A) and $(\tilde{A})$. This, however, is not generally true.

b) Structurally stable systems on closed surfaces.

The conditions of structural stability of dynamic systems (of class 1) on closed surfaces of non-zero kind, both oriented and non-oriented, were considered by M. Peixoto /7/. These conditions amount to the following: a dynamic system (A) of class 1 defined on a surface of the kind $p \geqslant 0$ is structurally stable if and only if

1) it has a finite number of equilibrium states, all of which are structurally stable;

2) it has no saddle-to-saddle separatrices;

3) it has a finite number of closed paths, all of which are simple limit cycles;

4) the α-limit (ω-limit) set of each path is either an equilibrium state or a limit cycle.

The only new condition added for structural stability on a closed surface of a nonzero kind is thus condition 4. On a sphere (or on plane), condition 4 automatically follows from conditions 1 through 3 by the Poincaré — Bendixson theorem (QT, §4.6). On a surface of a nonzero kind, however, dynamic systems may exist whose paths, say, are everywhere dense. Condition 4 rules out the possibility of the existence of these paths.

c) Structurally stable systems in the space of dynamic systems.

We have already noted (§6.1) that systems which are structurally stable in a certain region form an open set in the space of dynamic systems. We are now in a position to prove this statement. Let $\bar{G}$ be a bounded plane region, R_1 the space of dynamic systems of class 1 defined in $\bar{G}$, and $\bar{G}^*$ a region with a normal boundary, $\bar{G}^* \subset G$.

Theorem 25. The dynamic systems of class 1 defined in $\bar{G}$ which are structurally stable in $\bar{G}^*$ constitute an open set in R_1.

Proof. Let (A) be a dynamic system structurally stable in $\bar{G}^*$ which belongs to the space R_1. We will show that all the dynamic systems which are sufficiently close to (A) are also structurally stable. This will prove the theorem.

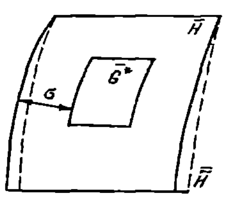

FIGURE 65

By Theorem 1, §18.1, and the remark to this theorem, there exists a region H with a normal boundary such that

$$\bar{G}^* \subset H \subset \bar{H} \subset G,$$

and system (A) has no equilibrium states and no closed paths in $\bar{H}$ other than those which are located in $\bar{G}^*$. Let σ be the distance from $\bar{G}^*$ to the boundary of $\bar{H}$. Evidently, $\sigma > 0$ (Figure 65).

Let ε be a positive number, $\varepsilon < \sigma/2$. In our proof of Theorem 23 we have shown that if $\delta_1 > 0$ is sufficiently small and system $(\tilde{A})$ is δ_1-close to system (A), then

$$(\bar{H}, A) \overset{\varepsilon}{\equiv} (\bar{\bar{H}}, \tilde{A}), \tag{4}$$

where $\bar{\bar{H}}$ is some region. We will take $\delta_1 > 0$ so that it satisfies this condition.

From (4) and the inequality $\varepsilon < \sigma/2$, we have

$$\bar{G}^* \subset \bar{\bar{H}}$$

(see footnote on p. 67).

The region $\bar{\bar{H}}$ was described in detail in our proof to Theorem 23. In particular, it has been established that if system $(\tilde{A})$ is δ_2-close to system (A), where δ_2 is sufficiently small, $(\tilde{A})$ has no equilibrium states and no closed paths in $\bar{\bar{H}}$ other than those which fall in the canonical neighborhoods U_i and V_j.

Finally, if $(\tilde{A})$ is δ_3-close to (A), where δ_3 is sufficiently small, the equilibrium states and the closed paths of $(\tilde{A})$ which are located in U_i and V_j are structurally stable (by Remark 2 to Theorem 18, §8.2, Remark 2 to Theorem 18, §9.2, and remark to Theorem 18, §14). From (4) it follows that $(\tilde{A})$ has no saddle-to-saddle separatrices in $\bar{\bar{H}}$.

We thus see from the above that if

$$\delta = \min \{\delta_1, \delta_2, \delta_3\},$$

and $(\tilde{A})$ is δ-close to (A), conditions I through III of Theorem 23 are satisfied for system $(\tilde{A})$ in a region $\bar{\bar{H}}$ with a normal boundary and, in virtue of this theorem, $(\tilde{A})$ is structurally stable in this region. But then, by (6) and Lemma 1, §6.1, we conclude that $(\tilde{A})$ is structurally stable in $\bar{G}^*$. This completes our proof.

An analogous theorem is evidently also true for a sphere.

*Theorem 25'. The structurally stable dynamic systems on a sphere form an open set in the space of dynamic systems.**

The validity of Theorem 25' follows almost immediately from Theorem 24.

Peixoto /7/ has shown that Theorem 25' is applicable to dynamic systems of class 1 on any closed surface, whether oriented or not.

We will now show that the structurally stable systems form an everywhere dense set in the space of dynamic systems. We will consider dynamic systems of class 1 defined in some region $\bar{G}$. Let H be a simply connected region limited by a simple closed curve Γ, such that $\bar{H} \subset G$. For simplicity, the proof will be given for the space of dynamic systems defined in $\bar{G}$ for which the curve Γ is a cycle without contact. Let this space be R^*. Closeness in this space is defined as closeness to rank 1 in $\bar{G}$.

Theorem 26. Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system in R^*. For any $\delta > 0$, there exists a system $(\tilde{A})$ δ-close to (A) which is structurally stable in H.

Proof. Let $\delta > 0$ be fixed. We may take δ so small that any system which is δ-close to (A) belongs to R^*, i.e., Γ is a cycle without contact of this system.

By the Weierstrass theorem on the approximation of continuous functions with polynomials, there exists a system

$$\frac{dx}{dt} = P_1(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{A$_1$}$$

$\delta/5$-close to (A), whose right-hand sides are polynomials.

Let P_1 and Q_1 be polynomials of degree m and n, respectively. In our proof to Theorem 10, §7.2, we have shown that there exist irreducible polynomials arbitrarily close to P_1 and Q_1, respectively, which are moreover of the same degree as P_1 and Q_1.

Let P_2 and Q_2 be such polynomials, and let the system

$$\frac{dx}{dt} = P_2(x, y), \quad \frac{dy}{dt} = Q_2(x, y) \tag{A$_2$}$$

be $\delta/5$-close to (A_1).

* Structural stability relative to one of the spaces $R_N^{(r)}$, $R_a^{(r)}$ is naturally meant here.

By the Bézout theorem (/12/, Ch. III, §3.1), system (A_2) may only have a finite number of equilibrium states, which does not exceed $m\cdot n$. Let O_i (x_i, y_i), $i=1, 2, \ldots, s$, be all the equilibrium states of (A_2) located in H $(s \leqslant m\cdot n)$.

Suppose that some of the equilibrium states O_i, e.g., O_1, is not simple, i.e.,

$$\begin{vmatrix} \frac{\partial P_2(x_1, y_1)}{\partial x} & \frac{\partial P_2(x_1, y_1)}{\partial y} \\ \frac{\partial Q_2(x_1, y_1)}{\partial x} & \frac{\partial Q_2(x_1, y_1)}{\partial y} \end{vmatrix} = 0.$$

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= \alpha(x-x_1) + P_2(x, y) = P_2^*(x, y), \\ \frac{dy}{dt} &= \beta(y-y_1) + Q_2(x, y) = Q_2^*(x, y). \end{aligned} \qquad (A_2^*)$$

For any choice of α and β, the point O_1 is an equilibrium state of (A_2^*), and P_2^* and Q_2^* are polynomials of degrees m and n, respectively. We take α and β sufficiently small, so that

$$\begin{vmatrix} \frac{\partial P_2^*(x_1, y_1)}{\partial x} & \frac{\partial P_2^*(x_1, y_1)}{\partial y} \\ \frac{\partial Q_2^*(x_1, y_1)}{\partial x} & \frac{\partial Q_2^*(x_1, y_1)}{\partial y} \end{vmatrix} \neq 0$$

(this is evidently always possible). System (A_2^*) is then arbitrarily close to (A_2), and O_1 is a simple equilibrium state of (A_2^*).

If the polynomials P_2^* and Q_2^* are not irreducible, we replace them with sufficiently close polynomials $\hat{P}_2$ and $\hat{Q}_2$ of the same degree (i.e., m and n, respectively) which are irreducible. We then obtain the system

$$\frac{dx}{dt} = \hat{P}_2(x, y), \quad \frac{dy}{dt} = \hat{Q}_2(x, y), \qquad (\hat{A}_2)$$

which also has at most $m\cdot n$ equilibrium states.

If ($\hat{A}_2$) is sufficiently close to (A_2^*), it has a simple equilibrium state $\hat{O}_1$ in a sufficiently small neighborhood of O_1 (see §2.2, Remark 3 to Theorem 6). Suppose that this is indeed so. System ($\hat{A}_2$) is thus arbitrarily close to (A_2) and has a finite number of equilibrium states (less than $m\cdot n$), at least one of which, $\hat{O}_1$, is simple.

Suppose that one of the equilibrium states of ($\hat{A}_2$) in H is multiple (we may take it as $\hat{O}_2$). Then, just as we have passed from (A_2) to ($\hat{A}_2$), we will pass from ($\hat{A}_2$) to an arbitrarily close system

$$\frac{dx}{dt} = \widetilde{P}_2(x, y), \quad \frac{dy}{dt} = \widetilde{Q}_2(x, y), \qquad (\widetilde{A}_2)$$

where the right-hand sides are irreducible polynomials of degrees m and n and which has a simple equilibrium state $\hat{O}_2$ in the neighborhood of $\widetilde{O}_2$. If ($\widetilde{A}_2$) is sufficiently close to ($\hat{A}_2$), ($\widetilde{A}_2$) also has a simple equilibrium state O_1 in the neighborhood of $\hat{O}_1$.

Continuing along the same lines, we obtain after a finite number of steps the system

$$\frac{dx}{dt} = P_3(x, y), \quad \frac{dy}{dt} = Q_3(x, y), \tag{A_3}$$

which is $\delta/5$-close to (A_2) and has a finite number of equilibrium states in H, all of which are simple.

If (A_3) has multiple foci or saddle-to-saddle separatrices in H, they can be eliminated by a suitable rotation of the vector field. Indeed, consider the system

$$\frac{dx}{dt} = P_3 - \mu Q_3 = P_4(x, y), \quad \frac{dy}{dt} = Q_3 + \mu P_3 = Q_4(x, y), \tag{A_4}$$

where $\mu \neq 0$. The vector field of this system is obtained by turning the vector field of (A_3) through the angle $\tan^{-1}\mu$.

The equilibrium states of (A_4) are all the equilibrium states of (A_3) and nothing but the equilibrium states of (A_3) (Lemma 3, §3.2). Since (A_3) only has simple equilibrium states in H, (A_4) will also have only simple equilibrium states in H, provided μ is sufficiently small. Suppose that (A_3) has a multiple focus (x_0, y_0). Without loss of generality, we may take $x_0 = y_0 = 0$. Let

$$\frac{\partial P_3(0, 0)}{\partial x} = a, \quad \frac{\partial P_3(0, 0)}{\partial y} = b, \quad \frac{\partial Q_3(0, 0)}{\partial x} = c, \quad \frac{\partial Q_3(0, 0)}{\partial y} = d.$$

Since $O(0, 0)$ is a multiple focus, we have for the equilibrium state O of (A_3)

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0, \qquad \sigma = a + d = 0.$$

For the same point considered as an equilibrium state of (A_4), we have

$$\Delta_\mu = \begin{vmatrix} a - \mu c & b - \mu d \\ c + \mu a & d + \mu b \end{vmatrix}, \qquad \sigma_\mu = \mu(b - c).$$

If $O(0, 0)$ is a multiple focus of (A_4), then $b = c$. Since also $d = -a$, we find $\Delta = -a^2 - b^2 \leqslant 0$, which contradicts the condition $\Delta > 0$. Thus, the point $O(0, 0)$ cannot be a multiple focus of (A_4). Clearly if $\mu \neq 0$ is sufficiently small, we have $\Delta^* > 0$, and σ^* is small, i.e., $O(0, 0)$ is a simple focus of (A_4). Thus, we have incidentally established that ***a rotation of the vector field of a dynamic system through a sufficiently small angle will reduce any multiple focus to a simple focus.***

We have shown in Chapter IV that a saddle-to-saddle separatrix of a dynamic system disappears when the vector field of the system is turned through a sufficiently small angle (it "decomposes" into two separatrices; see the lemma in §11.1 and the proof of Theorem 16 in §11.2). Since system (A_3) may only have a finite number of equilibrium states and separatrices, we conclude from the above that for a sufficiently small $\mu \neq 0$ the following conditions are satisfied:

1) System (A_4) is $\delta/5$ close to (A_3).

2) System (A_4) has only a finite number of equilibrium states in H, all of which are simple and which do not include multiple foci (in other words, H includes a finite number of equilibrium states all of which are structurally stable).

3) System (A_4) has no saddle-to-saddle separatrices in H.

Another condition follows from conditions 1 through 3 combined with the analyticity of (A_4):

4) System (A_4) may only have a finite number of closed paths in H.

The validity of condition 4 is proved along the same lines as in Theorem 21 (§16.1). There is only one distinctive feature in this proof which is worth considering separately. Using the notation introduced in the proof to Theorem 21, we will show that K_α or K_ω cannot be a closed path (i.e., case 5 in the proof to Theorem 21 is inapplicable to our conditions). Indeed, let K_ω be a closed path. Then, by the definition of K_ω, this path is a non-isolated closed path. The presence of a non-isolated closed path in an analytical system leads to the existence of a cell which is completely filled with closed paths (§12.3). Let W be such a cell. As we known (see QT, §23.2), the boundary of the cell W should be made up of two zero-limit continua, each of which is either a) a closed path, or b) an equilibrium state classified as a center, or c) a continuum of saddle-to-saddle separatrices and equilibrium states. However, in our case, no such zero-limit continua exist, since system (A_4) does not have any centers or saddle-to-saddle separatrices in H, and a closed path of an analytical system cannot be a zero-limit continuum (any closed path of an analytical system is either an isolated or an interior path in a cell). Thus, K_ω cannot be a closed path. In all other respects, the proof of proposition 4 does not differ from the proof of Theorem 21.

Let the closed paths of (A_4) lying in H be $L_1, L_2, \ldots, L_n$ (they are all limit cycles). If all these limit cycles are simple, (A_4) is structurally stable (by Theorem 23, §18.2) and our theorem is proved. Suppose now that some of the cycles L_i $(i = 1, 2, \ldots, r)$ of (A_4) are multiple. We may then rotate the vector field and consider the system

$$\frac{dx}{dt} = P_4 - \mu Q_4 = P_5(x, y), \qquad \frac{dy}{dt} = Q_4 + \mu P_4 = P_5(x, y). \tag{A_μ}$$

In §12.3 we defined the *multiplicity of a limit cycle* and established that every limit cycle of an analytical system has a definite multiplicity. We shall now use some results whose proof will only be given later on (Theorems 60 and 61, §32.4). By these theorems, if L_i is a cycle of finite miltiplicity of an analytical system (A_4), there exist $\varepsilon_i > 0$ and $\mu_i^* > 0$ with the following property: any system (A_μ) for which $|\mu| < \mu_i^*$ has at most two closed paths in $U_{\varepsilon_i}(L_i)$, and these paths are simple limit cycles.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$. We denote by V_i $(i = 1, 2, \ldots, r)$ a canonical neighborhood of the close paths L_i lying in $U_\varepsilon(L_i)$, and by γ_i' and γ_i'' the cycles without contact of (A_4) which form the boundary of V_i. We may take all the V_i to be nonintersecting neighborhoods in H. The set $\overline{H} \setminus \bigcup_1^r V_i$ (i.e., the complement of the union of the neighborhoods V_i in $\overline{H}$) will be designated F.

Let μ^*, $0 < \mu^* < \min\{\mu_1^*, \mu_2^*, \ldots, \mu_r^*\}$, be so small that if $|\mu| < \mu^*$, the following conditions are satisfied:

a) The system (A_μ) is $\delta/5$ close to (A_4).

b) The system (A_μ) has only a finite number of equilibrium states in H, which are all structurally stable.

c) The system (A_μ) has no saddle-to-saddle separatrices in H.

d) The cycles without contact γ_i' and γ_i'' ($i = 1, 2, \ldots, r$) of (A_4) are cycles without contact of (A_μ).

e) In every neighborhood V_i, there are at most two closed paths of (A_μ), and these paths are structurally stable limit cycles.

The validity of conditions a through d for small μ is self-evident. Condition e follows from the definition of the number ε, the numbers μ_i^*, and the neighborhoods V_i.

We will now show that if μ is sufficiently small in absolute magnitude, conditions a through e are supplemented by an additional condition:

f) Any closed path of (A_μ) lying in $\bar{H}$ is completely contained in one of the neighborhoods V_i.

To prove this proposition, note that the set F is a closed region limited by a finite number of cycles without contact of (A_4), and that (A_4) has only a finite number of equilibrium states in F, all of which are structurally stable, and has no saddle-to-saddle separatrices and no closed paths in this region. Therefore (A_4) is structurally stable in F. But then any system sufficiently close to (A_4) has the same topological structure in F (see §18.4a).

Hence it follows that for a sufficiently small μ, (A_μ) cannot have closed paths which are completely contained in F. Furthermore, if a closed path of (A_μ) has a point which belongs to one of the V_i, the entire path belongs to that V_i, since otherwise the path would intersect one of the cycles without contact γ_i' or γ_i'', which is impossible. This proves condition f.

We have thus shown that if μ is sufficiently small, the system (A_μ) satisfies conditions b, c, d, e. Then this system is structurally stable in H by the fundamental theorem of structural stability (Theorem 23, §18.2). Since system (A_μ) is δ-close to (A), the proof of Theorem 26 is complete.

Peixoto has shown that Theorem 26 is also true when R^* is the space of the dynamic systems on any closed surface, whether oriented or not (see /7/).

It follows from Theorem 26 that structurally stable systems form an everywhere dense set in the space R^* of systems of class 1.

In our proof of Theorem 26, we have actually established that any system in R^* has an arbitrarily close *analytical* structurally stable system. Hence it follows that in the space of *analytical* systems, the structurally stable systems are also everywhere dense.

Theorems 25 and 26 signify that structurally stable systems constitute, so to say, the "bulk" of the space of dynamic systems. Structurally unstable systems, on the other hand, constitute "partitions" partitioning the space into regions, each filled with structurally stable systems of the same topological type.

d) *Remark regarding the conditions of structural stability of a dynamic system relative to the spaces $R_N^{(r)}$ and $R_a^{(r)}$.*

In Theorem 23, which formulates the necessary and sufficient conditions of structural stability of a dynamic system, structural stability is naturally understood in the sense of Definition 10 (§6.1), i.e., relative to the space R_1 (see §6.3). Let now (A) be a system of class $N > 1$ or an analytical system, which is considered as a point of the space R^*, where R^* is one of the spaces $R_N^{(r)}$ ($r \leqslant N$) or $R_a^{(r)}$ (see §5.1). As before, we consider a region $\bar{G}^*$ with a

normal boundary, $\overline{G}^* \subset G$. Let (A) be structurally stable in $\overline{G}^*$ relative to the space R^*. Then:

(A) may only have a finite number of equilibrium states, all of which are simple (in virtue of Theorems 10 and 11, §7.3);

(A) does not have equilibrium states with pure imaginary characteristic numbers (by virtue of Remark 1 to Theorem 15, §10.4);

(A) does not have saddle-to-saddle separatrices (by virtue of the remark to Theorem 16, §11.2);

(A) may only have closed paths which are simple limit cycles (by virtue of Theorem 20, §15.3).

All this means that conditions I through III of Theorem 23 are the necessary conditions of structural stability of system (A) in $\overline{G}^*$ relative to the space R^*.

Conversely, if these conditions are satisfied, (A) is structurally stable relative to the space R_1, and therefore relative to the space R^* (see §6.3).

We have thus established that conditions I through III of Theorem 23 are the necessary and sufficient conditions of structural stability of a system (A) in $\overline{G}^*$ relative to any of the spaces $R_N^{(r)}$, $R_a^{(r)}$ which contain (A) as a point. Hence, if we are only dealing with the spaces $R_N^{(r)}$ and $R_a^{(r)}$, there is no need to indicate explicitly that (A) is structurally stable (or unstable) relative to one of these spaces.

Chapter VII

CELLS OF STRUCTURALLY STABLE SYSTEMS. AN ADDITION TO THE THEORY OF STRUCTURALLY STABLE SYSTEMS

INTRODUCTION

The present chapter comprises three sections. The first, §19, deals with cells of structurally stable systems. The concept of a cell of a dynamic system was introduced in QT, Chapter VII, §16. If all the singular elements, i.e., boundary arcs, equilibrium states, limit cycles, and orbitally unstable paths and semipaths, are removed from the region $\bar{G}$ (or the sphere) where the dynamic system is considered (assuming that there is only a finite number of such elements), the remaining points of $\bar{G}$ form an open set comprising a finite number of components. These components are the cells of the dynamic system. In every cell, the paths of the system show an identical behavior in a certain sense. The general properties of cells are investigated in QT, Chapter VII, and for general dynamic systems, there are infinitely many different types of cells. However, dynamic systems which are structurally stable in $\bar{G}^*$ and have a normal boundary are characterized by relatively few cell types. An investigation of these types is the subject of §19. It gives a complete listing of all the different types of simply connected interior cells of structurally stable systems (i.e., cells which do not touch the boundary of the region), and also of all the different types of doubly connected cells (both interior and adjoining the boundary).

Some examples of structurally stable systems with their cells are considered in §20.

The last section, §21, has no relation to the subject of cells. It proves that we can do without the requirement of ε-identity of close systems (see §18.4,a) in the definition of a structurally stable system within a cycle without contact (or on a sphere), i.e., the following definition can be advanced:

System (A) is said to be structurally stable in a region enclosed by a cycle without contact if any sufficiently close system has the same topological structure like (A) in the relevant region.

The equivalence of this definition and the original Definition 10 (§6.1), incorporating the requirement of ε-identity, is proved in §21, which thus constitutes an addition to the theory of structurally stable systems.

Note that the definition incorporating the condition of ε-identity is quite natural and is more convenient for the derivation of the necessary conditions of structural stability, since it permits restricting the analysis

to local considerations (e.g., in the neighborhood of a particular path), rather than proceeding with global treatment. On the other hand, the definition without ε-identity is intrinsically simpler and it directly shows that structurally stable systems constitute an open set in the space of dynamic systems.

§19. CELLS OF STRUCTURALLY STABLE DYNAMIC SYSTEMS

1. General considerations pertaining to cells of dynamic systems

The concept of a cell of a dynamic system and the properties of cells are considered in detail in QT, Chapter VII. In this section, we will reiterate without proof the essential information about cells that will be needed in our treatment of the cells of structurally stable systems.

The analysis can be carried out either on a sphere or in a bounded plane region $\bar{G}$. In the case of a sphere, we will assume that the dynamic system (A) has a finite number of singular paths (see QT, §16.9). If (A) is defined in a plane region $\bar{G}$, we will consider the system in a subregion G^* with a *normal boundary*, assuming that (A) has a finite number of singular paths in $\bar{G}^*$. To fix ideas, let us consider the case of a system defined in a plane region.

Let E be the set of all points which belong to the singular elements of the system in $\bar{G}^*$.* Since we assume a finite number of singular elements, E is a closed set. Its complement, the set $\bar{G}^* \setminus E$, is therefore open and consists of disjoint components. These components are the cells of the dynamic system (A).

The following general propositions are proved in QT, Chapter VII, §16:

I. The number of cells is finite.

II. Any cell is either simply connected or doubly connected.**

III. Paths belonging to a single cell are either all whole paths, or all positive (negative) semipaths, or all arcs of paths.

If the cell consists of whole paths, its paths are either all closed paths, or all loops, or all nonclosed paths, i.e., α- and ω-limit continua without common points.

IV. All the nonclosed whole paths which belong to the same cell have the same α-limit continuum and the same ω-limit continuum.

V. If a cell consisting of nonclosed whole paths is doubly connected, one of its boundary continua is an α-limit continuum, and the other is an ω-limit continuum of the cell.

* The set E comprises the points of all orbitally unstable paths and semipaths, the points of corner semipaths and corner arcs of paths, the points of boundary arcs and cycles without contact and boundary arcs of paths, and all the equilibrium states. See §16.2.

** We recall that a bounded region is said to be simply connected if its boundary consists of one connected set (the boundary continuum) and doubly connected if its boundary consists of two nonintersecting connected sets. In doubly connected regions, one of the two continua is an exterior boundary continuum, and the other is an interior boundary continuum. The interior boundary continuum, in particular, may comprise a single point.

VI. If a point P is a boundary point of a cell consisting of whole paths, all the points of a path L_P through P are boundary points of the cell.

VII. The boundary of each cell consists of points which belong to singular elements. If some point P is part of the boundary of a cell Z and belongs to a singular path L completely contained in G^*, or to a singular semipath $L^{()}$ (orbitally unstable or corner), or to a corner arc, the entire path L (semipath $L^{()}$ or arc l respectively) belongs to the boundary of the cell Z.

If the point P is part of the boundary of the cell Z and belongs to a boundary arc l of a path and P is not a corner point of the boundary, the entire arc l also belongs to the boundary of the cell Z.*

VIII. If a path or a semipath L is part of the boundary of the cell Z, all the limit points of L also belong to the boundary of the cell Z.

We will use the term a *singular arc* to designate a part of a boundary arc without contact where all the points, except the end points, belong to nonsingular paths, and each end point is either a corner point or belongs to a singular arc or a singular semipath (a singular arc may coincide, in particular, with a boundary arc without contact). A singular arc is called a *singular α- or ω-arc* depending on whether the paths of the system enter into G^* or leave G^* through this arc. Similarly, a limit cycle without contact with all its points belonging to nonsingular paths is called a singular α-cycle (or ω-cycle) depending on whether the paths of the system enter into $\bar{G}^*$ (or leave G^*) through this cycle.

IX. If a semipath (or an arc of a path) of cell Z crosses a singular ω-arc (α-arc) λ, all the semipaths (arcs of paths) of the cell Z cross the same arc λ at its inside points and do not cross any other ω-arc (α-arc) except λ.

If a semipath (or an arc of a path) of the cell Z crosses a singular ω-cycle (α-cycle), all the semipaths (or arcs of paths) of the cell Z cross this cycle.

X. A cell whose boundary contains an ω-arc or an α-arc is simply connected.

XI. A cell whose boundary contains a singular ω- or α-cycle is doubly connected.

Our problem is to identify all the various types of cells which are allowed for structurally stable systems. Since structurally stable systems have only a finite number of equilibrium states and closed paths and have no saddle-to-saddle separatrices (the equilibrium states being only simple nodes, saddle points, and foci), there are relatively few different types of cells in structurally stable systems. For example, structurally stable systems may not have cells filled with closed paths or loops.

2. Doubly connected cells of structurally stable systems

We will now proceed with a detailed analysis of the cells of structurally stable systems. In this subsection, we shall identify all the different types of doubly connected cells, which can be done very easily.

* If the point P belongs to the boundary of the cell Z and is a corner point which belongs to a boundary arc of a path, this arc will either belong entirely to the boundary of the cell Z or not belong altogether. This can be verified by simple examples.

First consider cells of structurally stable systems filled with whole paths. From proposition IV of §19.1 and from the properties of structurally stable systems we obtain the following theorem which is valid for both simply and doubly connected cells.

Theorem 27. If Z is a cell of a structurally stable system filled with whole paths, all these paths for $t \to +\infty$ go to the same sink (i.e., a stable node, focus, or limit cycle), and for $t \to -\infty$ they all go to the same source (an unstable node, focus, or limit cycle).

Theorem 27 evidently can be restated as follows: any cell of a structurally stable system filled with whole paths has one source and one sink.

Doubly connected cells filled with whole paths. Theorem 27 and propositions IV and V of §19.1 enable us to establish directly all the different types of doubly connected cells of a structurally stable system which are filled with whole paths.

Indeed, the boundary of such a cell consists of a single sink and a single source, i.e., either two limit cycle, or a limit cycle and an equilibrium state, or else two equilibrium states. The last alternative, however, has to to be rejected, since the boundary of a bounded cell cannot consist of two points. The boundary of a doubly connected cell thus consists either of two limit cycle or a limit cycle and an equilibrium state. Evidently, in either case, the exterior boundary continuum is necessarily a limit cycle, enclosing all the paths of the cell. The interior boundary continuum may be a limit cycle, a node, or a focus.

Before we can proceed with our classification of the different types of cells, we should decide on a criterion for inclusion of two cells in one type. Different criteria are available, and we will use the following definition.

*Definition 22. Cells Z_1 and Z_2 are said to be of the same type (or to belong to the same type)** if there exists an orientation-conserving, path-conserving topological mapping T of $\bar{Z}_1$ onto $\bar{Z}_2$ which does not reverse the direction of the paths in t. Otherwise, we shall say that the cells Z_1 and Z_2 are of different types.*

We are now in a position to describe all the different types of doubly connected cells of a structurally stable system which are filled with whole paths.

I. Cells bounded by a limit cycle and an equilibrium state. The boundary of each cell in this category consists of a limit cycle L_0 and an enclosed equilibrium state — a node or a focus. The cycle L_0 may be either stable or unstable, and the increase in t may correspond to motion in positive (counterclockwise) or negative sense along the cycle. Accordingly, there may be four different cell types in this category, which are shown in Figure 66.

* To structurally unstable systems, this theorem in general is inapplicable. For example, in a cell filled with loops, all the paths go to the same equilibrium state for $t \to -\infty$ and $t \to +\infty$, i.e., the source of this cell coincides with its sink.

** The condition stated in this definition could be replaced by a requirement of the existence of a mapping T with the relevant properties moving the cell Z_1 into Z_2 (i.e., without considering the cell closures). Alternatively, we could omit the requirement that T should be an orientation-conserving mapping or conserve the direction of motion in t along the paths. The choice of the particular criterion assigning two cells to the same type is largely arbitrary. Sometimes, the actual choice depends on the problem being considered.

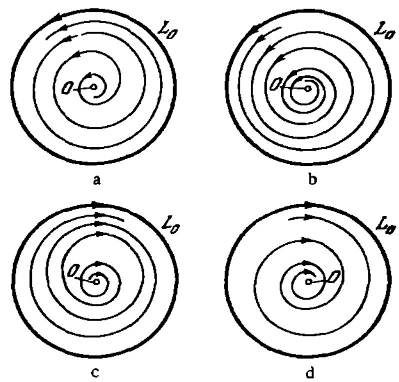

FIGURE 66. a) Stable cycle; b) unstable cycle; c) stable cycle; d) unstable cycle.

It is readily seen* that the four cell types are different in the sense of Definition 22, and that every doubly connected cell consisting of whole paths and bounded by a limit cycle and an equilibrium state corresponds to one of the types shown in Figure 66.

II. Cells bounded by two limit cycles. Let L_1 be the exterior limit cycle of the boundary and L_2 the interior cycle. As t increases, the point may move along each of the cycles L_1 and L_2 in positive or negative sense. Moreover, the cycle L_1 is either stable or unstable. All this leads to eight different possibilities, which are depicted in Figures 67 and 68.

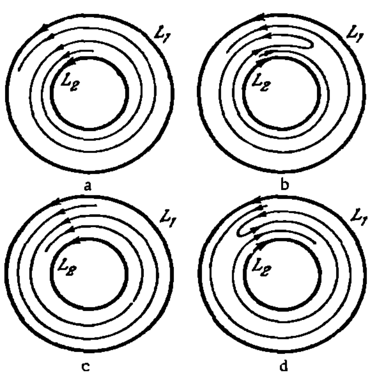

FIGURE 67. L_1 — limit cycle: a) and b) stable, c) and d) unstable.

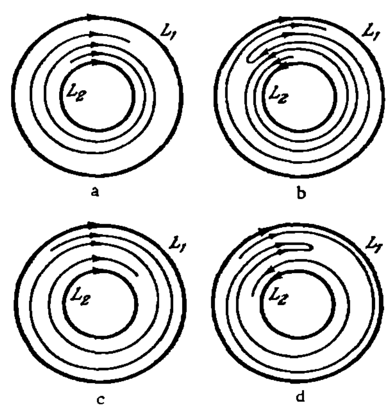

FIGURE 68. L_1 — limit cycle: a) and b) stable, c) and d) unstable.

* Using a scheme of a dynamic system, say (QT, §29).

It is readily seen that all the four cells shown in Figure 67 belong to different types. On the other hand, each cell in Figure 68 is of the same type as the corresponding cell in Figure 67.* There is therefore a total of four types of cells bounded by two limit cycles.

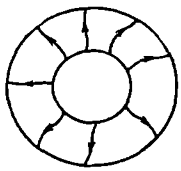

FIGURE 69

III. Doubly connected cells filled with semipaths or arcs of paths. The boundary of a cell filled with semipaths or arcs of paths evidently should comprise limit arcs or cycles without contact (see §19.1). By proposition X in §19.1, the boundary of a doubly connected cell cannot comprise singular arcs, and it therefore incorporates at least one singular cycle without contact. Hence it follows directly that the boundary of a doubly connected cell filled with semipaths consists of a singular cycle without contact and a source or a sink (a limit cycle, a node, or a focus). From these considerations, one can readily find the different types of these cells, and this assignment is left to the reader as an exercise. Doubly connected cells filled with arcs of paths are bounded by two singular cycles, one inside the other. Only one type of such cells is possible (Figure 69).

3. Interior cells of structurally stable systems. Simply connected interior cells

A cell Z in region $\bar{G}^*$ is said to be interior if $\bar{Z} \subset G^*$, i.e., the boundary of cell Z has no common points with the boundary of $\bar{G}^*$.

Lemma 1. Every interior cell consists of whole paths.

Proof. If cell Z consists of semipaths, their end points evidently lie on the boundary of the cell and therefore belong to certain singular elements. The only suitable singular elements are boundary arcs or cycles without contact. The boundary of a cell consisting of semipaths thus intersects the boundary of region G^*, i.e., this is not an interior cell. It is similarly proved that an interior cell cannot consist of arcs of paths.

Lemma 2. The boundary of an interior cell has no points which belong to corner semipaths or corner arcs of paths.

Proof. Let L be a corner semipath or an arc of a path, P a point of the semipath which belongs to the boundary of an interior cell Z. By proposition VII, §19.1, the end point M_0 of the semipath (or arc of path) L which is a point of the boundary of G^* also belongs to the boundary of cell Z, i.e., Z is notand interiorcell. This completes the proof.

From the definition of an interior cell and from Lemma 1, supported by proposition VI in §19.1, it follows that the boundary of an interior cell of a structurally stable system consists of whole paths, which are limit cycles, separatrices, or equilibrium states. Doubly connected cells were considered in the previous subsection, where we saw that their boundaries contain neither saddle points nor separatrices. We will now prove that the reverse situation applies to interior cells.

* The mapping T moving cell a in Figure 67 into cell a in Figure 68, while conserving the orientation, the paths and the sense of motion, clearly maps the exterior limit cycle of the cell into the interior limit cycle, and vice versa. This, however, does not contradict Definition 22.

Theorem 28. The boundary of a simply connected interior cell of a structurally stable system comprises a saddle point and at least two separatrices.

Proof. Let Z be an interior cell of a structurally stable system whose boundary contains neither saddle points nor separatrices. By Lemma 2, the boundary of cell Z then consists of nodes, foci, and limit cycles only, i.e., of sinks and sources. Since these elements are disjoint in pairs and at least two of these elements — namely the α- and ω-limit coninua of the paths of the cell Z — are included in the boundary of Z, Z cannot be a simply connected cell, which contradicts the original assumption. The boundary of Z thus must contain a saddle point or a separatrix. Now, if the boundary of a cell filled with whole paths contains the separatrix of some saddle point, it also contains the saddle point (§19.1, VIII). On the other hand, if the boundary of a cell contains a saddle point, it contains at least two separatrices of this saddle, which are continuation of each other. This completes the proof.

Let us now consider a regular system of canonical neighborhoods of a structurally stable system (A) in $\bar{G}^*$ (§17.1). Any α-separatrix (ω-separatrix) of a structurally stable system entirely contained in G^* invariably crosses a cycle without contact which belongs to a sink (a source).

In Chapter VI, §17.1, we introduced the concept of free and non-free cycles without contact, elementary α- and ω-arcs, simple and cyclic elementary arcs. Let us reiterate some of the propositions corresponding to these concepts.

1) Any nonsingular path crosses either precisely one α-cycle or precisely one α-arc (simple or cyclic) at an interior point, or it crosses either one ω-cycle or one ω-arc.

2) Nonsingular paths crossing an α-arc (ω-arc) cannot cross a free ω-cycle (α-cycle), and all these paths cross the same ω-arc (α-arc). If the α-arc and the ω-arc are such that a nonsingular path crossing one of the arcs inevitably crosses the other arc, the two arcs are said to be conjugate. Similarly, free α- and ω-cycles which are crossed by the same paths are called conjugate free cycles.

3) Any α-separatrix (ω-separatrix) of a saddle point of system (A) passes either through the common end point of two simple ω-arcs (α-arcs) or through the end point of a cyclic ω-arc (α-arc).

4) The paths of the same cell cross a single pair of conjugate α- and ω-arcs or a single pair of conjugate α- and ω-cycles (QT, Chapter XI, §27, Lemma 5).

We will now prove that no two cyclic arcs of a structurally stable system are conjugate. Note that this proposition does not apply to structurally unstable systems.

Lemma 3. Of two conjugate elementary arcs of a structurally stable system, at least one is a simple (not cyclic) arc.

Proof. Let a be a cyclic arc, M_0 its end point. To fix ideas, let a be a cyclic α-arc. The arc a and the point M_0 constitute a cycle without contact C which either belongs to a source or is a boundary cycle. Let L_0 be a path through M_0. According to the definition of a cyclic arc, L_0 is a singular path, i.e., it is either a separatrix, or a corner semipath, or a corner arc of a path.

First assume that L_0 is a separatrix of some saddle point O. It is clearly an ω-separatrix, i.e., it goes to O for $t \to +\infty$. Let L_1 and L_2 be α-separatrices of the saddle O (Figure 70). Let b be the ω-arc conjugate with a. All the paths passing through points of the arc a, other than its end point M_0, cross the arc b at its interior points as t increases.

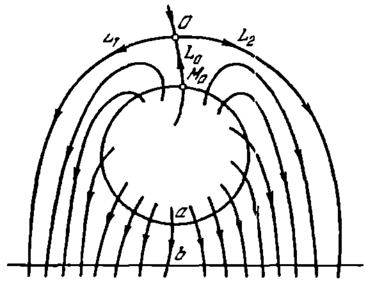

FIGURE 70

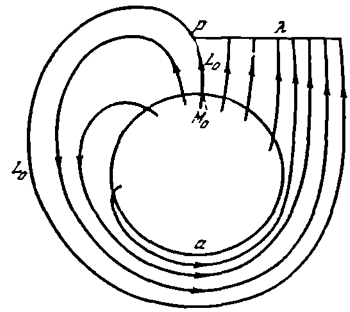

FIGURE 71

The separatrices L_1 and L_2 should pass through the end points of b. Indeed, if the separatrix L_1 does not pass through an end point of b, it passes through the common end point of two ω-arcs b_1 and b_2 or through the end point of a cyclic ω-arc b_3. But then paths passing through the points of the arc a near the point M_0 also cross the arcs b_1, b_2, or b_3 as t increases, i.e., they do not cross the arc b, contrary to the assumption that a and b are conjugate arcs.

The separatrices L_1 and L_2 thus must pass through the end points of b. Since these separatrices have no common points, the end points of b do not coincide, i.e., b is a simple arc.

Let now L_0 be a corner arc or a semipath. Let P be a corner point, which is the common end point of the semipath (or arc) L_0 and the boundary arc without contact λ (Figure 71). The paths through the points of the arc a sufficiently close to the point M_0, which lie on the same side from L_0 as the arc λ, will evidently cross the arc λ as t increases. It is thus readily seen that λ and a are conjugate arcs. Since λ is a boundary arc without contact, it is a simple ω-arc. This completes the proof of the lemma.

We will now consider the different types of simply connected interior cells of structurally stable systems.

Let Z be such a cell. By Theorem 28, the boundary of Z comprises a saddle point and its separatrix. Paths of the cell Z clearly do not intersect the free cycle C which belongs to a source or a sink. Indeed, if the paths of Z cross this cycle, all the points of the cycle belong to Z. Therefore, boundary points of the cell, and hence the boundary continua, lie both inside and outside the cycle C. These boundary continua should not have any common points, i.e., the cell Z is not simply connected, contrary to the original assumption.

Hence it follows that in virtue of property 4 the paths of a simply connected interior cell cross the single pair of conjugate α- and ω-arcs. By Lemma 3, at least one of these arcs is simple (not cyclic). Let the α-arc be simple. We will denote it by Q, and the cycle without contact incorporating this arc will be designated C_a. Clearly C_a belongs to a source (unstable node, focus, or limit cycle).

Let M_1 and M_2 be the end points of the simple arc a, and L_1 and L_2 the paths through M_1 and M_2. These paths enter the boundary of cell Z, i.e., by Lemma 2, they are separatrices. Suppose that the arc a lies on the positive side of the separatrix L_1 and therefore on the negative side of the separatrix L_2. All the paths crossing the cycle without contact C_a emerge from the canonical neighborhood of the corresponding source (to which C_a belongs) as t increases. The source may lie either inside the cycle C_a or outside the cycle. Let us consider the two possibilities separately.

A) *The source which belongs to cycle C_a lies inside C_a.* The source may be an unstable node, focus, or limit cycle. The paths crossing the cycle C_a, the separatrices L_1 and L_2 included, will emerge with increasing t from the canonical neighborhood enclosed by the cycle C_a. Let O_1 be a saddle point to which the separatrix L_1 goes for $t \to +\infty$ (see Figures 72, 73, 75, 77, 79, 80). This saddle point and its α-separatrix L_1', which is a continuation of the ω-separatrix L_1 in the positive direction, also enter the boundary of the cell Z (see proof of Theorem 28).

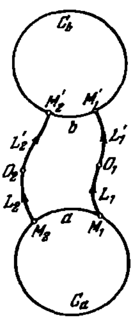

FIGURE 72

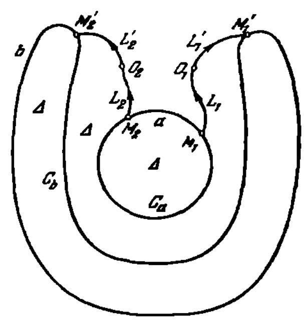

FIGURE 73

The separatrix L_2 through the end point M_2 of the arc a also goes to some saddle point O_2 for $t \to +\infty$. Two cases are possible:

A₁) O_1 and O_2 are different saddle points.

A₂) Saddle point O_2 coincides with O_1.

Let us consider the first of the two cases.

A₁) *O_1 and O_2 are different saddle points* (Figures 72 and 73). The arc a lies on the negative side of L_2, and the α-separatrix L_2', which is an ω-continuation of the separatrix L_2 in the negative direction, also enters the boundary of the cell Z. The separatrices L_1' and L_2' evidently pass through the end points M_1' and M_2' of the ω-arc b conjugate with the arc a.

Since M_1' and M_2' are two different points, b is a simple arc. Let C_b be a cycle without contact comprising the arc b. C_b belongs to the sink to which the separatrices L_1' and L_2' go for $t \to \mp \infty$.

Here again we should consider two alternatives:

A_{11}) The cycles without contact C_a and C_b do not enclose one another.

A_{12}) The cycle C_a lies inside the cycle C_b.

Consider case A_{11} first.

A_{11}) *The cycles without contact* C_a *and* C_b *do not enclose one another.* Let γ be a simple closed curve consisting of the arcs a and b, the segments M_2O_2 and M_1O_1 of the separatrices L_2 and L_1, respectively, the segments O_2M_2' and O_1M_1' of the separatrices L_2' and L_1', and the equilibrium states O_1 and O_2. Let Δ be the simply connected region bounded by the curve γ. The points of the curve γ which are the interior points of the arcs a and b belong to the cell Z, and all the other points of γ are boundary points of Z. All the paths of the cell cross the arc a and emerge from the cycle C_a with increasing t. These paths should either enter the region Δ or leave it. We will show that they necessarily enter this region.

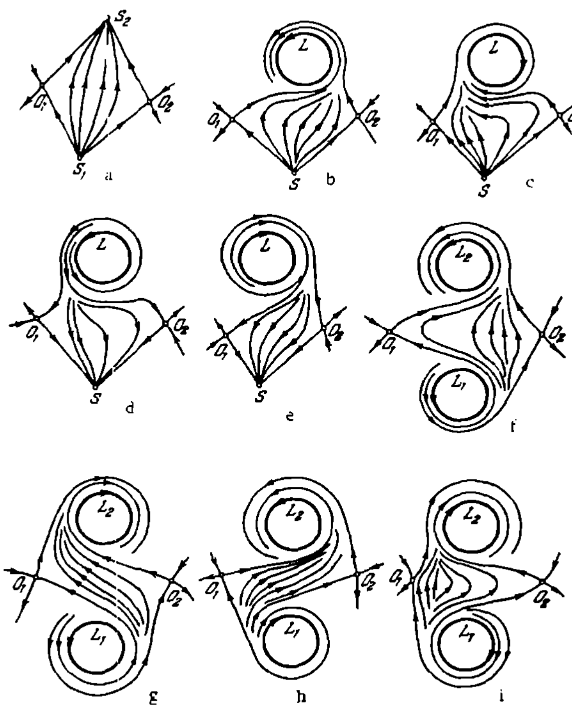

FIGURE 74. a) S_1 unstable node, S_2 stable node; b) S unstable node; c) S unstable node; d) S stable node; e) S stable node. In cases f, g, h, i, L_1 is an unstable cycle.

Indeed, let the paths of the cell Z which cross the arc a leave the region Δ as t increases (Figure 73). Then there are points of the cell Z which lie outside the curve γ, and therefore also boundary points of Z with this property. Let E be the set of these boundary points (which lie outside γ). The set E consists of the points of singular paths. It is readily seen that the regions enclosed by the limit cycles C_a and C_b lie inside the curve. Therefore, the separatrices L_1, L_1', L_2, and L_2' have no points which lie outside γ, i.e., E is the set of points of singular paths, other than the separatrices L_1, L_1', L_2, L_2' and clearly other than the equilibrium states O_1 and O_2. But all the points which are sufficiently close to the curve γ from the outside belong to nonsingular paths. The set E is therefore at a positive distance outside the curve γ. This means that E does not intersect with the boundary continuum which contains the separatrices L_1, L_2, L_1', L_2' and their limit points, i.e., the cell Z has at least two boundary continua. This evidently contradicts the assumption that Z is simply connected.

We have thus established that in the case A_{11} all the paths of the cell Z crossing the arc a leave the cycle C_a and enter into the region Δ as t increases (Figure 72). The cell will be of one of the types shown in Figure 74 according as the source or respectively the sink to which the paths of the cell go is an equilibrium state or a limit cycle, and in the latter case, according as the direction of motion along the limit cycle is positive (clockwise) or negative. In case A_{11} the boundary of the cell Z evidently consists of the separatrices L_1, L_1', L_2, and L_2' and their α- and ω-limit points.

A_{12}) C y c l e C_a l i e s i n s i d e c y c l e C_b. In this case, the cycle without contact C_b clearly belongs to some stable limit cycle L_0, and C_b lies inside L_0 (Figure 75). Considering, as before, a simple closed curve γ and the region Δ enclosed by this curve, we readily see that Δ is part of the ring region enclosed between the cycles C_a and C_b. The paths of the cell Z cross the arc a and with increasing t enter into Δ. As t further increases, these paths cross the arc b, enter into a canonical neighborhood of the cycle L_0 and go to L_0. The separatrices L_1' and L_2', with increasing t, also go to L_0. The limit cycle L_0 is therefore part of the boundary of the cell Z. As before, this boundary consists of the separatrices L_1, L_2, L_1', and L_2' and their α- and ω-limit points. The cell Z belongs to one of the types depicted in Figure 76. The exact type depends on the character of the source to which C_a belongs and on the direction of motion along this source (if it is a cycle) or along the cycle L_0.

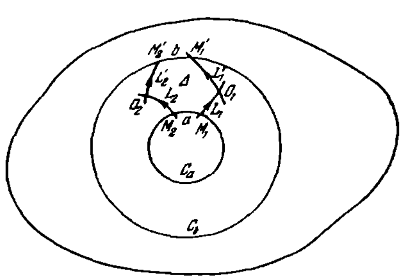

FIGURE 75

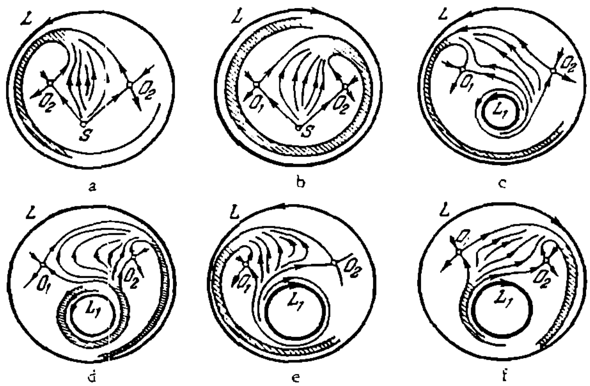

FIGURE 76

Let us now consider case A_2.

A_2) T h e s a d d l e p o i n t O_2 c o i n c i d e s w i t h O_1 (Figures 77, 79, 80). In this case, the separatrix L_1', which is an ω-continuation of the separatrix L_1 in the positive direction, is also an ω-continuation of the separatrix L_2 in the negative direction and enters the boundary of the cell Z. Let b be an arc without contact conjugate with the arc a, and C_b a cycle without contact comprising the arc b. The separatrix L_1' passes through the end point M_1' of the arc b. It is readily seen that paths crossing the arc a pass through the points of the cycle C_b sufficiently near the point M_1', on either side of this point. Therefore b is a cyclic arc.

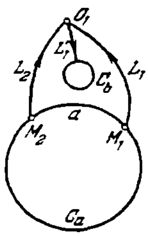

FIGURE 77

Let γ be a simple closed curve consisting of the segments M_1O_1 and M_2O_1 of the separatrices L_1 and L_2, respectively, of the arc a, and the point O_1; let Δ be the region enclosed by the curve γ. There are two possibilities:

A_{21}) T h e p a t h s o f t h e c e l l Z c r o s s t h e a r c a a n d e n t e r i n t o Δ a s t i n c r e a s e s (Figure 77). In this case, the separatrix L_1', and also the cycle without contact C_b, lie inside the cycle γ, and the sink to which C_b belongs lies inside C_b. The cycles C_a and C_b do not enclose one another. The boundary of the cell Z consists of the separatrices L_1. L_2. L_1' and their α- and ω-limit points. The cell Z belongs to one of the types shown in Figure 78. As in the previous cases, the exact type of the cell depends on the character of the sources and sinks to which the paths of the cell go and on the direction of motion along the source (or the sink), assuming that it is a limit cycle.

A_{22}) T h e p a t h s o f t h e c e l l Z c r o s s t h e a r c a a n d l e a v e Δ a s t i n c r e a s e s (Figures 79 and 80). In this case, either the cycles C_a and C_b do not enclose one another (Figure 79) or cycle C_a lies inside C_b (Figure 80).

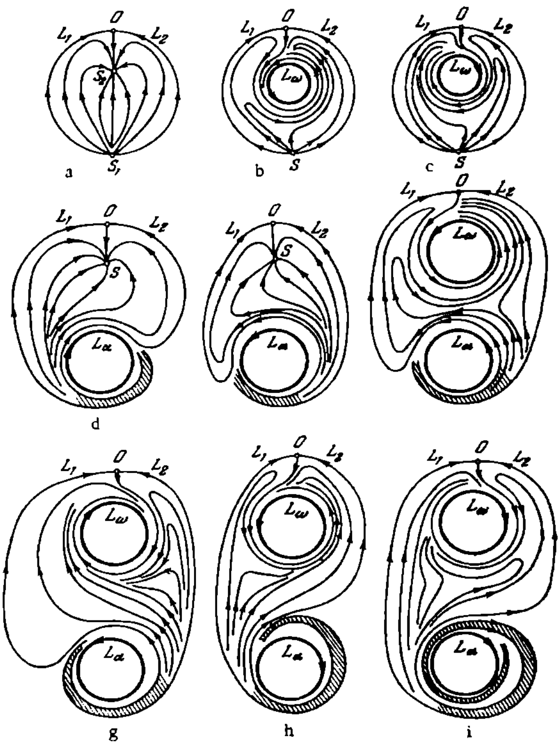

FIGURE 78

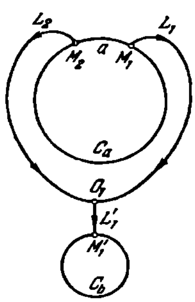

FIGURE 79

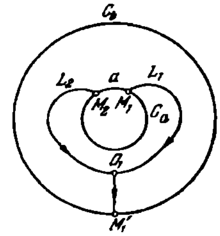

FIGURE 80

However, reasoning along the same lines as for case A_{11}, we can show that the case depicted in Figure 79 is inapplicable. Thus, cycle C_a lies inside C_b. Hence it follows that the source to which C_b belongs is a limit

cycle L_0, and C_b lies inside L_0. The boundary of the cell Z consists of the separatrices L_1, L_2, L_1' and their limit points. The cell Z may belong to one of the types shown in Figure 81.

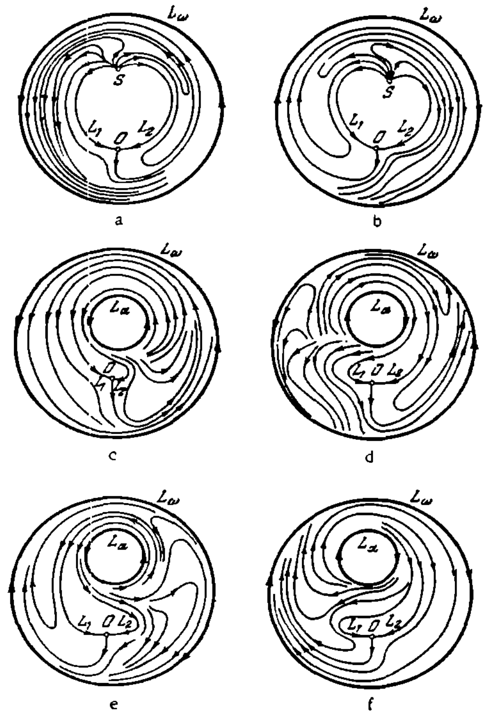

FIGURE 81

We have fully covered case A, when the source belonging to the cycle C_a lies inside C_a.

B) The source which belongs to cycle C_a lies outside C_a.

An unstable limit cycle L_0, with C_a lying inside L_0, is evidently a source of this kind. As in case A, the separatrices L_1 and L_2 through the end points M_1 and M_2 of the arc a may

B1) go to different saddle points O_1 and O_2 (Figure 82);

B2) go to the same saddle point O_1 (Figure 83).

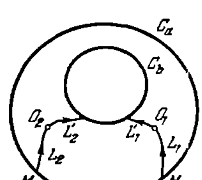

FIGURE 82

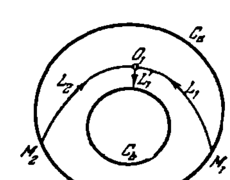

FIGURE 83

It is readily seen that case B_1 is analogous to case A_{12}, differing from the latter only in the direction of arrows along the paths, and case B_2 is analogous to case A_{22}.* Correspondingly, there is no need to analyze these cases in particular detail. The types of cells corresponding to case B_2 are shown in Figure 84. These cells are analogous to those shown in Figure 81.

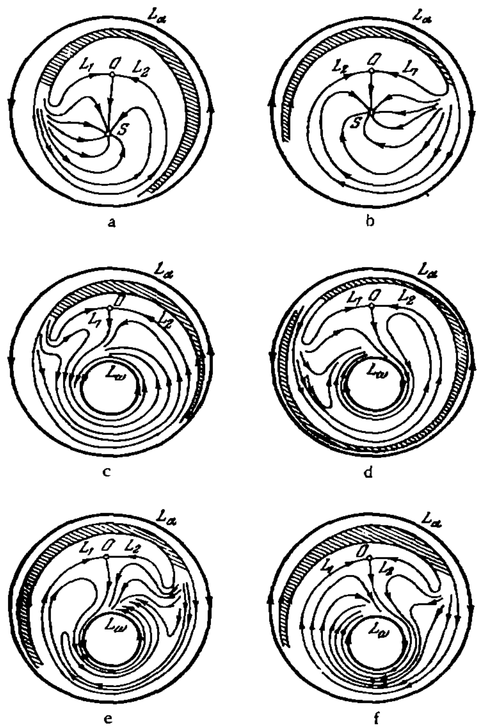

FIGURE 84

We have so far assumed that the arc a is a simple arc, and its conjugate arc b is either simple or cyclic. The cases when b is a simple arc and its conjugate arc a is cyclic are obtained from A_2 and B_2 by reversing the direction of the arrows along the paths.

We can now count the total number of different — in the sense of Definition 22 — simply connected interior cells which may exist in structurally stable systems. We will not do this, however. It should only be stressed that the above analysis does not show that each of the different cell types described actually exists for one of the structurally stable systems. We have only established that structurally stable systems

* In case A_{22}, the source lies inside the sink, and in case B_2 the sink lies inside the source.

can have no other types of simply connected interior cells in addition to those described. It can be proved without difficulty that all the cell types described do exist, however. This can be done, say, by constructing a dynamic system corresponding to each cell type in terms of its vector field.

We have established all the different types of doubly connected cells (§19.2), and all the types of interior simply connected cells which may occur in structurally stable systems. It remains to consider simply connected cells which are not internal cells, i.e., cells whose boundaries have points in common with the boundary of the region. We will not analyze these cells, as this can be done along the same lines as before. Figure 85 depicts the different types of simply connected cells of a structurally stable system which touch the boundary of $\bar{G}^*$ for the case when this boundary consists of a single cycle without contact.

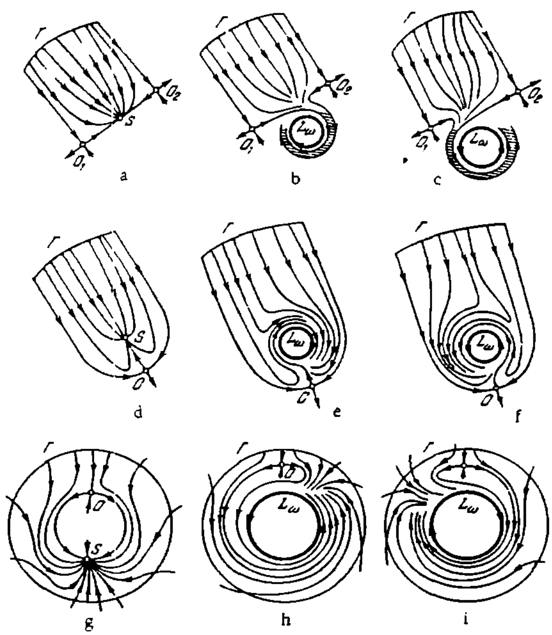

FIGURE 85

The concept of the *region of stability in the large* of a given sink is of considerable importance in various applied problems. This region is defined as the set of all the cells for which the given singular element is the sink. Examples of regions of stability in the large will be found in the next section.

§20. EXAMPLES OF STRUCTURALLY STABLE SYSTEMS

In this section we will consider some examples of structurally stable systems.

Example 4. Consider the system

$$\begin{aligned}\dot{x} &= ax + by - x(x^2+y^2) = P(x, y),\\ \dot{y} &= cx + dy - y(x^2+y^2) = Q(x, y),\end{aligned} \tag{1}$$

assuming that

$$b \neq 0, \tag{2}$$

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} < 0 \tag{3}$$

and

$$D = \sigma^2 - 4\Delta = (a-d)^2 + 4bc > 0 \qquad (\sigma = a + d). \tag{4}$$

Since for system (1)

$$P(-x, -y) = -P(x, y) \text{ and } Q(-x, -y) = -Q(x, y),$$

then together with any solution $x = \varphi(t)$, $y = \psi(t)$ of (1), $x = -\varphi(t)$, $y = -\psi(t)$ is also a solution of (1). Geometrically this signifies that *the reflection of any path of system (1) at the origin is also a path* or, alternatively, *the phase portrait of the dynamic system (1) is symmetrical about the origin.*

Let us consider the system (1) inside the circle

$$x^2 + y^2 \leqslant R^2, \tag{5}$$

with the boundary

$$x^2 + y^2 = R^2. \tag{6}$$

The path

$$x = \varphi(t), \qquad y = \psi(t) \tag{7}$$

of system (1) is tangent to the circle (6) at a point (x, y) when

$$\begin{vmatrix} \dot{x} & \dot{y} \\ -y & x \end{vmatrix} = x\dot{x} + y\dot{y} = 0.$$

From equations (1), we have for sufficiently large R

$$x\dot{x} + y\dot{y} = ax^2 + (b+c)xy + dy^2 - (x^2+y^2)^2 < 0. \tag{8}$$

The last relation shows that for sufficiently large R, circle (6) is a cycle without contact for paths of system (1). Moreover, since

$$x\dot{x}+y\dot{y}=\frac{1}{2}\frac{d}{dt}(x^2+y^2),$$

it follows from (8) that, as t increases, x^2+y^2 diminishes, i.e., all the paths crossing the circle (6) enter into the cycle without contact (6) as t increases. We will take R to be so large that the last condition is satisfied.

It follows from (2), as is readily seen, that the paths through the points of the axis $x=0$ cross from one side of the axis to the other side (with the exception of the path which is the equilibrium state $O\,(0,0)$).

Let $P(x, y)\neq 0$. We then change over from system (1) to a single equation

$$\frac{dy}{dx}=\frac{cx+dy-y(x^2+y^2)}{ax+by-x(x^2+y^2)}\,. \tag{9}$$

Let us determine for what k the straight line

$$y=kx \tag{10}$$

or part of this line is an integral curve of equation (9). From (9) and (10) we have

$$k=\frac{c+dk-kx^2(1+k^2)}{a+bk-x^2(1+k^2)}$$

or

$$bk^2+(a-d)\,k-c=0. \tag{11}$$

Hence

$$k=\frac{d-a\pm\sqrt{(d-a)^2+4bc}}{2b}\,. \tag{12}$$

From (2) and (4) it follows that there exist precisely two values of k satisfying equality (12). Let these two roots be k_1 and k_2, where k_1 corresponds to the plus in (12) and k_2 to the minus.

Let us now find the states of equilibrium. One of these is the point $O\,(0, 0)$. There are obviously no other equilibrium states on the vertical axis. The coordinates of any equilibrium state other than the point $O\,(0, 0)$ therefore have the form (x_0, kx_0). Inserting these coordinates in the equations $P(x, y)=0$, $Q(x, y)=0$, we find that k satisfies equation (11), i.e., every equilibrium state of system (1), other than the point O, lies on one of the straight lines $y=k_1x$ and $y=k_2x$, where k_1 and k_2 are defined by (12).

From the equation $P(x_0, kx_0)=0$ we now find that

$$x_0=\pm\sqrt{\frac{a+bk}{1+k^2}} \tag{13}$$

so that

$$y_0=kx_0=\pm k\sqrt{\frac{a+bk}{1+k^2}}\,. \tag{14}$$

The characteristic equation of the equilibrium state $O(0, 0)$ is

$$\lambda^2 - \sigma\lambda + \Delta = 0. \tag{15}$$

It follows from (3) that $O(0, 0)$ is a simple saddle point. Let λ_1 and λ_2 be the roots of equation (15). Direct calculations show that

$$a + bk_1 = \lambda_1, \quad a + bk_2 = \lambda_2, \tag{16}$$

and, by (3), $\lambda_1 > 0$, $\lambda_2 < 0$. From (13) and (16) it follows that the straight line $y = k_2x$ has no equilibrium states (except the point O) and therefore the rays $y = k_2x$ $(x > 0)$ and $y = k_2x$ $(x < 0)$ are paths of the system, i.e., separatrices of the saddle point O. The straight line $y = k_1x$, on the other hand, contains precisely two equilibrium states in addition to the origin, $O_1(x_0, y_0)$ and $O_2(-x_0, -y_0)$, where x_0 and y_0 are obtained from (13) and (14) for $k = k_1$. The equilibrium states O, O_1, and O_2 partition the line $y = k_1x$ into four parts, each of which is a path of the system. The segments OO_1 and OO_2 of the line $y = k_1x$ are separatrices of the saddle point O.

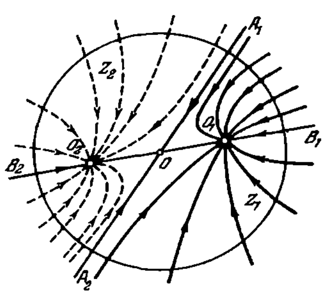

FIGURE 86

Fairly simple calculations show that $\Delta(x_0, y_0) = \Delta(-x_0, -y_0) = 2\lambda_1\sqrt{D} > 0$.

Therefore the points O_1 and O_2 are simple nodes (these cannot be foci, since they are limit points of paths which are segments of the line $y = k_1x$).

System (1) has no closed paths. Indeed, a closed path should enclose at least one of the equilibrium states O, O_1, O_2 and it should therefore cross the line $y = k_1x$, which is unfeasible, since this line is made up of a number of paths of the system. System (1) thus has no closed paths and no saddle-to-saddle separatrices; it has three equilibrium states — a simple saddle and two simple nodes. Therefore by Theorem 23, system (1) is structurally stable in the circle (5). The configuration of the paths is shown in Figure 86. The direction of the arrows along the paths is chosen so that all the paths enter into the circle (5) as t increases.

The circle (5) contains two cells of system (1), Z_1 and Z_2. Both cells are filled with semipaths and are simply connected. The boundary of the cell Z_1 (Z_2) is made up of the separatrices OA_1, OA_2, and OO_1 (OO_2), equilibrium states O and O_1 (O_2), and a simple arc without contact $A_1B_1A_2$ $(A_1B_2A_2)$. The conjugate arcs of $A_1B_1A_2$ and $A_1B_2A_2$ are clearly cyclic. The cell Z_1 (Z_2) is the region of stability in the large of the stable node O_1 (O_2).

Example 5. Consider the system (1) as before

$$\dot{x} = ax + by - x(x^2 + y^2), \quad \dot{y} = cx + dy - y(x^2 + y^2),$$

assuming that

$$\sigma = a + d > 0, \tag{17}$$

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0, \tag{18}$$

$$D = \sigma^2 - 4\Delta > 0. \tag{19}$$

As in the previous example, circle (6) of a sufficiently large radius R is a cycle without contact and all the paths of our system enter into it as t increases.

Consider the system inside circle (5) of sufficiently large radius. The lines $y = k_1x$ and $y = k_2x$, where k_1 and k_2 are the roots of (11), are made up of paths of our system.

The characteristic roots λ_1 and λ_2 of the equilibrium state $O\,(0, 0)$ are positive and different in virtue of conditions (17), (18), (19), so that O is an unstable node. From (13), (14), (16) and from the fact that both λ_1 and λ_2 are positive we conclude that each of the straight lines $y = k_1x$, $y = k_2x$ contains, besides the point O, two other equilibrium states. Let these equilibrium states be A_1, B_1 and A_2, B_2, respectively (Figure 87). Calculations show that for the equilibrium states A_1 and B_1 lying on the straight line $y = k_1x$, the determinant $\Delta = 2\lambda_1\sqrt{D} > 0$, and for the equilibrium states A_2 and B_2, $\Delta = -2\lambda_2\sqrt{D} < 0$. Thus, A_1 and B_1 are simple nodes, and A_2 and B_2 are simple saddle points. The system has no closed paths for the same reasons as before, and its phase portrait is symmetrical relative to the origin O. The configuration of the paths is shown in Figure 87. The direction of the arrows along the paths and the direction of the separatrices are determined so as to ensure that all the paths enter into the boundary cycle without contact as t increases.

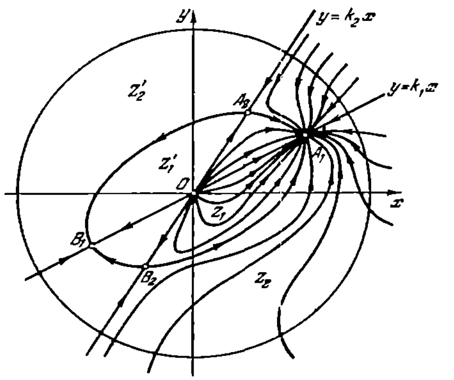

FIGURE 87

By Theorem 23, the system is structurally stable inside the circle (5). This circle consists of four cells: Z_1, Z_2 and the two cells Z_1', Z_2' in symmetrical position relative to the origin. Z_1 is an interior cell. It consists of whole paths extending from the unstable node O to the stable node A_1, and belongs to the same type as cell a in Figure 74. Cell Z_2 consists of semipaths and belongs to the same type as cell a in Figure 85. The stability region of the node A_1 comprises the cells Z_1 and Z_2, and that of the node B_1 the cells Z_1' and Z_2'.

R e m a r k . System (1) may be investigated by analogous methods for any other combination of the coefficients a, b, c, d. See QT, §30, Example 18.

E x a m p l e 6. The system

$$\begin{aligned}\frac{dx}{dt} &= 2y + xy + x^2 + y^2 - 1 = P(x, y),\\ \frac{dy}{dt} &= -2x - x^2 = Q(x, y)\end{aligned} \tag{20}$$

has two equilibrium states, $A(0, -1+\sqrt{2})$ and $B(0, -1-\sqrt{2})$. Here, A is an unstable focus and B is a saddle point. It is readily verified that

the circle

$$x^2+y^2-1=0 \tag{21}$$

is a path of the system (see QT, §30, Example 14). The focus A lies inside this circle, and the saddle point B outside the circle.

We will now show that the circle (21) is a structurally stable limit cycle of system (20). To prove this, we have to evaluate the integral

$$J=\int_0^{\tau}[P'_x(\varphi(t),\psi(t))+Q'_x(\varphi(t),\psi(t))]\,dt, \tag{22}$$

where

$$x=\varphi(t),\quad y=\psi(t)$$

is a solution of the system corresponding to path (21), and τ is the period of the functions φ and ψ ($\tau>0$). By (34), §13

$$J=\int_{(L_0)}\frac{P'_x(x,y)+Q'_y(x,y)}{\sqrt{P(x,y)^2+Q(x,y)^2}}\,ds, \tag{23}$$

where L_0 designates the circle (21). The last integral is readily evaluated by expressing the circle L_0 in parametric form $x=\cos t$, $y=\sin t$ and inserting for P and Q their expressions from (20). As a result, we find $J=\int_0^{2\pi}\frac{\sin t+2\cos t}{2-\cos t}\,dt=\frac{4}{3}\pi(3-2\sqrt{3})<0$. Hence it follows that circle (21) is a stable structurally stable limit cycle for our system.

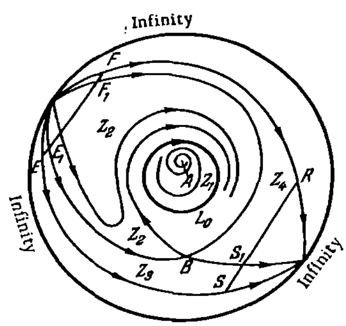

FIGURE 88

The topological structure of system (20) in a plane was investigated in QT, §30, Example 14.* It was established there that system (20) has no other closed paths, except the circle (21), and has no saddle-to-saddle separatrices. The configuration of the paths of system (20) is shown in Figure 88.

Let $\bar{G}^*$ be a region with a normal boundary bounded by the arcs without contact EF and RS and the arcs of paths FR and ES (Figure 88), which encloses the equilibrium states A and B of the system and its limit cycle (21). By Theorem 23, system (20) is structurally stable in $\bar{G}^*$. $\bar{G}^*$ contains four cells. Z_1 is an internal doubly connected cell consisting of whole paths which unwind from the focus A and wind onto the limit cycle L_0. The other three cells are simply connected. The cell Z_2 consists of semipaths and is bounded by an arc without contact E_1F_1, three separatrices

* This investigation analyzes the behavior of the paths of system (20) at infinity and considers the paths of an auxiliary system

$$\frac{dx}{dt}=2y+x^2+y^2-1,\quad \frac{dy}{dt}=-2x,$$

which have a general integral $(x^2+y^2-1)e^y=C$. The closed paths of the auxiliary system form a topographic system of curves for system (20).

of the saddle point B, limit cycle L_0, and saddle point B. The other two cells Z_3 and Z_4 consist of arcs of paths. The conjugate of the arc without contact E_1F_1 entering the boundary of the cell Z_2 is clearly a cyclic arc without contact. The stability region of the cycle L_0 comprises the cells Z_1 and Z_2.

§21. A DEFINITION OF STRUCTURAL STABILITY FOREGOING THE REQUIREMENT OF ε-IDENTITY

We will consider in this section dynamic systems structurally stable in W which are bounded by a cycle without contact Γ. We have seen in Chapter VI (§18.4, a) that structural stability of system (A) in such a region can be defined as follows:

I. System (A) is structurally stable in region W bounded by a cycle without contact Γ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any system $(\tilde{A})$ δ-close to (A) the following relation is satisfied:

$$(\overline{W}, \tilde{A}) \overset{\varepsilon}{\equiv} (\overline{W}, A).$$

Definition I not only requires that systems sufficiently close to the structurally stable system (A) have the same topological structure as (A) in $\overline{W}$, but also imposes a further condition, namely that the paths of any such system can be moved into the paths of system (A) by an arbitrarily small translation. All these factors combine into a fairly complex definition of structural instability of a system. And yet, according to the most natural and straightforward definition, a system is structurally unstable if its topological structure can be altered by infinitesimally small increments. We thus arrive at a better definition of a structurally stable system:

II. System (A) is structurally stable in region $\overline{W}$ bounded by a cycle without contact Γ if there exists $\delta > 0$ such that any dynamic system $(\tilde{A})$ δ-close to (A) has the same topological structure as (A) in $\overline{W}$.*

Definition II is simpler than Definition I, since it imposes fewer restrictions on structurally stable systems. It is clear that if system (A) is structurally stable in the sense of I it is also structurally stable in the sense of II. The reverse is not immediately obvious: it would seem that there might be dynamic systems which would be structurally stable in the sense of II and structurally unstable in the sense of I. In fact, however, this is not so, and the two definitions are equivalent. This fact was proved by Peixoto in /14/, and the present section is devoted to Peixoto's proof.

To fix ideas, we will consider structural stability in relation to the space $R_a^{(1)}$, i.e., the space of analytical functions with the distance defined in terms of the first derivatives only.

In case of structural stability relative to the spaces $R_a^{(r)}$, $R_N^{(r)}$, the proof is completely analogous (or even simpler).

* We should specify in Definitions I and II in relation to what space, $R_a^{(r)}$ or $R_N^{(r)}$, the structural stability is being considered. However, by virtue of remark d in §18.4, this is immaterial for Definition I. We will see in what follows that it is immaterial for Definition II, either.

Conditions I, II, III of Theorem 23 (§18.2) will be designated CSS (conditions of structural stability). These conditions postulate

1) existence of only a finite number of equilibrium states of system (**A**) in region W, these equilibrium states being simple nodes, saddle points, or foci;

2) absence of saddle-to-saddle separatrices;

3) absence of closed paths with characteristic index equal to zero.

To prove the equivalence of Definitions I and II, we will establish that the CSS follow from Definition II. We will require a number of lemmas in the process.

In Lemmas 1 throught 8 it is assumed that system (A) with the right-hand sides $P(x, y)$ and $Q(x, y)$ is an analytical dynamic system structurally stable in $\overline{W}$ in the sense of Definition II, and δ is the number mentioned in this definition. Analytical dynamic systems δ-close to system (A) will be called feasible systems. By Definition II, feasible systems have the same topological structure in $\overline{W}$ as the system (A). We will assume that all the paths of system (A) which cross the cycle without contact Γ enter into W as t increases.

Lemma 1. Structurally stable systems in the sense of Definition II form an open set in the space of all (analytical) dynamic systems.

The validity of Lemma 1 follows directly from Definition II, since by this definition feasible systems are structurally stable.*

Lemma 2. System (A) ***has only a finite number of equilibrium states, which are all simple.***

Proof. By Weierstrass's theorem, there exists a feasible system $(\tilde{\mathbf{A}})$ whose right-hand sides are irreducible polynomials (see proof of Theorem 10, §7.2). From the Bézout theorem it follows that $(\tilde{\mathbf{A}})$ only has a finite number of equilibrium states. But then (A) may also have only a finite number of equilibrium states. The first proposition of the lemma is thus proved.

Let us now prove that all the equilibrium states are simple. Suppose that one of the equilibrium states of (A) is multiple. Let this be the point $O(0, 0)$, i.e., $P(0, 0) = Q(0, 0) = 0$ and $\Delta(0, 0) = 0$.

Let $\varphi(x, y)$ be a polynomial which is equal to 1 at the point $O(0, 0)$ and vanishes for all the other equilibrium states of (A). Consider the system

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) + \varepsilon a x \varphi(x, y) = \tilde{P}(x, y), \\ \frac{dy}{dt} &= Q(x, y) + \varepsilon b y \varphi(x, y) = \tilde{Q}(x, y), \end{aligned} \tag{1}$$

where a and b are real numbers smaller than 1 in absolute value, and $\varepsilon > 0$ is so small that system (1) is feasible. All the equilibrium states of (A) are clearly at the same time equilibrium states of system (1). Since (1) is a feasible system, the number of its equilibrium states in $\overline{W}$ is the same as that of (A). This means that the two systems (A) and (1) have the same equilibrium states in $\overline{W}$.

Let C be a circle centered at $O(0, 0)$ which lies in W so that all the equilibrium states of system (**A**), except the point O, fall outside this

* Similar propositions for structurally stable systems in the sense of Definition I were advanced in §18.4, c, Theorems 25 and 26. Their proof, however, far from following directly from Definition I, conversely requires a preliminary derivation of the conditions of structural stability.

circle. We choose $\varepsilon > 0$ to be so small that for any two systems of the form (1), the vectors of the corresponding fields do not point in opposite directions at any point of the circle C. Then by QT, §10.2, Lemma 2, the rotations of the vector fields of all these systems on the circle C are equal to one another. Since C encloses a single equilibrium state O of (A), and hence of any system (1), the Poincare index of the equilibrium state O is constant for all the systems (1).

On the other hand, it follows from equations (1) that

$$\tilde{\Delta}(0, 0) = \left[\frac{D(\tilde{P}, \tilde{Q})}{D(x, y)}\right]_{\substack{x=0 \\ y=0}} = \varepsilon\,[aP'_x(0, 0) + bQ'_y(0, 0) + \varepsilon ab].$$

An appropriate choice of the numbers a and b will make $\tilde{\Delta}(0, 0)$ either positive or negative. By QT, §11.4, Theorem 30, in the former case the Poincare index of the equilibrium state O is $+1$, and in the latter case it is -1. This, however, contradicts the previous result. The proof is thus complete.

Lemma 3. None of the equilibrium states of system (A) *is a center.*

Proof. $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is the divergence of the vector field (P, Q), designated $\operatorname{div}(P, Q)$. We will use this notation in the following.

Let $O_i(a_i, b_i)$, $i = 1, 2, \ldots, n$, be all the equilibrium states of system (A). Consider the linear functions

$$\varphi_j(x, y) = m(x - a_j) + p(y - b_j)$$

$(j = 1, 2, \ldots, n)$, where the numbers m and p are chosen so that

$$\operatorname{div}(\varphi_j, \varphi_j) = m + p \neq 0$$

and for $j \neq k$, $\varphi_j(a_k, b_k) \neq 0$. Let $\varphi(x, y) = \varphi_1\varphi_2 \ldots \varphi_n$.

Clearly, $\varphi(x, y) = 0$ at any of the points O_i. Now from the equality

$$\operatorname{div}(\varphi, \varphi) = \sum_{j=1}^{n} \varphi_1 \ldots \varphi_{j-1} \operatorname{div}(\varphi_j, \varphi_j)\, \varphi_{j+1} \ldots \varphi_n$$

and from the previous conditions it follows that $\operatorname{div}(\varphi, \varphi)$ does not vanish at any of the equilibrium states of system (A).

Consider the system

$$\frac{dx}{dt} = P(x, y) + \varepsilon\varphi(x, y) = P^*, \quad \frac{dy}{dt} = Q(x, y) + \varepsilon\varphi(x, y) = Q^*, \qquad (\mathrm{A}^*)$$

where $\varepsilon \neq 0$ is so small that system (A*) is feasible. The set of the equilibrium states of (A*) evidently coincides with the set of the equilibrium states of (A), i.e., it consists of the points O_j. Since

$$\operatorname{div}(P^*, Q^*) = \operatorname{div}(P, Q) + \varepsilon \operatorname{div}(\varphi, \varphi),$$

for sufficiently small ε, $\operatorname{div}(P^*, Q^*) \neq 0$ at those of the points O_j where $\operatorname{div}(P, Q) \neq 0$. At those of the points O_j where $\operatorname{div}(P, Q) = 0$, we have

$$\operatorname{div}(P^*, Q^*) = \varepsilon \operatorname{div}(\varphi, \varphi) \neq 0.$$

Thus, div $(P^*, Q^*) \neq 0$ for all O_j, i.e., none of the equilibrium states of system (A^*) is a center. Therefore, system (A) has no centers either, and the proof of the lemma is complete.

R e m a r k . The above proof remains in force for systems of class 1, too. The fact that structurally stable systems of class 1 cannot have center-foci follows directly from the existence of feasible analytical systems.

L e m m a 4. System (A) *has no saddle-to-saddle separatrices.*

P r o o f . Let η, $0<\eta<\pi$, be so small that for any $\lambda \in I$, where I is the segment $0 \leqslant \lambda \leqslant 1$, the system

$$\begin{aligned} \frac{dx}{dt} &= P\cos(\lambda\eta) - Q\sin(\lambda\eta), \\ \frac{dy}{dt} &= P\sin(\lambda\eta) + Q\cos(\lambda\eta) \end{aligned} \qquad (A_\lambda)$$

is feasible. The vector field of the system (A_λ) is clearly obtained from the vector field of (A) by rotation through the angle $\lambda\eta$ in the positive sense. The equilibrium states of (A_λ) and (A) coincide. Finally, the Jacobians Δ of (A_λ) and (A) are equal at every point, as is readily seen. Therefore the saddle points of (A_λ) and (A) coincide. Let these saddle points be O_i $(i = 1, 2, \ldots, m)$; the separatrices of each saddle point O_i will be denoted L_{i1}, L_{i2}, L_{i3}, L_{i4}.

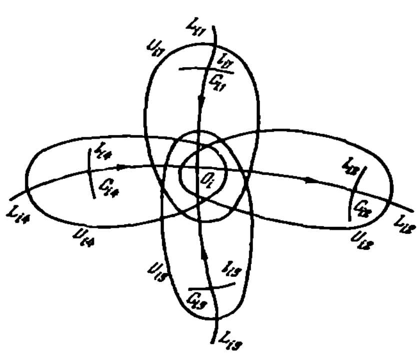

FIGURE 89

On each separatrix L_{ij} $(j = 1, 2, 3, 4;$ $i = 1, 2, \ldots, m)$ we choose a point C_{ij} which is sufficiently close to the corresponding equilibrium state O_i, so that no two segments O_iC_{ij} of the separatrices intersect. Let U_{ij} be the neighborhoods of these segments satisfying the following conditions: each neighborhood U_{ij} contains only one equilibrium state of system (A), namely O_i, and only one of the points C, namely C_{ij} (Figure 89). If the points C_{ij} are sufficiently close to the respective O_i, such neighborhoods clearly exist.

Through each point C_{ij}, we now pass a segment without contact l_{ij} (e.g., a segment of a normal to the path l_{ij}) which is entirely contained in U_{ij} and is so small that the segments l_{ij} have no common points with one another and each segment l_{ij} has only one common point with all the segments O_iC_{ij} of the separatrices, namely the point C_{ij}. If system (A_λ) is sufficiently close to (A), e.g., if η is sufficiently small, we conclude from §9.2, remark to Lemma 3, that to every segment O_iC_{ij} of the separatrix L_{ij} of the saddle point O_i corresponds a segment $O_iC_{ij}^{(\lambda)}$ of the separatrix $L_{ij}^{(\lambda)}$ of the saddle point O_i of system (A_λ) which is entirely contained in U_{ij} and is such that $C_{ij}^{(\lambda)} \in l_{ij}$. Furthermore, as in the case of system (A), every segment l_{ij} has a single common point with all the segments $O_iC_{ij}^{(\lambda)}$ of the separatrices of the system (A_λ). We will assume that η is so small that this condition is satisfied for all (A_λ), $\lambda \in I$.

In view of the above, every neighborhood U_{ij} thus naturally corresponds to a single separatrix of any system (A_λ), $0 < \lambda \leq 1$, namely the separatrix $L_{ij}^{(\lambda)}$. We will refer to this separatrix as *the separatrix corresponding to the neighborhood* U_{ij}.

Let now system (A) have a saddle-to-saddle separatrix γ. To fix ideas, let γ extend from saddle point O_1 to O_2. The case of a separatrix extending from some saddle point to the same saddle point, i.e., forming a loop, is treated analogously.

Let T_λ be a homomorphism of $\overline{W}$ into itself, which moves the paths of (A) into the paths of (A_λ) (this homomorphism exists, since (A) is a structurally stable system and (A_λ) is a feasible system; see Definition II). Evidently, $T_\lambda(\gamma)$ is a separatrix extending from the saddle point $T_\lambda(O_1)$ to the saddle point $T_\lambda(O_2)$. Being a separatrix of two saddle points, it clearly corresponds to two (different) neighborhoods U_{ij}.

There are uncountably many separatrices $T_\lambda(\gamma)$ (since $\lambda \in I$), whereas the pairs of neighborhoods U_{ij} form a countable set. Consequently, there exists an *uncountable* set $I^* \subset I$ of the values of λ and also two neighborhoods U_{ij}, e.g., the neighborhoods U_{33} and U_{44} of the saddle points O_3 and O_4, such that if $\lambda \in I^*$, the separatrix $T_\lambda(\gamma)$ extends from saddle point O_3 to saddle point O_4 and corresponds to the neighborhoods U_{33} and U_{44}.

Since I^* is an uncountable set of points of the segment $I = [0, 1]$, I^* cannot consist entirely of isolated points. Hence there exists at least one $\lambda \in I^*$ and a sequence of λ_i, $i = 1, 2, 3, \ldots$, such that $\lambda_i \in I^*$ and $\lim\limits_{i \to \infty} \lambda_i = \lambda_0$. Without loss of generality, we may take λ_i to be a monotonically decreasing sequence.

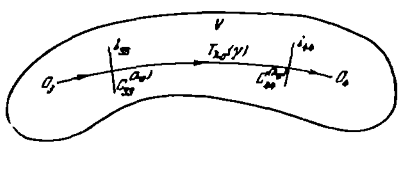

FIGURE 90

The separatrix $T_{\lambda_0}(\gamma)$, and likewise the separatrices $T_{\lambda_i}(\gamma)$, extend from saddle point O_3 to saddle point O_4 and correspond to the neighborhoods U_{33} and U_{44} (Figure 90). Let V be a neighborhood of the separatrix $T_{\lambda_0}(\gamma)$. From §9.2, remark to Lemma 3, and from §4.2, Lemma 7 it follows that for all sufficiently large n, $T_{\lambda_n}(\gamma) \subset V$. But the vector field of (A_{λ_0}) is obtained from the vector field of (A) by rotation through a positive angle. In our proof of Theorem 16 (§11.2) we established that if there exists a saddle-to-saddle separatrix of system (A) and a sufficiently small neighborhood of the separatrix, the system generated from (A) by a rotation of the vector field through a sufficiently small angle cannot have a saddle-to-saddle separatrix in that neighborhood.* We have thus reached a contradiction, which proves the lemma.

Remark. The proof of the last lemma actually shows that if a dynamic system (A) with a finite number of simple saddle points has saddle-to-saddle separatrices, the system obtained by rotating the vector field of (A) through a sufficiently small angle no longer has any such separatrices.

* In our proof of Theorem 16 we did not consider systems (A_λ), but systems of the form

$$\frac{dx}{dt} = P - \mu Q, \quad \frac{dy}{dt} = \mu P + Q. \qquad (\widetilde{A})$$

To move from (A) to $(\widetilde{A})$, we have to rotate the vector field through an angle α, $\tan \alpha = \mu$, and stretch the field vectors by a factor $\sqrt{1 + \mu^2}$.

Lemma 5. Every closed path L of system (A) is isolated, i.e., it is a limit cycle.

Proof. Since (A) is an analytical system, a closed path L either is a limit cycle or is enclosed in an open ring E entirely consisting of closed paths (see Chapter V, §12.3). We will show that no such ring can exist.

Indeed, suppose that ring E does exist. By QT, §23.2, the interior boundary of this ring, which is a zero-limit continuum, is either a closed path L_0, or consists of a finite number of saddle-to-saddle separatrices, or finally is a center. The last two alternatives are ruled out by Lemmas 3 and 4. The first alternative is unacceptable because all the paths which extend outside L_0 and sufficiently close to it are closed. But all the paths inside L_0 and sufficiently close to it are also closed, i.e., L_0 consists of the interior points of the ring E, which contradicts the original assumption. Lemma 5 is thus proved.

Remark. The proposition of the lemma is clearly also valid for systems of class 1.

Lemma 6. System (A) may only have a finite number of closed paths.

Proof of Lemma 6 is entirely analogous to the proof of Theorem 21 (§16.1).

Lemma 7. System (A) has no multiple limit cycles.

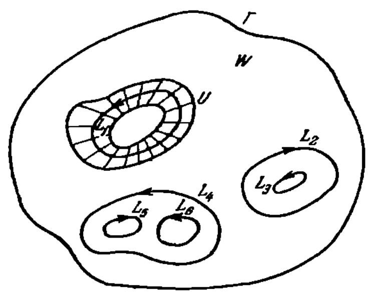

FIGURE 91

Proof. By Lemma 6, system (A) may only have a finite number of limit cycles. Let $L_1, L_2, \ldots, L_p$ be all the cycles of the system. Suppose that one of these, L_1 say, is a multiple cycle.

Let U be a sufficiently small neighborhood of L_1 which does not intersect with the paths $L_2, \ldots, L_p$ (Figure 91). Using the theorem of the creation of a closed path from a multiple limit cycle (§15.2, Theorem 19) and applying the same construction as in our proof of Lemma 2, §15.2, we obtain a system (A_1) of class 1 which is $\frac{\delta}{3}$-close to (A), coincides with (A) outside the neighborhood $U(L)$, and has in this neighborhood at least two closed paths. Let L_1' and L_1'' be two such paths (in U).

By Lemma 2, §15.2, there exists a system (A_2) of class 1, $\frac{\delta}{3}$-close to system (A_1), for which the curves L_1', L_1'', $L_2, L_3, \ldots, L_p$ are structurally stable limit cycles.

Finally, let (A_3) be an analytical system providing an adequate approximation to (A_2). By Theorem 18 and the remark to this theorem (the theorem of the structural stability of a limit cycle, §14), system (A_3) has one limit cycle in the neighborhood of each of the curves L_1', L_1'', $L_2, \ldots, L_p$, i.e., it has at least $p+1$ limit cycles. If (A_3) is also $\frac{\delta}{3}$-close to (A_2), it is δ-close to (A), i.e., it is a feasible system. But then, because of the structural stability of (A), (A_3) should have the same number of limit cycles as (A) does, i.e., p. The assumption of a multiple cycle

among the limit cycles of (A) thus has led to a contradiction. This completes the proof of the lemma.

Lemma 8. System (A) ***has no multiple foci.***

Proof. Suppose that one of the equilibrium states of (A), $O_1(a_1, b_1)$ say, is a multiple focus (we assume that it is a stable focus). At the point O_1, the Jacobian $\Delta > 0$, and div $(P, Q) = 0$. Let (A^*) be the system introduced in the proof to Lemma 3. We may take $\varepsilon = \varepsilon'$ to be so small and of such a sign that for the corresponding system (A^*) — we can denote it by $A^*(\varepsilon')$ — the point O_1 is an unstable focus. Simple standard reasoning (see, e.g., the proof of Theorem 14 about the creation of a closed path from a multiple focus, §10.3) shows that in this case, for some ε'' between 0 and ε', the system $A^*(\varepsilon'')$ will have a closed path contained entirely in a neighborhood of O_1. By Lemmas 6 and 7, system (A) has a finite number of closed paths, which are simple limit cycles. Seeing that a transformation to a sufficiently close system produces only a slight translation of every simple limit cycle,* we conclude immediately that the system $A^*(\varepsilon'')$ has at least one closed path more than system (A) does. This, however, contradicts the condition that $A^*(\varepsilon'')$ is a feasible system. The lemma is thus proved.

Remark. The proof of this lemma is also greatly simplified if we consider systems of class 1, and not analytical systems. In this case, we can easily find a feasible system whose right-hand sides in the neighborhood of a multiple focus O_1 are linear parts of P and Q. O_1 is evidently a center of this feasible system, which is again impossible.

Theorem 29. ***Definitions I and II of structural stability of a dynamic system in a region bounded by a cycle without contact are equivalent.***

Proof. If system (A) is structurally stable in the sense of Definition I, it is structurally stable in the sense of Definition II. This is obvious. If system (A) is structurally stable in the sense of Definition II, Lemmas 2 through 7 show that it satisfies the conditions of structural stability (CSS). Finally, if system (A) satisfies the CSS, it is structurally stable in the sense of Definition I by Theorem 23, §18.2. We thus see that I→II→CSS→I. This means that Definitions I and II and CSS are all equivalent. Q. E. D.

Theorem 29 is also valid for dynamic systems on a sphere. However, in the case of a sphere, we will consider only systems of class 1. Let the W in Definitions I and II be identified with a sphere S^2. We thus obtain two definitions of structural stability on a sphere, where Definition I coincides with Definition 12 (§6.2), and Definition II is free from the requirement of ε-identity.

Theorem 30. ***Definitions I and II of structural stability of a dynamic system of class 1 on a sphere S^2 are equivalent.***

Proof of Theorem 30 is conducted along the same lines as the proof of Theorem 29. Our proofs of Lemmas 1 through 8 remain in force for dynamic systems of class 1 on a sphere; the only difference is that some additional arguments, analogous to those adopted in the proof of Theorem 24 (§18.3), have to be used when dealing with a sphere.

Theorem 30 can be generalized without difficulty to systems of class $r > 1$. The case of analytical systems on a sphere is more complicated, and it will not be considered here.

Remark. Theorem 30 is valid for dynamic systems of class 1 on any closed surface, whether oriented or unoriented, and not only on a sphere [7].

* This follows from the structural stability of a simple limit cycle (see §14, Theorem 18).

Chapter VIII

BIFURCATIONS OF DYNAMIC SYSTEMS. DECOMPOSITION OF A MULTIPLE EQUILIBRIUM STATE INTO STRUCTURALLY STABLE EQUILIBRIUM STATES

INTRODUCTION

The first seven chapters of the book dealt with the theory of structurally stable systems. Chapter VIII begins the second part of the book, devoted to certain aspects of the so-called theory of bifurcations of dynamic systems. The present chapter contains two sections, §22 and §23. In §22, the main problems of the theory of bifurcations are formulated and a link is established between the first and the second part of the volume. In particular, the concepts of bifurcations and degree of structural instability of a dynamic system are defined. Since §22 is "narrative" (without any lemmas, theorems, and proofs), we will not summarize its contents here. It should be noted, however, that the theory of bifurcations is concerned with the changes which occur in the topological structure of a dynamic system in a particular region when the system itself (i.e., its right-hand sides) is altered, and the term bifurcation generally refers to these changes in topological structure.

Bifurcations of a multiple isolated equilibrium state (i.e., bifurcations of the dynamic system in the neighborhood of such an equilibrium state) are the subject of §23. The discussion is confined to analytical systems, and only the simplest multiple equilibrium state is considered, i.e., such that the series expansions of the functions P and Q in its neighborhood contain at least one linear term. Furthermore, the topic of bifurcations of these equilibrium states does not receive a fully general treatment in §23, as we only investigate the number and the character of the structurally stable equilibrium states into which the multiple state decomposes on passing to close systems*.

The topological structure of these equilibrium states is treated in detail in QT, Chapter IX (§21 and §22). It is established in QT that if for an equilibrium state O (0, 0)

$$\sigma = P'_x(0, 0) + Q'_y(0, 0) \neq 0,$$

the point O is a topological node, or a topological saddle point, or a saddle-node. If, however, $\sigma = 0$, six different possibilities arise: a

* The problem of bifurcations of a multiple equilibrium state in its general form is formulated as follows: establish the changes in the topological structure of a dynamic system in the neighborhood of an equilibrium state on passing to close systems. A relatively narrow segment of this general problem is considered in §23.

topological node, a topological saddle point, a focus or a center, an equilibrium state with an elliptical region, a degenerate equilibrium state, a saddle-node. In §23, a relationship is established between the topological structure of a multiple equilibrium state, on the one hand, and the number and character of the structurally stable equilibrium states into which the multiple state decomposes when passing to close systems, on the other. For example, if $\sigma = P'_x(0, 0) + Q'_y(0, 0) \neq 0$ and the multiple equilibrium state O is a topological saddle point, it may only decompose into an odd number of structurally stable nodes and saddle points, the number of structurally stable nodes being of necessity one less than the number of structurally stable saddle points. The other results of the chapter are contained in Theorems 35, 37 through 39. Note, however, that for $\sigma = 0$, the type of the multiple singular point is entirely determined by the number of structurally stable equilibrium states into which it decomposes and by their topological structure. If, on the other hand, $\sigma = 0$, the difference (other than topological) between nodes and foci has to be taken into consideration in some cases, and in other cases it is altogether impossible to establish the character of the multiple equilibrium state using the component structurally stable equilibrium states.

A reader wishing to speed up his progress through the book may omit the proof of Theorems 37 — 39, and familiarize himself with the statement of the theorems only.

In conclusion note that §23.3 contains a theorem by Poincaré (Theorem 36) which states that if a dynamic system has only simple equilibrium states and the isocline $P(x, y) = 0$ (or $Q(x, y) = 0$) has no singular points (i.e., points at which $P'_x = P'_y = 0$), then saddle points alternate with nodes and foci along this isocline. This theorem is used in the proof of Theorem 38, but it is also of considerable independent interest.

§22. THE DEGREE OF STRUCTURAL INSTABILITY AND BIFURCATIONS OF DYNAMIC SYSTEMS

The previous chapters dealt with *structurally stable* dynamic systems on a sphere or in a plane region. The definition of a structurally stable system in a plane region W was formulated assuming that W is an *arbitrary bounded* region (§6.1, Definition 10). A certain additional restriction was imposed at a later stage on W, e.g., in the derivation of the necessary and sufficient conditions of structural stability, and W was treated as a region with a normal boundary (§16.2, Definition 19). This restriction is not fundamental, although it simplifies some proofs.

The definition of structural stability in an arbitrary plane region W (including a region with a normal boundary) has one distinct disadvantage: together with W, we are forced to consider other regions close to W. To avoid the difficulties (again not of fundamental nature) associated with this approach and to achieve a more plastic description of the concepts that follow, we will assume in this section that the boundary Γ of W is a *cycle without contact*.*

* The simplest and most complete picture is obtained for dynamic systems on a sphere, where the definition of structural stability is markedly simpler. We again wish to emphasize that the restriction imposed on the relevant region is solely intended to simplify the presentation. The concepts of bifurcation and degree of structural instability can be defined analogously for dynamic systems in any bounded plane region.

Thus, let

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y) \tag{A}$$

be a dynamic system defined in $\bar{G}$ and considered in a closed region $\bar{W}$ bounded by a cycle without contact Γ ($\bar{W} \subset G$). The exact space of dynamic systems in which our analysis is carried out should be indicated. Let this be one of the spaces $R_N^{(r)}$ $(1 \leqslant r \leqslant N)$ or $R_a^{(r)}$ $(r \geqslant 1)$, defined for the region $\bar{G}$,* which we will denote by R^*. System (A) clearly should belong to the space R^*. In the following, structural stability (or structural instability) of system (A) in $\bar{W}$ will be understood as structural stability (or structural instability) relative to the space R^*.

The structural instability of a dynamic system may be considered using the concept of *ε-identity of partitions* (§6.1, Definition 10 and §18.4, remark a). In Chapter VII (§21, Theorem 20) we have seen, however, that the ε-identity can be dropped when dealing with a region bounded by a cycle without contact. Indeed, *system (A) is structurally stable in region $\bar{W}$ if all sufficiently close dynamic systems $(\tilde{A})$ have the same topological structure as (A) in $\bar{W}$.* In other words, if (A) is a structurally stable system in $\bar{W}$, all the points of a certain neighborhood $U_\delta((A) \mid R^*)$ of (A) in the space R^* are systems with the same topological structure in $\bar{W}$.

Conversely, if (A) is structurally unstable in $\bar{W}$, there always exist systems $(\tilde{A})$ arbitrarily close to (A) whose topological structure in $\bar{W}$ is different from the topological structure of (A).

In Chapter VI (§18.4) we established that the structurally stable systems form an open set in the space R^* and that this set is everywhere dense in R^*. Structurally unstable systems form in R^* "partitions" separating between regions filled with structurally stable systems. Each of these regions consists of dynamic systems with the same topological structure in $\bar{W}$.

In QT we investigated the different topological structures of a dynamic system in a particular region and the factors determining this structure. Our present topic deals with *the changes in the topological structure of a system in $\bar{W}$ when the dynamic system* (i.e., the right-hand sides, the functions P and Q) *is altered.* The remaining chapters of the book are concerned with the applications of this topic to a number of important particular cases.

It is naturally assumed that, despite the changes, the dynamic system always remains in the space R^*.

In applications, this question is sometimes considered in a restricted, particular form. Indeed, given a certain set E, $E \subset R^*$, one considers the changes in the topological structure in $\bar{W}$ as the dynamic system runs through the points of this set. The particular choice of the set E is determined by the problem being considered. E is often chosen as a small neighborhood of a given dynamic system (A_0) or as some line, surface, or hypersurface in the space R^*. Dynamic systems related to real physical problems generally contain one or several parameters, i.e., they have

* See §5.1. $R_N^{(r)}$ is the space of dynamic systems of class N with a metric defined by the distance (in $\bar{G}$) of the functions P and Q and their derivatives to r-th order inclusive. $R_a^{(r)}$ is the space of analytical functions with the same metric.

the form

$$\frac{dx}{dt} = P(x, y, \lambda_1, \lambda_2, \ldots, \lambda_m), \quad \frac{dy}{dt} = Q(x, y, \lambda_1, \lambda_2, \ldots, \lambda_m) \qquad (A_{\lambda, m})$$

In these problems we generally have to consider the changes in the topological structure of the dynamic system as the parameters λ_i vary in a certain region. The set E in this case is clearly an m-dimensional hypersurface (or some region of this hypersurface). If the system depends on a single parameter, the corresponding set E is a *line* in the space R^*.

The question of the changes in the topological structure of a partition into paths with changes in the dynamic system is of great independent theoretical interest. Two further factors greatly enhance its importance. The first is that the study of the changes in the topological structure associated with changes in a dynamic system, i.e., *the theory of bifurcations*, provided the main tool in the investigation of particular dynamic systems. As we have noted above, no general regular methods are available for this investigation, and without exaggeration we can state that almost all the available results in this direction have been obtained using the theory of bifurcations. The theory of bifurcations therefore plays a leading role in the study of particular systems.

The second factor is associated with the importance of the theory of bifurcations in applied problems, in particular in physical and engineering applications. Dynamic systems corresponding to these problems always contain a certain number of parameters. The changes in topological structure following changes in these parameters are of the utmost importance in the analysis of the properties of physical systems related to the topological structure of the corresponding dynamic system (e.g., when considering sustained oscillations in a given physical system). The theory of bifurcations in one form or another is therefore applied virtually to every dynamic system corresponding to a physical problem.

We will consider in what follows the case of *small changes* of a dynamic system. This is the key to the study of "large" changes, and it also has numerous important applications, e.g., in problems of stability of physical systems. Our problem is thus formulated in the following form: *investigate the changes in the topological structure of the partition of region $\overline{W}$ into paths following small changes in the corresponding dynamic system.*

Only *structurally unstable systems* should be considered, since if (A) is structurally stable in $\overline{W}$, its topological structure does not change as a result of small changes in the system. If, however, (A) is *structurally unstable*, dynamic systems of different topological structures always exist in any arbitrarily small neighborhood of (A) (in the space R^*).

We say in this case that *the point (A) of the space R^* is a bifurcation point of a dynamic system.** *Bifurcation* is generally understood as the *change in the topological structure* of a dynamic system occurring when it passes through a bifurcation point.

Structurally unstable systems, and only these systems, are bifurcation points in the space of dynamic systems, and the problem

* A more precise statement would be the following: (A) is a bifurcation point of the topological structure of a dynamic system in $\overline{W}$. We will nevertheless use the more concise form given in the main text.

reduces to the following: investigate the changes in the topological structure of a structurally unstable system on passing to sufficiently close systems.

The next topic to consider is the classification of structurally unstable systems. A structurally unstable system may be "more structurally unstable" or "less structurally unstable." These imprecise terms can be imbued with precise mathematical meaning by introducing the concept of the degree of structural instability of a system. Originally this concept was introduced in /9/. For the sake of simplicity, we will only give a definition of the degree of structural instability in a region $\overline{W}$ bounded by a cycle without contact. The corresponding definition for a general bounded region will be given at a later stage (§31, Definition 30).

We shall assume that the relevant systems are either analytical in $\overline{G}$ or of class N, where N is a natural number whose magnitude, as will be seen from the definition, depends on the degree of structural instability of the system. $\overline{W}$ is a subregion of G bounded by a cycle without contact. Structurally stable systems in $\overline{W}$ will be called systems of zero degree of structural instability. As we know (§18.4, a), system (A) is structurally stable in $\overline{W}$ if it has the following property: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $(\tilde{A})$ is λ-close to (A), then

$$(\overline{W}, \tilde{A}) \overset{\varepsilon}{\equiv} (\overline{W}, A).$$

We will define the degrees of structural instability by induction.

Definition 23. A dynamic system (A) *of class* $N \geqslant 3$ *is said to be a system of 1st degree of structural instability (or to have a degree of structural instability 1) in* $\overline{W}$ *if it is not structurally stable in this region and satisfies the following condition: for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any structurally unstable system* $(\tilde{A})$ δ*-close to* (A) *to rank 3 we have*

$$(\overline{W}, \tilde{A}) \overset{\varepsilon}{\equiv} (\overline{W}, A).$$

System (A) *of class* $N \geqslant 5$ *is said to be a system of 2nd degree of structural instability in* $\overline{W}$ *if it is not a system of zero or first degree of structural instability and the following condition is satisfied: for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that any system* $(\tilde{A})$ δ*-close to* (A) *to rank 5 is either a system of zero or first degree of structural instability in* $\overline{W}$ *or satisfies the relation*

$$(\overline{W}, \tilde{A}) \overset{\varepsilon}{\equiv} (\overline{W}, A).$$

System (A) *of class* $N \geqslant 2k+1$ *is said to be a system of* k*-th degree of structural instability in* $\overline{W}$ *if it is not a system of lower degree of structural instability (i.e., zero, 1st, 2nd, ...* $(k-1)$*-th) and the following condition is satisfied: for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that any system* $(\tilde{A})$ *which is* δ*-close to rank* $2k+1$ *to* (A) *either has a degree of structural instability of at most* $k-1$ *in* $\overline{W}$ *or satisfies the relation*

$$(\overline{W}, \tilde{A}) \overset{\varepsilon}{\equiv} (\overline{W}, A).$$

A few remarks concerning degrees of structural instability. We see from Definition 23 that for a system to have a definite degree of structural instability, it must be a system of a sufficiently high class. This is not an unexpected conclusion: we have encountered a similar situation in

Chapter I. Indeed, in defining the multiplicity of a root of a function, say (§1.3, Definition 2), we could speak of a root of multiplicity r of a function $F(x)$ only if $F(x)$ was a function of class $N \geqslant r$. A somewhat puzzling point in our definition of degrees of structural instability is the fact that only systems of class $N \geqslant 2k+1$ can be of a k-th degree of structural instability. We will not discuss here the factors responsible for this restriction. Note, however, that this fact is related to the properties of a multiple focus discussed in Chapter IX (§25.1, Theorem 40).

A significant shortcoming of Definition 23 is that it does not assign a definite degree of structural instability to each and every dynamic system. For systems of a finite class this is obvious. Indeed, let $N = 2k+1$, $k \geqslant 1$. We can easily construct a system (A) of class N, which is not a system of class $N+1$, with $k+1$ equilibrium states, each of multiplicity 2, in some region $\overline{W}$. It follows from Definition 23 and from the definition of the multiplicity of an equilibrium state (§7.3) that this (A) cannot have a degree of structural instability less than or equal to k. On the other hand, Definition 23 states that (A) cannot have a degree of structural instability greater than k. The system (A) that we have constructed thus does not have a definite degree of structural instability.

Let us now consider the analytical case. It is readily seen that analytical systems of any finite degree of structural instability exist. Furthermore, there exist analytical systems with other analytical systems of all finite degrees of instability contained in any of their neighborhoods. These systems are naturally assigned an infinite degree of structural instability. However, it does not follow from Definition 23 that each analytical system may have a definite finite or infinite degree of structural instability.*

Systems of the 1st degree of structural instability are *relatively* structurally stable in the set of all structurally unstable systems.** Similarly, systems of k-th degree of structural instability are relatively structurally stable in the set of all structurally unstable systems of degree of structural instability $\geqslant k$.

We have seen before that dynamic systems associated with physical problems generally contain one or several parameters. Therefore, systems dependent on parameters are of particular interest. Let, for simplicity, the right-hand sides contain a single parameter, i.e., we are considering a system of the form

$$\frac{dx}{dt} = P(x, y, \lambda), \qquad \frac{dy}{dt} = Q(x, y, \lambda). \tag{A_λ}$$

The parameter λ may vary over a certain set of real numbers or run through the entire real axis.

If we are interested only in systems which are obtained for various values of the parameter λ, we should specifically consider the concept of structural stability and degrees of structural instability in relation to systems (A_λ). The following definition of structural stability can be given (for a region $\overline{W}$ bounded by a cycle without contact): system (A_{λ_0}) is structurally stable in $\overline{W}$ relative to systems (A_λ) if there exists $\delta > 0$ such

* This can be shown without much trouble. See Gudkov /8/, p. 485.

** In the following sense: if we consider the set of structurally unstable systems and closeness to rank 3, systems of the 1st degree of structural instability are stable relative to this set.

that every dynamic system (A_λ) for which $|\lambda - \lambda_0| < \delta$ has the same topological structure as (A_{λ_0}) in $\overline{W}$. The degrees of structural instability are similarly defined.

If system (A_{λ_0}) is structurally stable in the usual sense, it is also structurally stable relative to system (A_λ). The converse, naturally, is not always true: a system may be structurally stable relative to the systems (A_λ), without being structurally stable in the usual sense.

The concept of bifurcation may also be redefined relative to systems (A_λ). We thus arrive at the following definition of the bifurcation value of the parameter:

Definition 24. The value λ_0 of the parameter λ is called a bifurcation value of the parameter if there exist values of the parameter λ arbitrarily close to λ_0 for which the topological structure of the dynamic system in the relevant region is different from the topological structure of (A_{λ_0}). Values of the parameter which are not bifurcation values are called ordinary values.

The situation is entirely analogous for a system dependent on several parameters. Thus, for instance, for a system

$$\frac{dx}{dt} = P(x, y, \lambda, \mu), \quad \frac{dy}{dt} = Q(x, y, \lambda, \mu), \qquad (A_{\lambda,\mu})$$

depending on two parameters λ and μ, we can speak of structural stability or degrees of structural instability relative to the systems $(A_{\lambda,\mu})$. We can similarly speak of the bifurcation pair of values of the parameters λ, μ and of the bifurcation point in the plane of the parameters. If the particular region in the plane of the parameters contains no bifurcation points, all the dynamic systems corresponding to this region have the same topological structure. A change in topological structure may occur (as a result of a continuous change in the parameters) only when the system crosses through a bifurcation point.

It is readily seen that the investigation of all the bifurcations of a dynamic system in a given region $\overline{W}$ reduces to an inspection of the changes which occur following small changes of the system in the neighborhood of the elements determining the topological structure. In other words, it suffices to investigate the changes in the topological structure in the neighborhood of the equilibrium states, closed paths, and limit continua in $\overline{W}$. Some of the pertinent topics will be treated fully or partially in this and next chapters.

In conclusion of this section, let us consider two examples.

Example 7. Consider a system

$$\begin{aligned} \frac{dx}{dt} &= P(x, y, \alpha) = x\cos\alpha + y\sin\alpha - (x\cos\alpha - y\sin\alpha)(x^2+y^2), \\ \frac{dy}{dt} &= Q(x, y, \alpha) = x\sin\alpha - y\cos\alpha - (x\sin\alpha + y\cos\alpha)(x^2+y^2), \end{aligned} \qquad (A_\alpha)$$

depending on a single parameter α. This system is generated by a rotation of the vector field of the system.

$$\frac{dx}{dt} = x - x(x^2+y^2), \quad \frac{dy}{dt} = -y - y(x^2+y^2) \qquad (A_0)$$

through the angle α. We may therefore take α to vary from 0 to 2π. We shall consider the system (A_α) over the entire plane (x, y). Since $(A_{\alpha+\pi})$ is

obtained from (A_α) by reversing the direction of the field vectors at every point of the plane, we need only consider the system (A_α) for $0 \leqslant \alpha < \pi$.

From the relations

$$P(-x, -y, \alpha) = -P(x, y, \alpha), \quad Q(-x, -y, \alpha) = -Q(x, y, \alpha)$$

it follows that the phase-plane representation of the dynamic system (A_α) for any α is symmetrical about the origin (§20, Example 4).

We will first apply the Bendixson criterion. Since

$$\frac{\partial P(x, y, \alpha)}{\partial x} + \frac{\partial Q(x, y, \alpha)}{\partial y} = -4(x^2 + y^2)\cos\alpha,$$

we see that for $\alpha \neq \frac{\pi}{2}$, $0 \leqslant \alpha < \pi$, the sum $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ does not reverse its sign anywhere in the plane. Therefore, by the Bendixson criterion (QT, §12.3), system (A_α) with $\alpha \neq \frac{\pi}{2}$ has neither closed paths (in particular, limit cycles) nor closed curves consisting of paths in the phase plane.

Let $x = x(t)$, $y = y(t)$ be paths of (A_α). Then, as it follows from the system equations,

$$x(t)\dot{x}(t) + y(t)\dot{y}(t) = \frac{1}{2}\frac{d}{dt}[x(t)^2 + y(t)^2] = x^2\cos\alpha + 2xy\sin\alpha - y^2\cos\alpha - (x^2 + y^2)^2\cos\alpha.$$

The last relation shows that for $0 \leqslant \alpha < \frac{\pi}{2}$, the infinity is absolutely unstable, and for $\frac{\pi}{2} < \alpha \leqslant \pi$ the infinity is absolutely stable (see §20, Example 4, and also QT, §13.1).

We can now easily establish the configuration of the paths of system (A_α) for various values of the parameter α, $0 \leqslant \alpha < \pi$. We will consider separately the four cases $\alpha = 0$, $0 < \alpha < \frac{\pi}{2}$, $\alpha = \frac{\pi}{2}$, $\frac{\pi}{2} < \alpha < \pi$.

1) $\alpha = 0$. From the equations of (A_0) we readily see that the positive and the negative coordinate semiaxes are paths of the system. Calculations show that (A_0) has three equilibrium states, the saddle point $O(0, 0)$ and two stable dicritical nodes $A(-1, 0)$ and $B(1, 0)$. Hence it follows that, in the absence of closed curves consisting of path segments, the partition of the phase plane into paths has the form schematically shown in Figure 92.

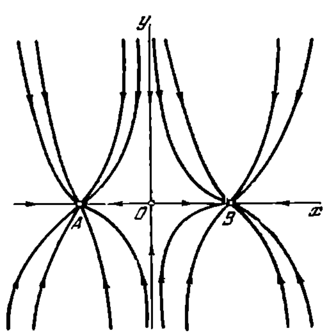

FIGURE 92

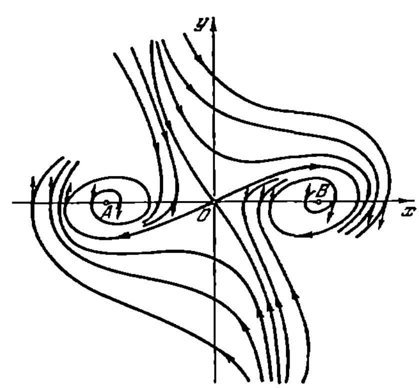

FIGURE 93

2) $0<\alpha<\frac{\pi}{2}$. System (A_α) again has the same three equilibrium states $O(0, 0)$, $A(-1, 0)$, $B(1, 0)$ as in the previous case.* $O(0, 0)$ is a simple saddle point for any α. The points A and B are stable structurally stable foci. There are no closed curves consiting of system paths, in particular, no limit cycles and saddle-to-saddle separatrices. Therefore the ω-separatrices of the saddle point O reach from infinity and the α-separatrices wind onto the two foci A and B. To define the direction of winding of the paths (spirals) around the foci, we have to establish the field of directions on the x axis. For $y=0$, $\frac{dy}{dt}=x\sin\alpha-x^3\sin\alpha=(x-x^3)\sin\alpha$. Since for the values of α being considered $\sin\alpha>0$, we have

$$\frac{dy}{dt}>0 \text{ for } \quad 0<x<1 \text{ and for } -\infty<x<-1.$$

$$\frac{dy}{dt}<0 \text{ for } -1<x<0 \quad \text{and for} \quad 1<x<+\infty.$$

Moreover, on the x axis, $\frac{dx}{dt}=(x-x^3)\cos\alpha$, so that $\frac{dy}{dt}=\tan\alpha$. The direction of the separatrices of the saddle point O is determined from the equation

$$\sin\alpha\cdot k^2+2\cos\alpha\cdot k-\sin\alpha=0$$

(see QT, §9.2, corollary of Lemma 1). Its solutions are

$$k_1=\tan\frac{\alpha}{2}, \quad k_2=\tan\left(\frac{\alpha}{2}+\frac{\pi}{2}\right).$$

We now readily see that the partition of the phase plane into paths should have the form shown schematically in Figure 93.

3) $\alpha=\frac{\pi}{2}$. The system takes the form

$$\frac{dx}{dt}=y+y(x^2+y^2), \quad \frac{dy}{dt}=x-x(x^2+y^2). \tag{$A_{\pi/2}$}$$

A direct substitution shows that $(A_{\pi/2})$ has the common integral

$$(x^2+y^2)^2-2(x^2-y^2)=C \tag{1}$$

(see QT, §1.13).

Since

$$(x^2+y^2)^2-2(x^2-y^2)\equiv(x^2-1)^2+2y^2(1+x^2)+y^4-1,$$

we conclude that $C\geqslant-1$. We may therefore take $C=a^4-1$, where $a\geqslant 0$. Equation (1) thus takes the form

$$(x^2+y^2)^2-2(x^2-y^2)=a^4-1. \tag{2}$$

The curves (2) constitute a family of Cassini ovals with the foci $A(-1, 0)$ and $B(1, 0)$. Elementary analysis shows that for $a\geqslant\sqrt{2}$ the curves (2) are convex ovals, for $1<a<\sqrt{2}$ these are "pinched" ovals, and for $a=1$ the curve is a lemniscate. For $0<a<1$ the curves break into two separate ovals,

* When a vector field is rotated, the number and the position of the equilibrium states do not change. Only their character may change (see QT, §1.14, remark preceding Example 7).

and finally for $a = 0$ they degenerate into two points A and B (Figure 94). Each of the ovals is a path of system $(A_{\pi/2})$, the lemniscate consists of three paths (the saddle point O and the two separatrices forming loops), and the points A and B are centers. The direction along the paths is readily defined by considering the sign of $\frac{dy}{dt}$ for $y = 0$. We thus obtain the configuration shown in Figure 94.

4) $\frac{\pi}{2} < \alpha < \pi$. This case is analyzed along the same lines as case 2. The configuration of paths in the phase plane is schematically shown in Figure 95.

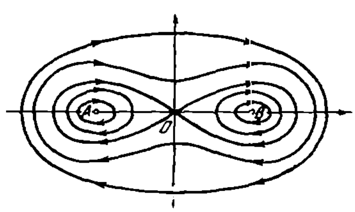

FIGURE 94

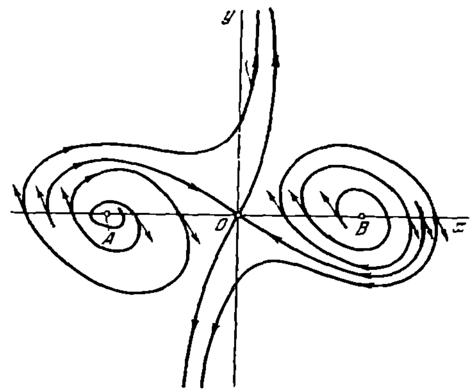

FIGURE 95

For $\alpha = \pi$, $\pi < \alpha < \frac{3}{2}\pi$, $\alpha = \frac{3}{2}\pi$, $\frac{3}{2}\pi < \alpha < 2\pi$, the configuration of the paths is the same as in Figures 92, 93, 94, and 95, respectively, but the direction of motion along the paths is reversed. For $\alpha = 2\pi$, we return to the original system (A_0).

Let $\overline{W}$ be the interior of the circle

$$x^2 + y^2 = R^2,$$

where the radius R is so large that the lemniscate

$$(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$$

is contained entirely inside W. From the fundamental theorem of structural stability (Theorem 23, §18.2) it follows that for α between the limits $0 < \alpha < 2\pi$, $\alpha \neq \frac{\pi}{2}$, $\alpha \neq \frac{3}{2}\pi$, (A_α) is structurally stable in $\overline{W}$,* and for $\alpha = \frac{\pi}{2}$ and $\alpha = \frac{3}{2}\pi$ it is structurally unstable. The **bifurcation values of the parameter** are thus $\frac{\pi}{2}$ and $\frac{3}{2}\pi$.

* Theorem 23 was proved for a region with a normal boundary, whereas $\overline{W}$ need not be such a region for (A_α). However, a circle of a sufficiently large radius $R_1 > R$ is a cycle without contact for (A_α), and therefore inside this circle (A_α) is structurally stable by Theorem 23, and then it is also structurally stable in $\overline{W}$ (§6.1, Lemma 1).

If in the determination of the topological structure we consider not only the configuration of the paths but also the direction of motion along the paths, bifurcation occurs when the parameter passes through the value $\frac{\pi}{2}$, say, since the stable foci A and B change into unstable foci, i.e., the topological structure changes. Note, however, that if the directions along the paths are ignored, the transition of the parameter through the bifurcation value does not lead to a bifurcation in our example.

Figures 92 through 95 enable us to trace the changes in the configuration of the paths of (A_0) during the rotation of the vector field. Originally, the points A and B are nodes, and the separatrices of the saddle point O are the coordinate semiaxes. As the vector field is rotated in the positive direction, the nodes change into foci, the paths wind onto the foci in the clockwise sense, and the tangents to the separatrices at the point O also rotate in the positive direction with half the rotation velocity of the field vectors.* For $\alpha = \frac{\pi}{2}$, all the paths are closed, except the foci (which are now centers) and the saddle point O with its separatrices. The separatrices join in pairs forming loops. As the field is further rotated, the centers again become foci, but the paths unwind in this case. The separatrices of the saddle point O separate, the α-separatrices extending to infinity for $t \to +\infty$ and the ω-separatrices going to the foci for $t \to -\infty$. The change of configuration with further rotation of the field is obvious.

Let us consider still another example which illustrates the variation of the topological structure as a result of field rotation.

Example 8. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= -x \sin\alpha - y\cos\alpha + (x^2+y^2-1)^2 (x\cos\alpha - y\sin\alpha), \\ \frac{dy}{dt} &= x\cos\alpha - y\sin\alpha + (x^2+y^2-1)^2 (x\sin\alpha + y\cos\alpha). \end{aligned} \qquad (B_\alpha)$$

This system is obtained by rotating the vector field of the system

$$\frac{dx}{dt} = -y + x(x^2+y^2-1)^2, \quad \frac{dy}{dt} = x + y(x^2+y^2-1)^2 \qquad (B_0)$$

through the angle α. We may therefore take α as varying between $-\pi$ and π. However, (B_α) and $(B_{\alpha+\pi})$ have identical paths, which only differ in the direction of motion, and we may therefore consider only the system (B_α) for $-\frac{\pi}{2} < \alpha \leqslant \frac{\pi}{2}$.

System (B_0), and therefore (B_α), has a single equilibrium state $O(0, 0)$. The characteristic equation of this equilibrium state

$$\begin{vmatrix} \cos\alpha - \sin\alpha - \lambda & -(\sin\alpha + \cos\alpha) \\ \cos\alpha + \sin\alpha & \cos\alpha - \sin\alpha - \lambda \end{vmatrix} = 0 \qquad (3)$$

has the roots

$$\lambda_{1,2} = \cos\alpha - \sin\alpha \pm \sqrt{-\sin 2\alpha - 1}.$$

* This follows from the relations $k_1 = \tan\frac{\alpha}{2}$, $k_2 = \tan\left(\frac{\alpha}{2} + \frac{\pi}{2}\right)$ (p. 210).

Therefore for α between the limits $-\frac{\pi}{2}<\alpha\leqslant\frac{\pi}{2}$, the equilibrium state $O(0, 0)$ of (B_α) is

an unstable focus for $-\frac{\pi}{2}<\alpha<-\frac{\pi}{4}$ and for $-\frac{\pi}{4}<\alpha<\frac{\pi}{4}$;

an unstable dicritical node for $\alpha=-\frac{\pi}{4}$;

a multiple focus or center for $\alpha=\frac{\pi}{4}$;

a stable focus for $\pi/4<\alpha\leqslant\frac{\pi}{2}$.

The paths $x=x(t)$, $y=y(t)$ of (B_α) are tangent to the circle

$$x^2+y^2=R^2 \tag{4}$$

when

$$\frac{1}{2}\frac{d\rho^2}{dt}=\dot{x}x+\dot{y}y=0 \qquad (\rho=\sqrt{x^2+y^2}). \tag{5}$$

From the equations of (B_α),

$$x\dot{x}+y\dot{y}=(x^2+y^2)\,[(x^2+y^2-1)^2-\tan\alpha]\cos\alpha, \tag{6}$$

if $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, and

$$x\dot{x}+y\dot{y}=-(x^2+y^2), \tag{7}$$

if $\alpha=\frac{\pi}{2}$.

Finally note that, in polar coordinates, (B_α) can be written in the form

$$\frac{d\rho}{dt}=\rho\,[(\rho^2-1)^2\cos\alpha-\sin\alpha], \tag{8}$$

$$\frac{d\theta}{dt}=(\rho^2-1)^2\sin\alpha+\cos\alpha. \tag{9}$$

Equality (8) is clearly equivalent to (6) for $\alpha\neq\frac{\pi}{2}$ and to (7) for $\alpha=\frac{\pi}{2}$.

The above relations enable us to investigate the configuration of the paths of (B_α) for various values of the parameter α.

1) $\alpha=\frac{\pi}{2}$. The equilibrium state $O(0, 0)$ is a stable focus. It follows from (5) and (7) that all circles (4) are cycles without contact. The system therefore has no closed paths. Since the focus O is the only equilibrium state, the system has no limit continua consisting of continued paths. By (7), the paths crossing the cycle without contact (4) enter into the cycle as t increases. Thus all the paths of the system wind onto the focus O $(0, 0)$ as t increases, and go to infinity as t decreases (Figure 96). Equation (9) shows that as t increases, the motion along the paths is counterclockwise.

2) $\frac{\pi}{2}>\alpha>\frac{\pi}{4}$. In this case, the equation

$$(\rho^2-1)^2\cos\alpha-\sin\alpha=0 \tag{10}$$

has a single real root $\rho_0 = \sqrt{1 + \sqrt{\tan\alpha}}$. By (8), the circle

$$\rho = \rho_0, \quad \text{i.e.,} \quad x^2 + y^2 = \rho_0^2, \tag{11}$$

is a path of (B_α). All the other circles (4), in virtue of (6), are cycles without contact; if $R > \rho_0$ $(R < \rho_0)$, the paths crossing the circle (4) leave the circle (enter into the circle) as t increases. The circle (11) is therefore an unstable limit cycle of the system. The point O (0, 0) is a stable structurally stable focus. The system has no other limit continua, and all the paths are therefore spirals unwinding from the limit cycle. It follows from (9) that the motion along these paths in the direction of increasing t is counterclockwise. The configuration of the paths is shown in Figure 97.

The analysis of the other cases proceeds along the same lines.

3) $\alpha = \frac{\pi}{4}$. The system has a single unstable limit cycle, the circle

$$x^2 + y^2 = 2. \tag{12}$$

The equilibrium state O (0, 0) is a multiple (structurally unstable) stable focus. The motion along the spirals with increasing t is in the counterclockwise sense (Figure 98).

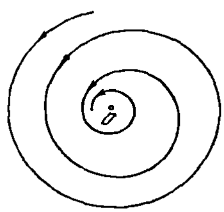

FIGURE 96. $\alpha = \frac{\pi}{2}$. Stable focus.

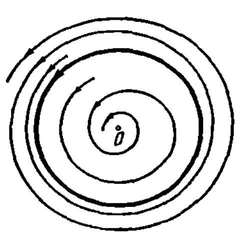

FIGURE 97. $\frac{\pi}{2} > \alpha > \frac{\pi}{4}$, $r > \sqrt{2}$.
Unstable cycle; stable focus.

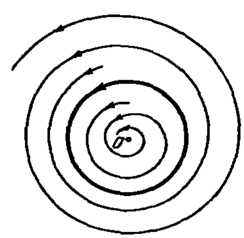

FIGURE 98. $\alpha = \frac{\pi}{4}$, $r = \sqrt{2}$.
Unstable cycle; multiple stable focus.

4) $\frac{\pi}{4} > \alpha > 0$. The system has two limit cycles, an unstable limit cycle

$$x^2 + y^2 = 1 + \sqrt{\tan\alpha}, \tag{13}$$

which is a circle of radius greater than 1, and a stable limit cycle

$$x^2+y^2=1-\sqrt{\tan\alpha}, \tag{14}$$

which is a circle of radius smaller than 1. The equilibrium state $O(0,0)$ is an unstable focus. The motion along the spiral is in the counterclockwise sense (Figure 99).

5) $\alpha = 0$. The system has one semistable, and therefore multiple (see §12.3), limit cycle, the circle

$$x^2+y^2=1, \tag{15}$$

and an unstable focus $O(0, 0)$. The motion along the paths is in the counterclockwise direction (Figure 100).

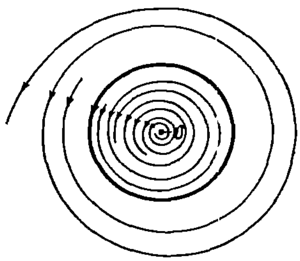

FIGURE 99. $\frac{\pi}{4}>\alpha>0$, $r_1>1$, unstable cycle; $r_2<1$, stable cycle, unstable focus.

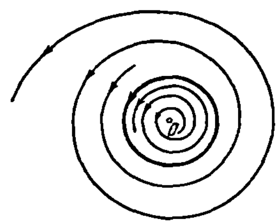

FIGURE 100. $\alpha=0$. No stable cycles; unstable focus.

6) $0>\alpha>-\frac{\pi}{4}$. All circles (4) are cycles without contact. The system therefore has no closed paths and no closed curves consisting of paths. All the paths go to infinity as t increases, and for $t\to-\infty$ they wind onto the unstable focus $O(0, 0)$.

We see from (9) that

$$\frac{d\theta}{dt}>0, \quad \text{if} \quad \rho^2<1+\sqrt{-\frac{\cos\alpha}{\sin\alpha}},$$

and

$$\frac{d\theta}{dt}<0, \quad \text{if} \quad \rho^2>1+\sqrt{-\frac{\cos\alpha}{\sin\alpha}}.$$

Therefore inside the circle

$$\rho^2=x^2+y^2=1+\sqrt{-\frac{\cos\alpha}{\sin\alpha}} \tag{16}$$

the motion along the paths is counterclockwise, and outside this circle the motion is clockwise (Figure 101). The circle (16) is evidently orthogonal to each crossing path.

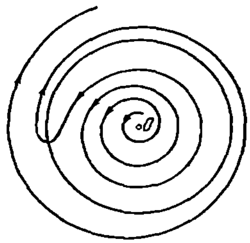

FIGURE 101. $0 > \alpha > -\frac{\pi}{4}$. No cycles. $R > \sqrt{2}$ unstable focus.

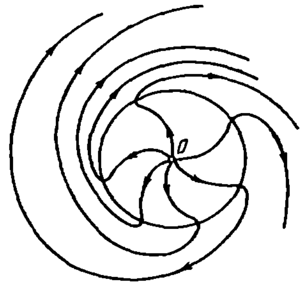

FIGURE 102. $\alpha = -\frac{\pi}{4}$. No cycles. $R = \sqrt{2}$ unstable dicritical node.

7) $\alpha = -\frac{\pi}{4}$. The configuration of the paths is the same as in the previous case, but the point O (0, 0) is an unstable dicritical node, and not a focus (Figure 102). Circle (16) is replaced by circle (12).

8) $-\frac{\pi}{4} > \alpha > -\frac{\pi}{2}$. The equilibrium state O(0, 0) is an unstable focus. All circles (4) are cycles without contact. The derivative $\frac{d\theta}{dt} = (\rho^2 - 1)^2 \sin\alpha + \cos\alpha$ vanishes on the circles

$$\rho^2 = 1 + \sqrt{-\frac{\cos\alpha}{\sin\alpha}} \tag{17}$$

and

$$\rho^2 = 1 - \sqrt{-\frac{\cos\alpha}{\sin\alpha}}. \tag{18}$$

In the ring between these circles, $\frac{d\theta}{dt} > 0$, and elsewhere in the plane $\frac{d\theta}{dt} < 0$. Therefore the motion along the paths with increasing t is in the counterclockwise direction inside the ring and in the clockwise direction outside the circle (17) and inside the circle (18) (Figure 103).

For $\alpha = -\frac{\pi}{2}$, the system has the same paths as for $\alpha = \frac{\pi}{2}$, but the direction of motion is reversed. As α further diminishes from $-\frac{\pi}{2}$ to $-\frac{3}{2}\pi$, we successively obtain the same configurations as in Figures 97 through 103, but with the direction of the arrows reversed. Finally, for $\alpha = -\frac{3}{2}\pi$, we return to the original system $(B_{\pi/2})$ (Figure 96).

The bifurcation values of the parameter α (for $\frac{\pi}{2} \geqslant \alpha \geqslant -\frac{\pi}{2}$) are clearly $\frac{\pi}{2}$, $\frac{\pi}{4}$, 0, and $-\frac{\pi}{2}$. Our analysis and the figures clearly show how the paths of the system evolve as the parameter α decreases (i.e., as the field vectors rotate clockwise) and what bifurcations the system experiences. On passing through the bifurcation value of the parameter $\frac{\pi}{2}$, an unstable

limit cycle appears — a circle of a large radius centered at O (this cycle, as we say, is created from infinity). As α decreases further, the cycle contracts (the radius of the circle monotonically diminishes). For the bifurcation value $\alpha = \frac{\pi}{4}$, the topological structure of the system is the same as before, but the stable structurally stable focus O changes into a multiple (structurally unstable) focus. When the system passes through the bifurcation value $\frac{\pi}{4}$, a stable limit cycle is created from this multiple focus, and the multiple stable focus changes into an unstable structurally unstable focus, while the existing unstable limit cycle continues contracting. After that, the stable limit cycle expands, and the unstable limit cycle contracts, and for the bifurcation value $\alpha = 0$ both cycles merge into a single multiple semistable cycle. When the system crosses the value $\alpha = 0$, this multiple cycle disappears, and further decrease of α between the limits $0 > \alpha > -\frac{\pi}{2}$ does not produce any additional change in the topological structure. We see from Figures 101—103 how the direction of motion along the paths is reversed. Indeed, when the system passes through $\alpha = 0$, a circle $\rho^2 = 1 + \sqrt{-\frac{\cos\alpha}{\sin\alpha}}$ appears from infinity on which this change occurs. This circle contracts as α diminishes. For $\alpha = -\frac{\pi}{4}$, the focus $O\,(0, 0)$ changes into a dicritical node (of the same stability). Further decrease of α produces a second circle near O on which the direction of rotation is again reversed, and the dicritical node changes back to a focus. Both circles move one toward the other. For $\alpha = -\frac{\pi}{2}$ they merge and the motion along the two circles is in the same direction, i.e., clockwise (Figure 96 with the direction of the arrows reversed).

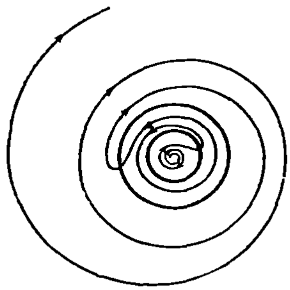

FIGURE 103. $-\frac{\pi}{4} > \alpha > -\frac{\pi}{2}$. No cycles. $R_1 < \sqrt{2}$, $R_2 < 1$ unstable focus.

Note that as α increases, the changes of topological structure occur in a reverse order. In particular, for small negative α, the system has no limit cycles (Figure 101). When α reaches the value 0, a semistable (multiple) limit cycle $x^2 + y^2 = 1$ appears (Figure 100). In this case we say that the limit cycle is created from path condensation (or path clustering). Further increase in α produces two limit cycles. In this case, we say that a multiple limit cycle decomposes into two cycles or an additional cycle is created from a multiple cycle. In our example we thus observed creation of cycles from infinity, from a multiple focus, from a multiple limit cycle, and from path condensation. In Chapters XI through XIII we shall encounter additional cases of limit cycle creation.

§23. DECOMPOSITION OF A MULTIPLE EQUILIBRIUM STATE INTO STRUCTURALLY STABLE EQUILIBRIUM STATES

1. The number of structurally stable equilibrium states obtained from a multiple equilibrium state

In the present section we will only consider analytical dynamic systems. Some propositions regarding multiple equilibrium states, however, will remain valid for systems of class N also. However, since the various proofs given for the analytical case either remain without change for systems of class N or are actually simplified, we will concentrate on analytical systems.

Let

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y) \tag{A}$$

be a dynamic system and $O(0, 0)$ its equilibrium state. We assume that $O(0, 0)$ is a multiple equilibrium state of multiplicity r, where r is a natural number, $r \geqslant 2$. By Definition 15 (§7.3) and Definition 5 (§2.1), an equilibrium state O of system (A) is of multiplicity r if the following conditions are satisfied:

(a) there exist numbers $\varepsilon_0>0$ and $\delta_0>0$ such that any system $(\tilde{A})$ δ_0-close to rank r to system (A) has at most r equilibrium states in $U_{\varepsilon_0}(O)$;

(b) for any $\varepsilon<\varepsilon_0$ and $\delta>0$, there exists a system $(\tilde{A})$ δ-close to rank r to (A) which has at least r equilibrium states in $U_\varepsilon(O)$.

From the definition of r-multiplicity of the equilibrium state O and from the condition $r \geqslant 2$ and Theorem 6 (§2.2) it follows that $O(0, 0)$ is an isolated equilibrium state in our case and that

$$\Delta(O)=\begin{vmatrix} P'_x(0, 0) & P'_y(0, 0) \\ Q'_x(0, 0) & Q'_y(0, 0) \end{vmatrix}=0. \tag{1}$$

In the following, $\varepsilon_0>0$ and $\delta_0>0$ are fixed numbers defined by condition (a). Moreover, we take ε_0 to be so small that $O(0, 0)$ is the only equilibrium state of (A) in $U_{\varepsilon_0}(O)$. If necessary, other conditions will also be imposed on ε_0 and δ_0, provided they do not clash with the basic requirement of smallness of these numbers.

Analytical dynamic systems δ_0-close to rank r to system (A) will be called *feasible* systems. Let C be the boundary of the neighborhood $U_{\varepsilon_0}(O)$. We take δ_0 to be so small that no feasible system $(\tilde{A})$ has equilibrium states on the circle C, and the vectors defined by systems (A) and $(\tilde{A})$ do not point in opposite directions at any point of this circle. Then by QT, §10.2, Lemma 2, the rotation $W_A(C)$ of the vector field of system (A) along the curve C is equal to the rotation $W_{\tilde{A}}(C)$ of the vector field of system $(\tilde{A})$ along the same curve, i.e.,

$$W_{\tilde{A}}(C)=W_A(C). \tag{2}$$

We will try to elucidate maximum information about the equilibrium states of a feasible system $(\tilde{A})$ which lie in $U_{\varepsilon_0}(O)$, if it is known that they all are structurally stable. We will first present some propositions which are

applicable to any r-tuple equilibrium state (Lemmas 1 and 2 and Theorems 31 and 32), and then restrict the range of equilibrium states by imposing one further condition.

If the feasible system $(\tilde{A})$ has precisely k equilibrium states $O_1, O_2, \ldots, O_k$ in the neighborhood $U_{\varepsilon_0}(O)$, and they are all structurally stable, we shall say that the multiple equilibrium state O (or the multiple singular point O) decomposes into structurally stable equilibrium states $O_1, O_2, \ldots, O_k$ on passing to system $(\tilde{A})$.

Lemma 1. If a feasible system $(\tilde{A})$ *has* r *equilibrium states in a neighborhood* $U_{\varepsilon_0}(O)$ *of an* r*-tuple equilibrium state* O*, all these equilibrium states are simple.*

Proof. Let the feasible system $(\tilde{A})$ have r equilibrium states $O_1, O_2, \ldots, O_r$ in $U_{\varepsilon_0}(O)$, and at least one of them, O_1 say, is a multiple equilibrium state. Let U_i be a neighborhood of O_i, $i = 1, 2, \ldots, r$, such that $U_i \subset U_{\varepsilon_0}(O)$ and no two U_i intersect. If the point O_i, $i = 2, 3, \ldots, r$, is not a simple equilibrium state of system $(\tilde{A})$, we can replace this system in a sufficiently small neighborhood V_i of the point O_i $(V_i \subset U_i)$ by a system $(\tilde{A}_i')$ as close as desired to $(\tilde{A})$ for which O_i is a simple equilibrium state.* Furthermore, in a sufficiently small neighborhood V_1 of the point $O_1, V_1 \subset U_1$, we can replace $(\tilde{A})$ by another system $(\tilde{A}_1')$, as close as desired to $(\tilde{A})$, which has in V_1 at least two equilibrium states O_1' and O_2'; this can be done since by Theorem 6 (§2.2) a multiple equilibrium state has a multiplicity higher than 1. By the first footnote to this page, O_1' and O_2' may be regarded as simple equilibrium states. We can further construct a system $(\tilde{A}')$ of class r, which would be as close to rank r as desired to system (A) and would coincide outside the neighborhoods U_i with $(\tilde{A})$ and inside each of the neighborhoods V_i with (A_i').** Let now $(\tilde{A}^*)$ be a dynamic system whose right-hand sides are polynomials providing a sufficiently good approximation to the right-hand sides of $(\tilde{A}')$. Evidently, $(\tilde{A}^*)$ is then a feasible system with at least $r+1$ equilibrium states in $U_{\varepsilon_0}(O)$. This contradicts the assumption that O is an r-tuple equilibrium state. The proof of the lemma is complete.

Lemma 2. If O *is an* r*-tuple equilibrium state of system* (A)*, there exist systems as close to rank* r *to system* (A) *as desired, which have* r *structurally stable equilibrium states in* $U_{\varepsilon_0}(O)$*.*

The validity of Lemma 2 follows directly from the definition of multiplicity of an equilibrium state and from the first footnote to this page.

Theorem 31. If $I_A(O) = I$ *is the Poincaré index of an* r*-tuple equilibrium state* O *of system* (A)*, then*

$$I \equiv r \pmod 2. \tag{3}$$

Proof. Let C be the boundary of the neighborhood $U_{\varepsilon_0}(O)$. By definition, the Poincaré index I is equal to the rotation $W_A(C)$ of the vector field

* Let the system have the form $\frac{dx}{dt} = a(x-x_i) + b(y-y_i) + \ldots$, $\frac{dy}{dt} = c(x-x_i) + d(y-y_i) + \ldots$ in the neighborhood of the point $O_i(x_i, y_i)$. If $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$, adding sufficiently small increments to a and d we obtain a close system with $\Delta \neq 0$, i.e., a system where O_i is a simple equilibrium state. Similarly, a structurally unstable point can be converted into a structurally stable point by an arbitrarily small change of the system.

** The possibility of constructing such a system of class r is established in Chapter VI (§18.3) in our proof to Theorem 24.

of system (A) along the curve C divided by 2π (QT, §11.2, Definition XIII and §10.2, Definition XI). According to the preceding lemma, there exists a feasible system $(\tilde{A})$ which has in $U_{\varepsilon_0}(O)$ precisely r equilibrium states, all of which are structurally stable. Furthermore, from equation (2)

$$W_{\tilde{A}}(C) = W_A(C) = 2\pi I.$$

On the other hand, by QT, §11.2, Lemma 1,

$$I = I_{\tilde{A}}(O_1) + I_{\tilde{A}}(O_2) + \ldots + I_{\tilde{A}}(O_r). \tag{4}$$

But every equilibrium state O_i of $(\tilde{A})$ is structurally stable, i.e., its index is either $+1$ or -1, so that it follows from the last equality that

$$I \equiv r \pmod 2.$$

The proof is complete.

Theorem 32. If an equilibrium state O of system (A) *is of multiplicity r, and a feasible system $(\tilde{A})$ has precisely k equilibrium states in $U_{\varepsilon_0}(O)$, all of which are structurally stable, then*

$$k \equiv r \pmod 2. \tag{5}$$

Proof. As in the previous theorem, we prove that

$$k \equiv I \pmod 2. \tag{6}$$

Relation (5) follows from (3) and (6). Q. E. D.

The previous lemmas and theorems are valid, as we have observed, for any r-tuple equilibrium state. We will now consider in some detail a relatively narrow, but extremely important class of equilibrium states. Throughout the remaining part of this section, we will assume that the series expansions of the right-hand sides — the functions P and Q — in the neighborhood of the relevant equilibrium state $O(0, 0)$ contain at least one linear term, i.e.,

$$|P'_x(0, 0)| + |P'_y(0, 0)| + |Q'_x(0, 0)| + |Q'_y(0, 0)| \neq 0. \tag{7}$$

To fix ideas, let

$$Q'_y(0, 0) \neq 0. \tag{8}$$

If $Q'_y(0, 0) = 0$, but $P'_x(0, 0)$ does not vanish, say, nothing changes significantly.

We will first derive the necessary and sufficient condition for the equilibrium state of the particular type to be of multiplicity r. Note that for $r = 1$ and $r = 2$, condition (7) is fulfilled automatically (Theorems 6 and 7, §2.2 and §2.3).

Theorem 33. Let $O(0, 0)$ be an equilibrium state of a system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

of class r (in particular, analytical system) and let at least one of the first derivatives of the functions P and Q, say $Q'_y(0, 0)$, not vanish at the point $O(0, 0)$. Let further $y = \varphi(x)$ be the solution of the equation

$$Q(x, y) = 0 \tag{9}$$

for y in some sufficiently small neighborhood of O, and*

$$\theta(x) = P(x, \varphi(x)). \tag{10}$$

*Then the point O is an r-tuple equilibrium state of system (A) if and only if the number 0 is an r-tuple root of the function $\theta(x)$.***

Proof of Theorem 33 is analogous to the proof of the corresponding proposition in Theorem 7 (§2.3). Using Lemma 1, §1.3 in our proof, we take for $\tilde{P}(x, y)$ the function

$$\alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{r-1} x^{r-1} + P(x, y)$$

with appropriately chosen coefficients α_i. Conditions (a) and (b) of Theorem 7 are represented by conditions (1) and (8), respectively, and are fulfilled in our case by assumption.

We now proceed to the next theorem. Let $O(0, 0)$ be an r-tuple equilibrium state of (A) for which condition (8) is fulfilled, and $\varepsilon_0 > 0$ and $\delta_0 > 0$ the numbers introduced in the preceding. We moreover assume that ε_0 is sufficiently small so that $Q'_y \neq 0$ in the neighborhood $U_{\varepsilon_0}(O)$, and the curve $Q(x, y) = 0$ may be defined in this neighborhood by an explicit equation $y = \varphi(x)$, where $\xi_1 < x < \xi_2$, $\xi_1 < 0$, $\xi_2 > 0$.

Theorem 34. *For any whole number k satisfying the inequalities $0 \leqslant k \leqslant r$, $k \equiv r \pmod 2$, and for any positive $\delta < \delta_0$ and $\varepsilon < \varepsilon_0$, there exists an analytical system ($\tilde{\mathrm{A}}$) which is δ-close to rank r to system (A) and has in $U_{\varepsilon_0}(O)$ precisely k, and at that structurally stable, equilibrium states, all of which moreover lie in $U_\varepsilon(O)$.*

Proof. Let, as before,

$$P(x, \varphi(x)) = \theta(x). \tag{10}$$

By Theorem 33, $x = 0$ is a root of multiplicity r of function $\theta(x)$. Therefore,

$$\theta(x) = Ax^r + \ldots = Ax^r(1 + f(x)), \tag{11}$$

where $A \neq 0$, and $f(x)$ is an analytical function, $f(0) = 0$.

Consider the system

$$\frac{dx}{dt} = \tilde{P}(x, y) = P(x, y) + p(x, y), \quad \frac{dy}{dt} = \tilde{Q}(x, y) = Q(x, y), \tag{12}$$

where

$$p(x, y) = A(x - x_1)(x - x_2) \ldots (x - x_k)(x^{r-k} + \alpha)(1 + f(x)) - \theta(x), \tag{13}$$

and k is an integer, $0 \leqslant k \leqslant r$, $k \equiv r \pmod 2$, $\alpha > 0$, $\xi_1 < x_i < \xi_2$, and all x_i are different ($p(x, y)$ is independent of y).

Since $\tilde{Q} \equiv Q$, the curve $\tilde{Q}(x, y) = 0$ coincides with the curve $Q(x, y) = 0$, i.e., in the neighborhood $U_{\varepsilon_0}(O)$ it may be expressed by an explicit equation

* The equation $Q(x, y) = 0$ has a single-valued solution $y = \varphi(x)$ in a sufficiently small neighborhood of O in virtue of condition (8) and the theorem of implicit functions; here, φ is a function of class r and $\varphi(0) = 0$.

** If $Q'_y(0, 0) = 0$, but $P'_x(0, 0)$, say, does not vanish, $\theta(x)$ should be replaced by a function $\theta^*(y) = Q(\varphi^*(y), y)$, where $x = \varphi^*(y)$ is the solution of the equation $P(x, y) = 0$ for x.

$y=\varphi(x)$, $\xi_1<x<\xi_2$. Therefore

$$\widetilde{\theta}(x)=\widetilde{P}(x, \widetilde{\varphi}(x))=\widetilde{P}(x, \varphi(x))=p(x, y)+P(x, \varphi(x))= \\ =A(x-x_1)(x-x_2)\ldots(x-x_k)(x^{r-k}+\alpha)(1+f(x)). \qquad (14)$$

By assumption, $O(0,0)$ is the only equilibrium state of (A) in $U_{\varepsilon_0}(O)$. It thus follows from (10) and (11) that $1+f(x)\neq 0$ for $\xi_1<x<\xi_2$. Moreover, $x^{r-k}+\alpha\neq 0$, since $\alpha>0$, and $r-k$ is even. But then we see from (14) that system (12) has in $U_{\varepsilon_0}(O)$ precisely k equilibrium states $O_i(x_i, \varphi(x_i))$, $i=1, 2, \ldots, k$. By (11) and (13),

$$p(x, y)=A(1+f(x))\{x^{r-k}[(x-x_1)(x-x_2)\ldots(x-x_k)-x^r]+ \\ +\alpha(x-x_1)(x-x_2)\ldots(x-x_k)\}.$$

If the numbers x_i, $i=1, 2, \ldots, k$, and α are sufficiently small, the function $p(x, y)$ in a finite interval is arbitrarily close to zero with its derivatives, and system (12) is therefore arbitrarily close to rank r to system (A). Moreover, for small x_i, all the points O_i lie in $U_\varepsilon(O)$.

We will now show that the equilibrium states $O_i(x_i, \varphi(x_i))$ of system (12) are all simple. Indeed, from the relations $\widetilde{P}(x, \varphi(x))=\widetilde{\theta}(x)$, $Q(x, \varphi(x))\equiv 0$ it follows that

$$\widetilde{\theta}'(x)=\frac{\begin{vmatrix}\widetilde{P}'_x(x, \varphi(x)) & \widetilde{P}'_y(x, \varphi(x))\\ Q'_x(x, \varphi(x)) & Q'_y(x, \varphi(x))\end{vmatrix}}{Q'_y(x, \varphi(x))}.^{*}$$

Therefore

$$\widetilde{\Delta}(O_i)=\widetilde{\Delta}(x_i, \varphi(x_i))=-\widetilde{\theta}'(x_i)Q'_y(x_i, \varphi(x_i)) \qquad (i=1, 2, \ldots, k).$$

But $Q'_y(x_i, \varphi(x_i))\neq 0$ by assumption, and $\widetilde{\theta}(x_i)$ does not vanish, since x_i is a simple root of the function $\widetilde{\theta}(x)$ (see (14)). Hence, $\widetilde{\Delta}(O_i)\neq 0$, i.e., the points O_i $(i=1, 2, \ldots, k)$ are simple equilibrium states of system (12).

If the points O_i are structurally stable equilibrium states, the proof is completed. If some of these are structurally unstable equilibrium states (i.e., multiple foci or centers) we can adopt the same technique as in Lemma 1 and change over to an arbitrarily close analytical system which has in $U_{\varepsilon_0}(O)$ precisely k equilibrium states, all of which are structurally stable and lie in $U_\varepsilon(O)$. This completes the proof of the theorem.

Remark. Theorems 32 and 34 naturally complement one another and completely determine the number of equilibrium states of a sufficiently close system which may lie in $U_{\varepsilon_0}(O)$ if they are all structurally stable. Theorem 32, however, has been proved for a general case, whereas in Theorem 34 we assumed condition (8) (or (7)). It is therefore interesting to try to establish whether or not Theorem 34 is applicable to the general case, i.e., when condition (7) is not satisfied.

2. The character of the structurally stable equilibrium states obtained from a multiple equilibrium state with $\sigma\neq 0$

In the previous section we determined the number of structurally stable singular points to which an r-tuple equilibrium state O of a dynamic

* We have previously encountered this formula in the proof of Theorem 7 (§2.3).

system (A) may decompose on passing to close systems. In this subsection we will try to elicit some information about the character of these structurally stable singular points. We assume that the multiple equilibrium state O is isolated and satisfies the condition

$$|P'_x(0, 0)|+|P'_y(0, 0)|+|Q'_x(0, 0)|+|Q'_y(0, 0)| \neq 0. \tag{7}$$

The topological structure of the dynamic system in the neighborhood of such an equilibrium state has been studied in detail in QT, §21 and §22. We will require the principal results from QT, which are reiterated below. We distinguish between two cases, $\sigma = 0$ and $\sigma \neq 0$.

(a) Let

$$\sigma = P'_x(0, 0)+Q'_y(0, 0) \neq 0. \tag{15}$$

In this case, system (A) is transformed by a non-singular linear transformation to the form

$$\frac{dx}{dt} = P_2(x, y), \qquad \frac{dy}{dt} = y + Q_2(x, y), \tag{16}$$

where P_2 and Q_2 are analytical functions whose series expansions in the neighborhood of the point $O(0, 0)$ consist of terms of not lower than second order.

Let

$$y = \varphi(x) \tag{17}$$

be the solution of the equation

$$y + Q_2(x, y) = 0 \tag{18}$$

in the neighborhood of $O(0, 0)$ and let the expansion of the function

$$\Psi(x) = P_2(x, \varphi(x)) \tag{19}$$

in powers of x have the form

$$\Psi(x) = \Delta_m x^m + \ldots, \tag{20}$$

where $m \geqslant 2$, and

$$\Delta_m \neq 0 \tag{21}$$

(the existence of these numbers m and Δ_m follows from the fact that the equilibrium state is isolated).

The following proposition applies:

I (QT, §21.2, Theorem 65)

1. If m is odd, and $\Delta_m > 0$, the equilibrium state O of system (16) is a topological node (Figure 104).
2. If m is odd, and $\Delta_m < 0$, O is a topological saddle point (Figure 105).
3. If m is even, the equilibrium state $O(0, 0)$ is a saddle-node, i.e., its canonical neighborhood consists of a parabolic and two hyperbolic sectors (Figures 106 and 107).

(b) Let

$$\sigma = P'_x(0, 0)+Q'_y(0, 0) = 0. \tag{22}$$

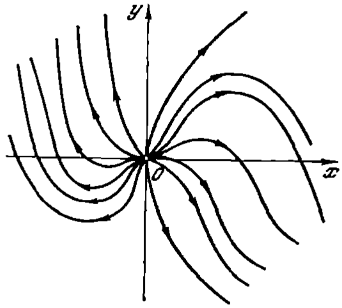

FIGURE 104. Odd m, $\Delta_m > 0$.

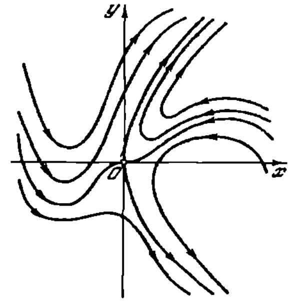

FIGURE 105. Odd m, $\Delta_m < 0$.

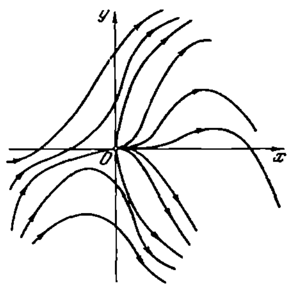

FIGURE 106. Even m, $\Delta_m > 0$.

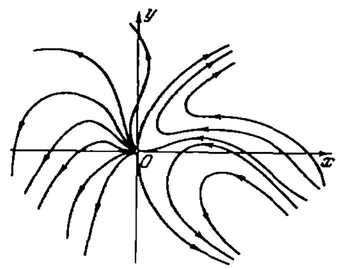

FIGURE 107. Even m, $\Delta_m < 0$.

In this case, system (A) can be transformed by a non-singular linear transformation to the form

$$\frac{dx}{dt} = y + P_2(x, y), \quad \frac{dy}{dt} = Q_2(x, y). \tag{23}$$

System (23) in its turn can be reduced to an even simpler form

$$\frac{d\xi}{dt} = \eta, \quad \frac{d\eta}{dt} = \widetilde{Q}_2(\xi, \eta)$$

by the transformation

$$\xi = x, \quad \eta = y + P_2(x, y),$$

which is one-to-one in the neighborhood of $O(0, 0)$. Reverting to the original notation, we may thus consider a system of the form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = Q_2(x, y), \tag{24}$$

where the series expansion of $Q_2(x, y)$ contains no linear terms.

Since $O(0, 0)$ is by assumption an isolated equilibrium state of system (24), this system can be written in the form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = ax^r[1 + h(x)] + bx^n y[1 + g(x)] + y^2 f(x, y), \tag{25}$$

where $h(x)$, $g(x)$, $f(x, y)$ are analytical functions; $h(0) = g(0) = 0$; $r \geqslant 2$; $a \neq 0$; b may be equal to zero; if $b \neq 0$, we have $n \geqslant 1$.

In this case, the following proposition applies:

II (QT, §22.2, Theorems 66 and 67)

1. If r is odd, the equilibrium state O is either a topological saddle point, or a topological node, or a focus (a center), or finally an equilibrium state with an elliptical region (whose neighborhood contains one hyperbolic and one elliptical sector, Figure 108).

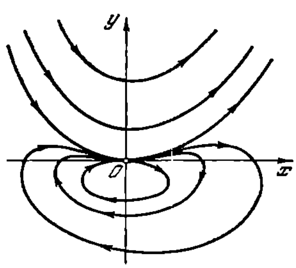

FIGURE 108

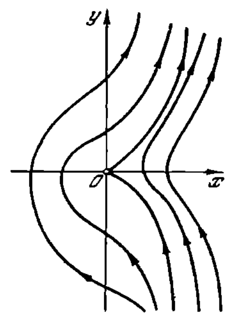

FIGURE 109

2. If r is even, O is either a degenerate equilibrium state (two hyperbolic sectors, Figure 109), or a saddle-node (Figure 110).

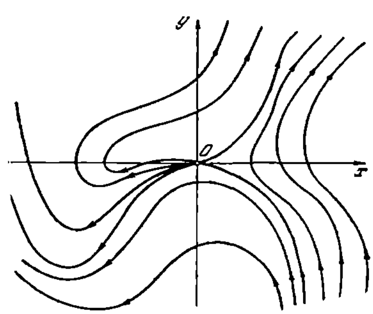

FIGURE 110

Note that by Theorem 33 and the condition $a \neq 0$, the number r is the multiplicity of the equilibrium state O.

We can now consider the character of the structurally stable equilibrium states obtained from a multiple singular point. First we consider the case when

$$\sigma = P'_x(0, 0) + Q'_y(0, 0) \neq 0. \tag{15}$$

The solution of the problem is very simple, since the character of the component structurally stable states is entirely determined by the Poincaré index of the singular point O. We may assume without loss of generality that the system has the form (16).

Let H be the number of hyperbolic, and E the number of elliptical sectors of a canonical neighborhood of the equilibrium state O. According to Bendixson's formula, the Poincaré index of the point O is

$$I = \frac{E - H}{2} + 1 \tag{26}$$

(see QT, Appendix, §10).

In our case, as we see from (16), $\sigma = 1$ and $\Delta = 0$. Let $(\tilde{A})$ be a dynamic system δ_0-close to (A) to the required rank, and $O_1, O_2, \ldots, O_k$ the equilibrium states of this system lying in $U_{\varepsilon_0}(O)$, all of which are structurally stable. If the numbers δ_0 and ε_0 are sufficiently small, the number $\sigma_i = \sigma(O_i)$ is close to 1 for each of the equilibrium states O_i, $i = 1, 2, \ldots, k$, and $\Delta_i = \Delta(O_i)$ is close to zero. Then, if $\Delta(O_i) > 0$, O_i is a structurally stable node, and if $\Delta_i(O_i) < 0$, O_i is a structurally stable saddle point. Thus, in our case, all the structurally stable equilibrium states into which a multiple equilibrium state O decomposes are structurally stable nodes and saddle points.

By Theorem 33 and relations (20) and (21), the multiplicity of the equilibrium state $O(0, 0)$ of system (16) is m. Evaluating the Poincaré index $I = I_A(O)$ of the point O from (26), we obtain the following results:

1) If O is a topological node, $I = 1$ $(E = H = 0)$.
2) If O is a topological saddle point, $I = -1$ $(E = 0,\ H = 4)$.
3) If O is a saddle-node, $I = 0$ $(E = 0,\ H = 2)$.*

On the other hand, we know (see the proof of Theorem 31) that

$$I_A(O) = I = I_{\tilde{A}}(O_1) + I_{\tilde{A}}(O_2) + \ldots + I_{\tilde{A}}(O_k). \tag{27}$$

Since the Poincaré indices of a structurally stable node and a structurally stable saddle point are +1 and −1, respectively, equation (27) directly gives the number of structurally stable nodes and structurally stable saddle points among the points $O_1, O_2, \ldots, O_k$ in each of the cases 1, 2, 3 above. The value of the number k is completely fixed by Theorems 32 and 34.

All the various results can now be summarized in the following theorem.

Theorem 35. Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

* Note that the Poincaré index of the equilibrium state $O(0, 0)$ of system (16) can be readily computed without using Bendixson's formula (26). To this end, we should count the number of times the field vector of the dynamic system crosses the direction of the y axis while moving along a circle of small radius centered at O. See Kranosel'skii et al. /19/, §7.

be a dynamic system, $O(0, 0)$ *an equilibrium state of this system for which* $\Delta = 0$ *and* $\sigma = P'_x(0, 0) + Q'_y(0, 0) \neq 0$, *and let* $m \geqslant 2$ *be the multiplicity of the equilibrium state* O. *Then the number* k *of structurally stable equilibrium states* $O_1, O_2, \ldots, O_k$ *into which the multiple equilibrium state* O *decomposes on passing to arbitrarily close systems* satisfies the conditions*

$$0 \leqslant k \leqslant m \quad \text{and} \quad k \equiv m \ (\mathrm{mod}\ 2)$$

and may be equal to any number satisfying these conditions. Each of the points O_i $(i = 1, 2, \ldots, k)$ *is either a structurally stable node or a structurally stable saddle point. Moreover,*

1) if O *is a topological node,* k *is odd and the number of structurally stable nodes among the points* $O_1, O_2, \ldots, O_k$ *is 1 more than the number of structurally stable saddle points;*

2) if O *is a topological saddle point,* k *is odd and the number of structurally stable nodes among the points* $O_1, O_2, \ldots, O_k$ *is 1 less than the number of structurally stable saddle points;*

3) if O *is a saddle-node,* k *is even and the number of structurally stable nodes among the points* $O_1, O_2, \ldots, O_k$ *is equal to the number of structurally stable saddle points.*

Remark. Theorem 35 fully determines the types of the structurally stable equilibrium states which are obtained from a multiple equilibrium state $O(0, 0)$ in the case $\sigma = P'_x(0, 0) + Q'_y(0, 0) \neq 0$. It follows from this theorem, in particular, that the topological structure of the equilibrium state O in this case is uniquely determined by its Poincaré index, which is equal to the difference between the number of structurally stable nodes and the number of structurally stable saddle points obtained from the multiple equilibrium state.

3. The character of the structurally stable equilibrium states obtained from a multiple equilibrium state with $\sigma = 0$

In our proof of the basic theorems of this subsection, we will make use of one proposition which belongs to Poincaré (/15/, p. 43, Theorem V) and is also of independent interest.

Theorem 36 (Poincaré theorem). If a dynamic system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

has only simple equilibrium states and if the isocline $P(x, y) = 0$ *(or* $Q(x, y) = 0$*) has no singular points (i.e., points at which both partial derivatives* P'_x *and* P'_y *vanish simultaneously), the equilibrium states with* $\Delta < 0$, *i.e., saddle points, alternate on this isocline with equilibrium states with* $\Delta > 0$, *i.e., nodes and foci.*

Proof. Let $O_1(x_1, y_1)$ and $O_2(x_2, y_2)$ be two simple equilibrium states of (A) and l a simple arc of the curve $P(x, y) = 0$ between these points, which contains no other equilibrium states except the end points. We have to show that $\Delta(O_1)$ and $\Delta(O_2)$ are of different signs. Suppose that this is not so and let $\Delta(O_1) > 0$ and $\Delta(O_2) > 0$, say.

* This number is defined before Lemma 1.

Consider the vectors $\mathbf{n}(M)=\mathbf{n}(x, y)$ with the coordinates $P'_x(x, y), P'_y(x, y)$, where $M(x, y)$ is a point of the curve $P(x, y) = 0$. These vectors form a continuous field of normals on the arc l, and no vector of this field is zero (i.e., the field has no singularities on the arc l). Consider the function $Q(x, y) = Q(M)$ and its gradient $\mathbf{grad}\, Q(M)$. Since O_1 and O_2 are equilibrium states of (A), we have $Q(O_1) = Q(O_2) = 0$. Furthermore, the conditions $\Delta_1 > 0$ and $\Delta_2 > 0$ clearly indicate that the vector $n(O_i)$ makes a positive angle with the vector $\mathbf{grad}\, Q(O_i)$, $i=1, 2$ (Figure 111).

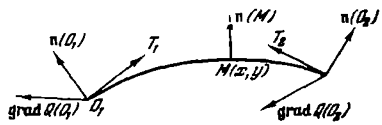

FIGURE 111

To fix ideas, suppose that the tangent O_1T_1 at the point O_1 of the curve l, corresponding to its direction from O_1 to O_2, makes a positive angle with the normal $\mathbf{n}(O_1)$. Then the tangent O_2T_2 at the point O_2 of the curve l, corresponding to the direction from O_2 to O_1, evidently makes a negative angle with the normal $\mathbf{n}(O_2)$. Computing the derivative of the function $Q(x, y)$ along the curve l in the direction O_1O_2 (O_2O_1) at the point O_1 (O_2) and remembering that this derivative is equal to the projection of the gradient of $Q(x, y)$ on the corresponding tangent, we find that this derivative is negative at the point O_1 and positive at the point O_2. This result, combined with the relations

$$Q(O_1) = Q(O_2) = 0$$

and with the definition of a derivative along an arc shows that the function $Q(x, y)$ is negative on the arc l near the point O_1 and positive on this arc near the point O_2. But then $Q(x, y) = 0$ at some interior point of the arc l, i.e., this point is an equilibrium state of (A), contrary to the original assumption. Q. E. D.

Let us now consider the equilibrium state $O(0, 0)$ assuming that

$$|P'_x(0, 0)| + |P'_y(0, 0)| + |Q'_x(0, 0)| + |Q'_y(0, 0)| \neq 0 \tag{7}$$

and

$$\sigma = P'_x(0, 0) + Q'_y(0, 0) = 0. \tag{22}$$

The problem is much more difficult now than in the previous case, and it entails a number of special algebraic propositions. We will therefore list a number of relevant results without proof. Detailed proof will be found in /16/.

The complications are associated with two factors. First, if a singular point O decomposes into structurally stable equilibrium states O_i $(i = 1, 2, \ldots, k)$, $\Delta(O_i)$ and $\sigma(O_i)$ may take any arbitrarily small values. In particular, the difference $\sigma(O_i)^2 - 4\Delta(O_i)$ may be either positive or negative.

This means that the points O_i may in general include structurally stable foci, and not only structurally stable saddle points of nodes. Second, the Poincaré index of the equilibrium state O no longer determines the topological structure of this equilibrium state (as it was in the previous case). Indeed, by proposition II of the previous subsection, if the multiplicity r of O is odd, this point may be

a) a topological saddle point ($E = 0$, $H = 4$);
b) a topological node ($E = 0$, $H = 0$);
c) an equilibrium state with an elliptical region ($E = 1$, $H = 1$);
d) a focus or a center ($E = 0$, $H = 0$).

If the miltiplicity r of O is even, the equilibrium state O may be

e) a degenerate equilibrium state ($H = 2$, $E = 0$);
f) a saddle-node ($H = 2$, $E = 0$).

By Bendixson's formula, the Poincaré index of the point O has the following values:

$$\begin{aligned} I &= -1 \text{ in case a;} \\ I &= +1 \text{ in cases b, c, d;} \\ I &= 0 \text{ in cases e, f.*} \end{aligned} \tag{28}$$

The topological structure of the equilibrium state is therefore completely determined by the Poincaré index only if it is equal to -1, i.e., O is a saddle point. The characteristic of the point O in terms of the component structurally stable equilibrium states for case a is the same as in Theorem 35, provided we do not distinguish between structurally stable nodes and structurally stable foci, i.e., between structurally stable equilibrium states with $\Delta > 0$.** Using Theorems 32 and 34, we obtain the following result.

Theorem 37. A multiple equilibrium state $O(0, 0)$ *of system* (A) *for which condition* (7) *is satisfied but* $\sigma = P'_x(0, 0) + Q'_y(0, 0) = 0$ *is a topological saddle point if and only if the number of structurally stable nodes and foci into which it decomposes is 1 less than the number of structurally stable saddle point. The total number* k *of structurally stable equilibrium states into which* O *decomposes may be any positive odd number not exceeding* r, *where* r *is the multiplicity of* O.

For the sake of completeness, the next step in our analysis should have been aimed at determining whether certain relations exist between the number of structurally stable nodes and the number of structurally stable foci or each number may take any value between 0 and $\frac{k-1}{2}$. We will not consider this problem, however.

In cases b, c, d, as it follows from (28) and from the previous results, the number of nodes and foci among the structurally stable equilibrium states $O_1, O_2, \ldots, O_k$ is 1 greater than the number of saddle points, and in cases e and f the number of nodes and foci is equal to the number of saddle points.† In terms of decomposition into structurally stable

* As for system (16), the Poincaré index of the equilibrium state O of system (24) can be readily computed without resorting to Bendixson's formula. See footnote on p. 226.

** Note that from a pure topological point of view, no such distinction is possible, since the node and the focus have an identical topological structure.

† We recall that $O_1, O_2, \ldots, O_k$ are the structurally stable equilibrium states into which the multiple equilibrium state O decomposes.

equilibrium states, we can thus distinguish between points b, c, d, on the one hand, and points e, f, on the other. In similar terms, we can distinguish between points of type e and points of type f, and also fully characterize each of the types b, c, d. However, the resulting classification is not pure topological, since we wish to differentiate between structurally stable nodes and structurally stable foci obtained from the multiple point O. Moreover, we are forced to consider in this case the decomposition into the maximum number of structurally stable equilibrium states (equal to the multiplicity of the original equilibrium state), and not into any possible number, as before. To emphasize the last factor, we will refer to feasible system $(\tilde{A})$ as a split system if on passing from (A) to $(\tilde{A})$ the singular point O of multiplicity r decomposes into r structurally stable equilibrium states.

Let us first formulate a result relating to a quadruple equilibrium state ($I = 0$, cases e and f).

Theorem 38. *Let* (A) *be a dynamic system,* $O(0, 0)$ *a multiple equilibrium state of the system of multiplicity* $r = 2m$, $m \geqslant 1$, *for which condition* (7) *is satisfied and* $\sigma = P'_x(0, 0) + Q'_y(0, 0) = 0$ *(i.e., a degenerate equilibrium state or a saddle-node). Then*

1) If the equilibrium state O *decomposes into* k *structurally stable equilibrium states* O_i, $i = 1, 2, \ldots, k$, *on passing to a feasible system,* k *is even and the number of structurally stable saddle points among* O_i *is equal to the number of structurally stable nodes and foci.*

2) If O *is a degenerate equilibrium state, there exist split systems arbitrarily close to* (A) *for which* O *decomposes only into structurally foci and saddle points.*

3) If O *is a saddle-node, the structurally stable equilibrium states into which* O *decomposes on passing to any sufficiently close split system include at least one structurally stable node.*

Proof. The validity of the first proposition follows from Theorem 32 and equation (28) and has in fact been established before.

Let us prove the second proposition. Without loss of generality, we may assume (see §3.2, Lemma 2) that system (A) has the form (25):

$$\frac{dx}{dt} = y = P(x, y), \quad \frac{dy}{dt} = ax^{2m}[1 + h(x)] + bx^n y[1 + g(x)] + y^2 f(x, y) = Q(x, y),$$

where $h(x)$, $g(x)$, $f(x, y)$ are analytical functions, $h(0) = g(0) = 0$, $m \geqslant 1$; $a > 0$; $n \geqslant 1$ if $b \neq 0$. Let $O(0, 0)$ be a degenerate equilibrium state. In QT (§22.2, Theorem 67) it is proved that in this case

$$\begin{aligned} &\text{either } b = 0, \\ &\text{or } b \neq 0 \text{ and } n \geqslant m. \end{aligned} \tag{29}$$

Let, first, $b \neq 0$. We take some positive number $\eta > 0$ and a sequence of arbitrary numbers x_i, $i = 1, 2, \ldots, n-1$, such that

$$0 < x_1 < x_2 < \ldots < x_{n-1} < \eta. \tag{30}$$

We draw up a polynomial

$$b(x) = bx(x - x_1)(x - x_2)\ldots(x - x_{n-1}) \equiv bx^n + b_1 x^{n-1} + \ldots + b_{n-1} x, \tag{31}$$

whose roots are the numbers $0, x_1, x_2, \ldots, x_{n-1}$.

By (29), $m-1\leqslant n-1$. Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ be any numbers satisfying the inequalities

$$0<\bar{x}_1<x_1<\bar{x}_2<x_2<\ldots<x_{m-2}<\bar{x}_{m-1}<x_{m-1}<\bar{x}_m<\eta. \tag{32}$$

We now construct a polynomial

$$a(x)=ax(x-x_1)\ldots(x-x_{m-1})(x-\bar{x}_1)\ldots(x-\bar{x}_m)=ax^{2m}+a_1x^{2m-1}+\ldots+a_{2m-1}x, \tag{33}$$

whose roots are the numbers 0, x_i, $\bar{x}_j$. Clearly, if η is sufficiently small, the coefficients b_i and a_j may be made as small as desired.

Consider a system

$$\begin{aligned}
\frac{dx}{dt}&=y=\tilde{P}(x, y),\\
\frac{dy}{dt}&=(ax^{2m}+a_1x^{2m-1}+\ldots+a_{2m-1}x)[1+h(x)]+\\
&\quad+(bx^n+b_1x^{n-1}+\ldots+b_{n-1}x)y[1+g(x)]+y^2f(x, y)=\\
&=a(x)[1+h(x)]+b(x)y[1+g(x)]+y^2f(x, y)=\tilde{Q}(x, y).
\end{aligned} \tag{$\tilde{\mathrm{A}}$}$$

We take $\eta>0$ to be so small that the following conditions are satisfied:

$$\left.\begin{array}{l}
\text{(a) } \eta<\varepsilon_0;\\
\text{(b) } (\tilde{\mathrm{A}}) \text{ is a feasible system;}\\
\text{(c) if } |x|<\eta, \text{ then } 1+h(x)>0.
\end{array}\right\} \tag{34}$$

All the equilibrium states of system $(\tilde{\mathrm{A}})$ lie on the axis Ox. From (33) and (34, c) it follows that all the equilibrium states of $(\tilde{\mathrm{A}})$ lying in $U_\eta(O)$ are

$$O(0, 0),\quad O_i(x_i, 0),\quad i=1, 2, \ldots, m-1, \text{ and } \bar{O}_j(\bar{x}_j, 0),\quad j=1, 2, \ldots, m. \tag{35}$$

System $(\tilde{\mathrm{A}})$ thus has $2m$ equilibrium states in $U_\eta(O)$. But $U_\eta(O)\subset U_{\varepsilon_0}(O)$ by (34, a). Therefore $(\tilde{\mathrm{A}})$ is a split system, and by Lemma 1 all the equilibrium states (35) are simple. Let us identify their character. To this end, we will compute the values of Δ and σ for each of these states.

Direct computations show that if $O^*(x^*, 0)$ is an equilibrium state of $(\tilde{\mathrm{A}})$, then

$$\Delta(O^*)=-\tilde{Q}'_x(x^*, 0)=-a'(x^*)[1+h(x^*)]-a(x)h'(x^*) \tag{36}$$

and

$$\sigma(O^*)=\tilde{Q}'_y(x^*, 0)=b(x^*)[1+g(x^*)]. \tag{37}$$

If O^* is one of the points in (35), we have $a(x^*)=0$ and

$$\Delta(O^*)=-a'(x^*)[1+h(x^*)]. \tag{38}$$

Computing $a'(x)$, we insert for x^* the abscissas of the points in (35), and using the inequalities (32), (34, c) and the condition $a>0$, we obtain from the last expression

$$\begin{aligned}
&\Delta(O)>0;\quad \Delta(O_i)>0,\quad i=1, 2, \ldots, m-1;\\
&\Delta(\bar{O}_j)<0,\quad j=1, 2, \ldots, m.
\end{aligned} \tag{39}$$

Furthermore, since $x=0$ and $x=x_i$, $i=1, 2, \ldots, m-1$, are roots of the polynomial $b(x)$, we see from (37) that

$$\sigma(0)=0, \quad \sigma(x_i)=0, \quad i=1, 2, \ldots, m-1. \tag{40}$$

Relations (39) and (40) show that the points $\bar{O}_j$, $j = 1, 2, \ldots, m$, are structurally stable saddle points, and the points O and O_i, $i = 1, 2, \ldots, \ldots, m-1$, are multiple foci of $(\tilde{\mathrm{A}})$. In accordance with the standard reasoning (see, e.g., the proof of Lemma 1), there exists a system $(\bar{\tilde{\mathrm{A}}})$ arbitrarily close to $(\tilde{\mathrm{A}})$ for which $\bar{O}_j$ remain structurally stable saddle points, whereas O and O_i are structurally stable foci. Clearly, $(\bar{\tilde{\mathrm{A}}})$ is a split system and, if η is sufficiently small, $(\bar{\tilde{\mathrm{A}}})$ is as close as desired to (A). This proves the second proposition of the theorem for the case $b\neq 0$.

If $b=0$, we do not need to construct the polynomial $b(x)$, and $x_1, x_2, \ldots, x_{m-1}$ can be taken as any positive numbers smaller than η.

We now proceed to the third proposition of the theorem. Let the equilibrium state $O(0, 0)$ of system (A) be a saddle-node. By QΓ, §22.2, Theorem 67, we have in this case

$$b\neq 0 \text{ and } 1\leqslant n<m. \tag{41}$$

We will divide the proof into two parts.

a) Consider a system

$$\begin{aligned} &\frac{dx}{dt}=y,\\ &\frac{dy}{dt}=\hat{Q}(x, y)=Q(x, y)+q(x, y)=\\ &\qquad=ax^{2m}[1+h(x)]+bx^n y[1+g(x)]+y^2 f(x, y)+q(x, y), \end{aligned} \tag{$\hat{\mathrm{A}}$}$$

where $b\neq 0$ and $1\leqslant n<m$. We will first prove that if $(\hat{\mathrm{A}})$ is sufficiently close to rank 2 -1 to system (A) and has in $U_{\varepsilon_0}(O)$ precisely $2m$ equilibrium states, which are all structurally stable, then at least one of these equilibrium states is a node.

Evidently, closeness to rank $2m-1$ of (A) and $(\hat{\mathrm{A}})$ indicates that the function $q(x, y)$ is close to rank $2m-1$ to zero.

Let $O_i(x_i, 0)$, $i = 1, 2, \ldots, 2m$, be the equilibrium states of $(\hat{\mathrm{A}})$ in $U_{\varepsilon_0}(O)$. Consider the values of Δ, σ, and $\lambda=\sigma^2-4\Delta$ corresponding to each of these equilibrium states. Clearly,

$$\Delta(x, 0)=-\hat{Q}'_x(x, 0)=-q'_x(x, 0)-2\,max^{2m-1}[1+h(x)]-ax^{2m}h'(x), \tag{42}$$

$$\sigma(x, 0)=\hat{Q}'_y(x, 0)=q'_y(x, 0)+bx^n[1+g(x)]. \tag{43}$$

It follows from these relations that

$$\lambda(x)=\sigma^2-4\Delta=b^2x^{2n}+\varphi_1(x)+\varphi_2(x)=b^2x^{2n}+\varphi(x), \tag{44}$$

where

$$\varphi_1(x)=2b^2x^{2n}g(x)+b^2x^{2n}g^2(x)+8\,max^{2m-1}[1+h(x)]+4ax^{2m}h'(x) \tag{45}$$

and

$$\varphi_2(x) = q_y'^2(x, 0) + 2q_y'(x, 0)\, bx^n\,[1 + g(x)] + 4q_x'(x, 0). \tag{46}$$

From $n < m$ it follows that $2n < 2m - 1$. Since, moreover, $g(0) = 0$, the function $\varphi_1(x)$ may be written in the form

$$\varphi_1(x) = x^{2n+1}\bar{\varphi}_1(x),$$

where $\bar{\varphi}_1(x)$ is an analytical function. Consequently, there exists a number ε_1 satisfying the following conditions:

$$\left.\begin{array}{l} \text{(a) } 0 < \varepsilon_1 < \varepsilon_0; \\ \text{(b) } b^2x^{2n} + \varphi_1(x) > 0 \quad \text{for } x = \varepsilon_1 \text{ and for } x = -\varepsilon_1; \\ \text{(c) } |\varphi_1^{(2n)}(x)| < \frac{(2n)!\, b^2}{2} \quad \text{for } -\varepsilon_1 \leqslant x \leqslant \varepsilon_2. \end{array}\right\} \tag{47}$$

We now choose a number δ_1, $0 < \delta_1 < \delta_0$, satisfying the following condition: if the function $q(x, y)$ is δ_1-close to rank $2m-1$ to zero, then

$$\left.\begin{array}{l} \text{1) } \lambda(-\varepsilon_1) > 0,\ \lambda(\varepsilon_1) > 0; \\ \text{2) } \varphi_2^{(2n)}(x) < \frac{(2n)!\, b^2}{2} \quad \text{for } -\varepsilon_1 \leqslant x \leqslant \varepsilon_2; \\ \text{3) all the equilibrium states } O_i,\ i = 1, 2, \ldots, 2m, \text{ lie in } U_{\varepsilon_1}(O). \end{array}\right\} \tag{48}$$

Condition (48,1) can be satisfied in virtue of (44) and (47, b). Condition (48, 2) holds true for a sufficiently small δ_1 in virtue of (46) and the inequality $2n + 1 \leqslant 2m - 1$. Finally, (48, 3) is automatically satisfied for a sufficiently small δ_1.

244 Let now $(\tilde{A})$ be δ_1-close to (A) to rank $2m - 1$, i.e., the function $q(x, y)$ is δ_1-close to zero to rank $2m - 1$. Consider the function

$$\lambda(x) = b^2x^{2n} + \varphi(x) \tag{49}$$

on the segment $[-\varepsilon_1, \varepsilon_1]$.

From the equality

$$\varphi(x) = \varphi_1(x) + \varphi_2(x) \tag{50}$$

and from (47,c) and (48, 2) it follows that on this segment

$$|\varphi^{(2n)}(x)| < (2n)!\, b^2. \tag{51}$$

The function $\lambda(x)$ therefore has at most $2n$ roots on the segment $[-\varepsilon_1, \varepsilon_1]$.* Let $\xi_1, \xi_2, \ldots, \xi_k$ be all the different roots of $\lambda(x)$ on this segment (some of them may be multiple roots). Consider the $k + 1$ intervals into which the roots ξ_j divide the segment $[-\varepsilon_1, \varepsilon_1]$. In each of these intervals, the function $\lambda(x)$ retains a constant sign. By (48,1), $\lambda(x) > 0$ in the first and the last of these intervals. The intervals where $\lambda(x) < 0$ will be called negative intervals and designated $J_1, J_2, \ldots, J_l$ (Figure 112). We will now prove that $l \leqslant n$. Indeed, both ends of each negative interval are roots of the function $\lambda(x)$. If the intervals J_j and J_{j+1} have a common end point ξ_s

* If the function has N roots on some segment (counting their multiplicities), its derivative has at least $N-1$ roots. This follows from the Rolle theorem and from the fact that each multiple root of a function is a root of the derivative of multiplicity smaller than 1.

(e.g., the intervals J_2 and J_3 in Figure 112), ξ_s is a root of even multiplicity of $\lambda(x)$, i.e., it has a multiplicity of at least 2. Therefore, the total number of roots of the function $\lambda(x)$ on $[-\varepsilon_1, \varepsilon_1]$ is at least $2l$. Since, on the other hand, the number of roots is at most $2n$, we see that $l \leqslant n$. Then by (41)

$$l < m. \tag{52}$$

All the equilibrium states O_i, $i = 1, 2, \ldots, 2m$, lie in $U_{\varepsilon_1}(O)$ (see (48)) and are structurally stable and therefore simple. If $O_i(x_i, 0)$ lies in one of the negative intervals, $\Delta(O_i) = \Delta_i > 0$, since otherwise $\lambda(x_i) = \lambda_i = \sigma_i^2 - 4\Delta_i > 0$. But then by Theorem 36 (Poincaré theorem), each negative interval J_j contains at most one point O_i. This signifies, as we see from inequality (52), that the number of equilibrium states O_i with $\lambda < 0$, i.e., the number of foci, is less than m. On the other hand, the first proposition of the theorem indicates that the total number of nodes and foci among the points O_i is equal to the number of saddle points m. Hence, there is at least one node among the points O_i, which proves the third proposition for system $(\hat{A})$.

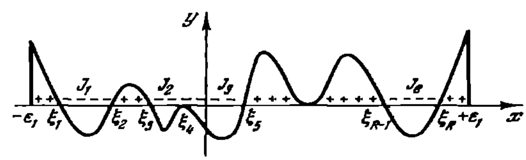

FIGURE 112

b) Let us now consider the general case. Let

$$\frac{dx}{dt} = y + p(x, y) = \tilde{P}(x, y), \quad \frac{dy}{dt} = Q(x, y) + q(x, y) = \tilde{Q}(x, y) \tag{$\tilde{A}$}$$

be a dynamic system δ-close to rank $2m$ to system (A) which has in $U_{\varepsilon_0}(O)$ precisely $2m$ equilibrium states, all of which are structurally stable. We have to prove that if δ is sufficiently small, these equilibrium states contain at least one node. Let the equilibrium states be $O_1, O_2, \ldots, O_{2m}$.

Consider the transformation

$$X = x, \quad Y = y + p(x, y). \tag{53}$$

We assume that the region $\overline{G}$ in which all the systems are treated is convex in y* and that $\delta < 1$. Then, as is readily seen, the mapping (53) is one-to-one and regular, moving $\overline{G}$ into some region $\overline{H}$ in the plane (X, Y). Indeed, if two different points (x_1, y_1) and (x_2, y_2) are mapped by (53) onto the same point, then $x_1 = x_2$, $y_1 \neq y_2$ and $y_1 + p(x_1, y_1) = y_2 + p(x_1, y_2)$. But then $y_2 - y_1 = p(x_1, y_1) - p(x_1, y_2) = (y_1 - y_2)\, p'_y(x_1, y^*)$, where $y_1 < y^* < y_2$ (or $y_1 > y^* > y_2$). Therefore $|p'_y(x, y^*)| = 1$, which contradicts the condition $\delta < 1$.

Transformation (53) changes $(\tilde{A})$ into the system

$$\frac{dX}{dt} = Y, \quad \frac{dY}{dt} = \hat{Q}(X, Y).$$

* Convexity in y indicates that if the end points of a segment parallel to the axis Oy lie in G, the entire segment lies in G.

Let us consider X and Y as coordinates in the plane (x, y), i.e., we replace X, Y with x, y, respectively. This gives a system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \hat{Q}(x, y), \tag{Â}$$

defined in $\overline{H}$.

Transformation (53) is δ-close to rank $2m$ to the identity transformation $X = x$, $Y = y$ (see §3.2). Let $\overline{G}_1$ be a closed region, such that $G \supset \overline{G}_1 \supset U_{\varepsilon_0}(O)$. If δ is sufficiently small, we see by Lemma 1, §3.2 that system (Â) is defined in $\overline{G}_1$ and is arbitrarily close to system ($\tilde{\mathrm{A}}$), and hence to system (A), to rank $2m-1$. To every equilibrium state O_i, $i = 1, 2, \ldots, 2m$, of system ($\tilde{\mathrm{A}}$) corresponds an equilibrium state $\hat{O}_i$ of system (Â). Since (53) is a regular mapping, $\hat{O}_i$ are structurally stable equilibrium states by Lemma 2, §6.1, and the corresponding pairs O_i and $\hat{O}_i$ are either both nodes, or both foci, or both saddle points. Finally, for a sufficiently small δ, all the equilibrium states $\hat{O}_i$ of system (Â) lie in $U_{\varepsilon_0}(O)$ and there are no other equilibrium states of ($\tilde{\mathrm{A}}$) in this neighborhood.

But then system (Â) satisfies all the conditions of proposition a (see p. 232), i.e., among the points $\hat{O}_i$, and therefore among O_i, there is at least one structurally stable node. This completes the proof of the theorem.

In Theorems 37 and 38 we considered cases when the Poincaré index of the original equilibrium state $O(0, 0)$ is -1 or 0. Let us now proceed to the last case, when the Poincaré index is $+1$. The equilibrium state $O(0, 0)$ of system (A), as we have noted before (see (28)), is then one of the following:

b) a topological node;

c) an equilibrium state with an elliptical region;

d) a focus or a center (see p. 229).

In §23.2 we saw that the relevant system can be reduced to the form (25):

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = ax^r[1 + h(x)] + bx^n y[1 + g(x)] + y^2 f(x, y).$$

It is established in QT, Chapter IX, §22.2, Theorems 66 and 67, that cases b, c, d obtain if $a < 0$ and r is an odd number. We may thus take system (A) in the form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = ax^{2m+1}[1 + h(x)] + bx^n y[1 + g(x)] + y^2 f(x, y), \tag{54}$$

where $a < 0$, $m \geqslant 1$, h, g, and f are analytical functions, and $h(0) = f(0) = 0$. Then (see QT, Theorem 66)

b) $O(0, 0)$ is a topological node if

$$b \neq 0, \quad n \text{ is even, and } n < m \tag{55}$$

or if

$$b \neq 0, \quad n \text{ is even, } n = m \text{, and } D = b^2 + 4(m+1)a > 0; \tag{56}$$

c) $O(0, 0)$ is an equilibrium state with an elliptical region if

$$b \neq 0, \quad n \text{ is odd, } n < m \tag{57}$$

or if

$$b \neq 0, \quad n \text{ is odd}, \; n = m, \; \text{and} \; D = b^2 + 4(m+1)a \geqslant 0; \tag{58}$$

d) $O(0, 0)$ is a focus or a center if

$$b = 0, \tag{59}$$

or if

$$b \neq 0 \text{ and } n > m, \tag{60}$$

or if

$$b \neq 0, \quad n = m \text{ and } D = b^2 + 4(m+1)a < 0. \tag{61}$$

The next theorem characterizes — in terms of decomposition into structurally stable equilibrium states — the difference between cases b and c, on the one hand, and case d, on the other. This classification, however, is formulated for the case when system (A) is given in the form (54) and if $n = m$, then $D = b^2 + 4(m+1)a \neq 0$. We will show somewhat later (see remark to Theorem 39) that the last condition is fundamentally not restrictive.

T h e o r e m 39. *1) If on passing from system* (54) *to a feasible system the equilibrium state* O (0,0) *decomposes into* k *structurally stable equilibrium states* O_i, $i = 1, 2, \ldots, k$, k *is odd and the number of structurally stable saddle points among* O_i *is 1 less than the number of structurally stable nodes and foci.*

2) If O (0, 0) *is a topological node or an equilibrium state with an elliptical region and if for* $n = m$, $D = b^2 + 4(m+1)a \neq 0$, *the structurally stable equilibrium states into which* O *decomposes on passing to a sufficiently close split system include at least one structurally stable node.*

3) If O (0, 0) *is a multiple focus or center, there exist split systems arbitrarily close to* (54) *on passing to which* O *decomposes into structurally stable foci and saddle points only.*

P r o o f. The first proposition of the theorem follows directly from the fact that the Poincaré index is +1 in cases b, c, d (see (28)). The second and the third proposition for $m \neq n$ are proved precisely in the same way as the corresponding propositions of Theorem 38. If, however, $m = n$, the proof is more complicated, and is omitted here.*

R e m a r k. If $n = m$, and $D = b^2 + 4(m+1)a = 0$, there exist systems of the same type, arbitrarily close to (54), for which $D > 0$ and $D < 0$. Therefore, no criterion is applicable in this case which would differentiate — in terms of decomposition into structurally stable equilibrium states — between a node or an equilibrium state with an elliptical region, on the one hand, and a focus or center, on the other.

For the sake of completeness, we require still another criterion, which would differentiate — in terms of decomposition into structurally stable equilibrium states — between case b and case c, i.e., between a node and an equilibrium state with an elliptical region. One such criterion has been

* A number of special algebraic lemmas have to be used in this case. See /16/, Theorem 5.

established in /16/ (p. 55, Theorem 6). It makes use, in particular, of the character of the stability of the structurally stable nodes and foci into which the multiple equilibrium state O decomposes.

This criterion, however, employs essentially nontopological concepts, and it is therefore inadequate from the topological point of view adopted in our treatment. The question of a satisfactory (topological) criterion of this kind thus remains open.

In conclusion of this section, note that our results indicate that the type of a multiple equilibrium state $O(0, 0)$ of system (A) satisfying conditions (7) and (22) (i.e., when $\sigma = P'_x + Q'_y = 0$) is not determined in general by the topological structure and the number of structurally stable points into which the multiple point decomposes. To obtain a full characteristic of points of types b, c, d, e, f (p. 229), we have to consider the (non-topological) difference between nodes and foci. If in (54), $n = m$ and $D = b^2 + 4(m+1)a = 0$, cases b and c cannot be distinguished from case d by considering the structurally stable points obtained from the multiple point.

Chapter IX

CREATION OF LIMIT CYCLES FROM A MULTIPLE FOCUS

INTRODUCTION

In the previous chapter we considered bifurcations of a multiple equilibrium state, concentrating only on the number and the character of the structurally stable equilibrium states into which a multiple state may decompose. In Chapter IX we will consider a simple equilibrium state, namely a structurally unstable focus ($\Delta \neq 0$, pure imaginary characteristic numbers) and determine the number of limit cycles which may be created in its neighborhood on passing to close systems. Once this problem is solved, we will be able to describe the possible bifurcations of a dynamic system in the neighborhood of a structurally unstable focus.

The chapter is divided into two sections. In §24, various auxiliary propositions are considered. It studies in more detail the properties of the succession function, and also defines the leading concepts of focal values and multiplicity of a multiple focus. The succession function was originally defined in §10: let O be a focus, l a ray issuing from the focus, M_0 a point on the ray, sufficiently close to O, L a path (spiral) through M_0, M_1 the next (after M_0, in terms of increasing t) intersection point of the path L and ray l. Let $OM_0 = \rho_0$, $OM_1 = \rho_1$ (Figure 26, p. 91). The function $\rho_1 = f(\rho_0)$ is called the succession function on the ray l. The focal values are defined as the values of the derivative of the function

$$d(\rho_0) = f(\rho_0) - \rho_0$$

at the point $\rho_0 = 0$.

In §24 we prove that if n exists such that

$$d'(0) = d''(0) = \ldots = d^{(n-1)}(0) = 0, \quad d^{(n)}(0) \neq 0,$$

then n is an odd number (Lemma 5). In this case

$$k = \frac{n-1}{2}$$

is called the multiplicity of the focus O. A simple focus has multiplicity zero ($n = 1$), whereas a multiple focus either has no definite multiplicity, or its multiplicity is $k \geqslant 1$. A number of formulas are derived in §24.3 which can be applied to compute the focal values. All these formulas lean

on the proof of the fundamental theorem of Chapter IX. Expressions for the focal values of an analytical system are derived in §24.4. These expressions are useful in treatment of particular systems.

The fundamental theorem of this chapter is proved in §25 — the theorem of the creation of limit cycles from a multiple focus (Theorem 40). Its formulation is very simple: if $O(0, 0)$ is a multiple focus of multiplicity $k \geqslant 1$ of a dynamic system (A), systems $(\tilde{A})$ sufficiently close to (A) to rank $2k+1$ can have at most k closed paths in a sufficiently small neighborhood of the focus. On the other hand, there exist systems $(\tilde{A})$ as close as desired to (A) (to rank $2k+1$) which have precisely k closed paths in an arbitrarily small neighborhood of the focus. Thus a k-tuple focus may create k, but no more than k, limit cycles.

Theorem 41 establishes that for any s, $1 \leqslant s \leqslant k$, there exist bifurcations in which precisely s limit cycles are created from a k-tuple focus.

Theorems 40 and 41 show that a dynamic system may only have a finite number of different bifurcations in the neighborhood of a focus of finite multiplicity. In §25.2 a classification of these bifurcations is given.

At the end of the chapter (§25.3) we consider one particular case often encountered in applications, namely a system dependent on a single parameter and its bifurcations in the neighborhood of a multiple focus of multiplicity 1 when the parameter is varied. When the system crosses the bifurcation value of the parameter, the stability of the focus changes and a limit cycle is created, or alternatively an existing limit cycle "contracts" into the focus.

§24. FOCAL VALUES

1. Some properties of the succession function

The contents of the present chapter is directly related to §10, Structural Instability of an Equilibrium State with Pure Imaginary Characteristic Roots. We will therefore use the notation and the concepts introduced in §10, and some of the previous results. We will consider a system of class N which has the following canonical form in the neighborhood of an equilibrium state $O(0, 0)$ with pure imaginary characteristic roots:

$$\frac{dx}{dt} = -\beta y + \varphi(x, y), \qquad \frac{dy}{dt} = \beta x + \psi(x, y), \tag{1}$$

where $\beta > 0$. This system is a particular case of the system

$$\frac{dx}{dt} = \alpha x - \beta y + \varphi(x, y), \qquad \frac{dy}{dt} = \beta x + \alpha y + \psi(x, y). \tag{2}$$

The functions φ and ψ are discussed in §10.1.

Changing over to the polar coordinates ρ, θ, we first obtain the system

$$\begin{aligned} \frac{d\rho}{dt} &= F(\rho, \theta) = \alpha\rho + \varphi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta, \\ \frac{d\theta}{dt} &= \beta + \Phi(\rho, \theta) = \beta + \frac{\psi}{\rho}\cos\theta - \frac{\varphi}{\rho}\sin\theta, \end{aligned} \tag{3}$$

and then the equation

$$\frac{d\rho}{d\theta} = R(\rho, \theta) = \frac{F(\rho, \theta)}{\beta + \Phi(\rho, \theta)}. \tag{4}$$

It is assumed here that $\Phi(0, \theta) \equiv 0$ for all θ, which ensures the continuity of *the function* Φ. Taking

$$\rho = f(\theta; \theta_0, \rho_0) \tag{5}$$

to be the solution of equation (4) satisfying the initial condition

$$f(\theta_0; \theta_0, \rho) \equiv \rho_0, \tag{6}$$

we define the succession function

$$\rho = f_{\theta_0}(\rho_0) = f(\theta_0 + 2\pi; \theta_0, \rho_0) \tag{7}$$

on the ray $\theta = \theta_0$, and also the function

$$d_{\theta_0}(\rho_0) = f_{\theta_0}(\rho_0) - \rho_0. \tag{8}$$

For $\theta_0 = 0$, we designate these functions $f(\rho_0)$ and $d(\rho_0)$, respectively. Thus,

$$f(\rho_0) = f(2\pi; 0, \rho_0), \tag{9}$$

$$d(\rho_0) = f(\rho_0) - \rho_0 = f(2\pi, 0, \rho_0) - \rho_0. \tag{10}$$

In §10 we considered the functions $f_{\theta_0}(\rho_0)$ and $d_{\theta_0}(\rho_0)$ for $\rho_0 \geqslant 0$. In the present section, we will allow negative values of ρ_0 also. We will assume, however, that

$$|\rho_0| < \delta, \tag{11}$$

where δ is a sufficiently small positive number.

First note that equation (4) is not affected by a simultaneous substitution of $-\rho$ for ρ and of $\theta + \pi$ for θ. More precisely, if $\rho = \rho(\theta)$ is a solution of equation (4) and if $\rho^* = -\rho$ and $\theta^* = \theta + \pi$, then $\frac{d\rho^*}{d\theta^*} = R(\rho^*, \theta^*)$. Indeed,

$$\frac{d\rho^*}{d\theta^*} = -\frac{d\rho}{d(\theta+\pi)} = -\frac{d\rho}{d\theta} = -R(\rho, \theta) = R(\rho^*, \theta^*)$$

(the last equality follows directly from (3) and (4)). Thus if $\rho = \rho(\theta)$ is a solution of equation (4), then

$$\frac{d(-\rho)}{d(\theta+\pi)} = R(-\rho, \theta+\pi) \equiv -R(\rho, \theta). \tag{12}$$

Lemma 1. The following equality holds true:

$$d_{\theta_0}(-\rho) = -d_{\theta_0+\pi}(\rho) = -f(\theta_0 + 3\pi; \theta_0 + \pi, \rho) + \rho. \tag{13}$$

Proof. By definition

$$d_{\theta_0}(-\rho_0) = f(\theta_0 + 2\pi;\ \theta_0,\ -\rho_0) - (-\rho_0), \tag{14}$$

where $\rho = f(\theta;\ \theta_0,\ -\rho_0)$ is the solution of equation (4) for the initial conditions $\theta = \theta_0,\ \rho = -\rho_0$.

Consider the solution of equation (12) for the initial conditions $\rho^* = -\rho = \rho_0,\ \theta^* = \theta + \pi = \theta_0 + \pi$. This solution evidently has the form

$$\rho^* = -\rho = f(\theta^*;\ \theta_0 + \pi,\ \rho_0) = f(\theta + \pi;\ \theta_0 + \pi,\ \rho_0).$$

Thus for the particular initial conditions chosen

$$\rho = -f(\theta + \pi;\ \theta_0 + \pi,\ \rho_0). \tag{15}$$

But the condition $\rho^* = \rho_0$ for $\theta^* = \theta_0 + \pi$ is equivalent to the condition $\rho = -\rho_0$ for $\theta = \theta_0$. Therefore, in virtue of the uniqueness theorem,

$$-f(\theta + \pi;\ \theta_0 + \pi,\ \rho_0) \equiv f(\theta;\ \theta_0,\ -\rho_0). \tag{16}$$

By (14), (16), (7), and (8), we have

$$d_{\theta_0}(-\rho_0) = f(\theta_0 + 2\pi;\ \theta_0,\ -\rho_0) - (-\rho_0) = -f(\theta_0 + 3\pi;\ \theta_0 + \pi,\ \rho_0) + \rho_0 =$$
$$= -[f_{\theta_0+\pi}(\rho_0) - \rho_0] = -d_{\theta_0+\pi}(\rho_0).$$

Substituting ρ for ρ_0, we obtain (13). This completes the proof of the lemma.

Geometrically, Lemma 1 is self-evident. Indeed, let M_0 be a point with polar coordinates $(\theta_0, -\rho_0)$, where $\rho > 0$, and let L be a path through this point which again crosses the ray OM_0 (as the polar angle θ is increased by 2π) at a point M_1 with the polar coordinates $(\theta_0 + 2\pi, -\rho_1)$, $\rho_1 > 0$ (Figure 113). By definition $d_{\theta_0}(-\rho_0) = -\rho_1 - (-\rho_0) = \rho_0 - \rho_1$. On the other hand, M_0 and M_1 can be considered as points with the polar coordinates $(\theta_0 + \pi, \rho_0)$ and $(\theta_0 + \pi, \rho_1)$, respectively. But then $d_{\theta_0+\pi}(\rho_0) = \rho_1 - \rho_0 = -d_{\theta_0}(-\rho_0)$.

Lemma 2. If there exists $r_1 > 0$ such that for all ρ, $0 < \rho \leqslant r_1$, $d_{\theta_0}(\rho) > 0$ $(d_{\theta_0}(\rho) < 0)$, there also exists $r_2 > 0$ such that for all ρ, $0 < \rho \leqslant r_2$, $d_{\theta_0}(-\rho) < 0$ $(d_{\theta_0}(-\rho) > 0$, respectively). Therefore, for all ρ, $0 < |\rho| \leqslant r = \min\{r_1, r_2\}$,

$$d_{\theta_0}(\rho)\cdot d\theta_0(-\rho) < 0. \tag{17}$$

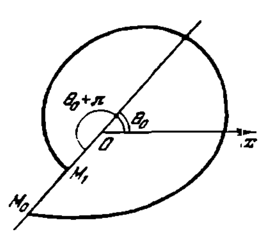

FIGURE 113

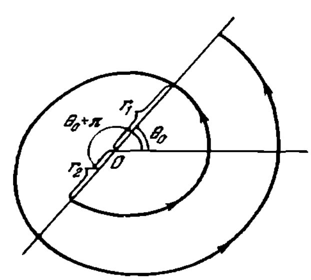

FIGURE 114

Proof. To fix ideas, consider the case when for all ρ, $0<\rho\leqslant r_1$,

$$d_{\theta_0}(\rho)>0. \tag{18}$$

In this case, all the paths of system (2) through the points (θ_0, ρ), $0<\rho\leqslant r_1$, of the ray $\theta=\theta_0$ are not closed: these are spirals winding onto the focus $O(0,0)$ for $t\to-\infty$ (§10.1). But then all these paths cross all the rays $\theta=$ $=$ const, in particular, the ray $\theta=\theta_0+\pi$. The succession function on this ray $f_{\theta_0+\pi}(\rho)=f(\theta_0+2\pi;\ \theta_0+\pi,\rho)$ and the function $d_{\theta_0+\pi}(\rho)=f_{\theta_0+\pi}(\rho)-\rho$ are a priori defined for all ρ, $0<\rho\leqslant r_2$, where r_2 is some positive number (Figure 114). By inequality (18) we conclude that for all ρ, $0<\rho\leqslant r_2$, $d_{\theta_0+\pi}(\rho)>0$. Then, by Lemma 1, the function $d_{\theta_0}(-\rho)$ is defined for all and is negative. Hence, for all ρ, $0<\rho\leqslant r_2$, $0<\rho\leqslant\min\{r_1, r_2\}$, we have $d_{\theta_0}(\rho)\,d_{\theta_0}(-\rho)<0$. Q.E.D.

2. Multiplicity of a multiple focus. Focal values

The following treatment leans on two lemmas which will now be given.

As in §24.1, we consider system (2) and the associated equation (4). The right-hand side $R(\rho,\theta)$ of equation (4) is defined for all θ and for all ρ, $|\rho|<\rho^*$, where ρ^* is a sufficiently small positive number; it is a continuous function periodic in θ with a period of 2π.

Lemma 3. If system (2) *is of class $N\geqslant 1$, the function $R(\rho,\theta)$ has continuous partial derivatives with respect to ρ to order N inclusive everywhere in the region $-\infty<\theta<\infty$, $0\leqslant|\rho|<\rho^*$.*

Proof of Lemma 3, although fundamentally simple, involves tedious mathematics. It is therefore deferred to Appendix, subsection 3.

Corollary. It follows from Lemma 3, by QT, Appendix, §8.3, Theorem B'', that the solution of equation (4), i.e., the function $\rho=$ $=f(\theta;\theta_0,\rho_0)$, has continuous partial derivatives with respect to ρ_0 up to order N inclusive.

Lemma 4. The partial derivatives

$$\frac{\partial f}{\partial\rho_0},\ \frac{\partial^2 f}{\partial\rho_0^2},\ \ldots,\ \frac{\partial^N f}{\partial\rho_0^N} \tag{19}$$

considered as functions of θ (i.e., for constant θ_0 and ρ_0) satisfy the system of differential equations

$$\left.\begin{aligned}
&\frac{d}{d\theta}\left(\frac{\partial f}{\partial\rho_0}\right)=\frac{\partial R(f(\theta;\theta_0,\rho_0),\theta)}{\partial\rho}\frac{\partial f(\theta;\theta_0,\rho_0)}{\partial\rho_0},\\
&\frac{d}{d\theta}\left(\frac{\partial^2 f}{\partial\rho_0^2}\right)=\frac{\partial R(f,\theta)}{\partial\rho}\frac{\partial^2 f}{\partial\rho_0^2}+\frac{\partial^2 R(f,\theta)}{\partial\rho^2}\left(\frac{\partial f}{\partial\rho_0}\right)^2=\\
&\qquad=\frac{\partial R(f,\theta)}{\partial\rho}\frac{\partial^2 f}{\partial\rho_0^2}+E_2(\theta;\theta_0,\rho_0),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d}{d\theta}\left(\frac{\partial^N f}{\partial\rho_0^N}\right)=\frac{\partial R(f,\theta)}{\partial\rho}\frac{\partial^N f}{\partial\rho_0^N}+\frac{\partial^N R}{\partial\rho^N}\left(\frac{\partial f}{\partial\rho_0}\right)^N+\ldots+N\frac{\partial^2 R}{\partial\rho^2}\frac{\partial^{N-1} f}{\partial\rho_0^{N-1}}\frac{\partial f}{\partial\rho_0}=\\
&\qquad=\frac{\partial R(f,\theta)}{\partial\rho}\frac{\partial^N f}{\partial\rho_0^N}+E_N(\theta;\theta_0,\rho_0)\ ^*
\end{aligned}\right\} \tag{20}$$

* The expressions for E_2, E_3, ..., E_N in (20) are not given in explicit form. Their structure can be guessed at without much difficulty, and they can be found if we notice that each equation in (20) is obtained by differentiating the preceding equation with respect to ρ_0.

with the initial conditions

$$\frac{\partial f}{\partial \rho_0}\Big|_{\theta=\theta_0} = 1, \ \frac{\partial^2 f}{\partial \rho_0^2}\Big|_{\theta=\theta_0} = 0, \ \ldots, \ \frac{\partial^N f}{\partial \rho_0^N}\Big|_{\theta=\theta_0} = 0. \tag{21}$$

Proof. The validity of (20) follows from QT, Appendix, §8.3, Theorem B''. Relations (21) follow directly from identity (6). Q. E. D.

In what follows we assume for simplicity that $\theta_0 = 0$ (this, evidently, does not lead to a loss in generality). For the functions $f_0(\rho_0) = f(2\pi; 0, \rho_0)$ and $d_0(\rho_0) = f_0(\rho_0) - \rho_0$ we will use the respective notation $f(\rho_0)$ and $d(\rho_0)$ (see (9) and (10)). Note that the function f entering (20) and (21) is a function of three variables, $f(\theta; \theta_0, \rho_0)$, whereas $f(\rho_0)$ is a function of a single variable. The relation between these two functions is fixed by (9).

Since $f(0; \theta_0, \rho_0)$ has continuous partial derivatives with respect to ρ_0 to order N inclusive, the functions $f(\rho_0)$ and $d(\rho_0)$ are continuously differentiable N times.

Definition 25. The value of the i-th derivative of the function $d(\rho_0)$ at the point O, i.e., $d^{(i)}(0)$, is called the i-th focal value of the focus O.

If (2) is a system of class N, the focal values $d^{(i)}(0)$, $1 \leqslant i \leqslant N$, a priori exist.

Lemma 5. If there exists k such that

$$d'(0) = 0, \quad d''(0) = 0, \ \ldots, \ d^{(k-1)}(0) = 0, \quad d^{(k)}(0) \neq 0, \tag{22}$$

k is an odd number.

Proof. By (3) and (4), $\rho \equiv 0$ is a solution of equation (4). Therefore

$$f(0) = d(0) = 0. \tag{23}$$

Applying Maclaurin's formula to the function $d(\rho_0)$ and using relations (22) and (23), we find

$$d(\rho_0) = \frac{d^{(k)}(\eta\rho_0)}{k!}\rho_0^k, \tag{24}$$

where $0 < \eta < 1$. Therefore, if k is even, $d(\rho_0)$ has the same sign for all sufficiently small ρ_0, both negative and positive (its sign coincides with the sign of the k-th focal value $d^{(k)}(0)$). This contradicts Lemma 2, however. Thus k must be odd. Q. E. D.

Definition 26. If conditions (22) are satisfied, and $k = 2m + 1$, $m \geqslant 0$, we shall say that the focus $O(0, 0)$ is a focus of multiplicity m.

In Chapter IV (§10.2) it is proved that the first focal value is

$$d'(0) = e^{2\pi\frac{\alpha}{\beta}} - 1. \tag{25}$$

If $m = 0$, then $k = 1$, $d'(0) \neq 0$, $\alpha \neq 0$. But then the focus $O(0, 0)$ has complex, though not pure imaginary, characteristic numbers, i.e., it is a simple focus. Conversely, if $m > 0$, then $k \geqslant 3$, $d'(0) = 0$, $\alpha = 0$, and the characteristic numbers are pure imaginary, i.e., the focus is multiple (§10.3, Definition 16). Thus for $m \geqslant 1$ we are dealing with a multiple focus of multiplicity m. Note that a multiple focus does not always have a definite multiplicity. Indeed, if (1) is a system of class N, but not of class $N+1$, and if $d'(0) = d''(0) = \ldots = d^{(N)}(0) = 0$, Definition 26 is inapplicable.

In what follows we will deal with multiple foci only, since a simple focus is structurally stable and the topological structure of the partition into paths in a sufficiently small neighborhood of a simple focus does not change on passing to close systems.

When $\beta > 0$, the focus is stable if for all sufficiently small positive ρ_0, $d(\rho_0) < 0$, and unstable if $d(\rho_0) > 0$.*

Hence and from (24) it follows that if $k = 2m + 1 \geqslant 1$ satisfies conditions (22) and $\beta > 0$, the focus O is

$$\text{stable when } d^{(k)}(0) < 0, \tag{26}$$

and

$$\text{unstable when } d^{(k)}(0) > 0. \tag{27}$$

Definition 27. The first (counted in the proper order) nonzero focal value of a multiple focus is called the Lyapunov value.

In other words, the Lyapunov value is the number $d^{(k)}(0)$ provided that relations (22) hold true and $k \geqslant 3$. If $k = 2m + 1$, the Lyapunov value will also be called the m-th Lyapunov value $(m \geqslant 1)$.**

From (26) and (27) it follows that for $\beta > 0$, a multiple focus is stable (unstable) if its Lyapunov value is negative (positive).

3. Calculation of the focal values of a multiple focus

Since we will be dealing with multiple foci only, we take in what follows $\alpha = 0$, i.e., we will consider system (1). It follows from the results of Chapter IV (§10.1, (9), and §10.2, (23)) that in this case

$$\left[\frac{\partial R(\rho, \theta)}{\partial \rho}\right]_{\rho=0} \equiv 0, \quad \left[\frac{\partial f(\theta; 0, \rho_0)}{\partial \rho_0}\right]_{\rho_0=0} = 1. \tag{28}$$

To compute the focal values, we use Lemma 4.

Let

$$\frac{1}{k!}\left[\frac{\partial^k f(\theta; 0, \rho_0)}{\partial \rho_0^k}\right]_{\rho_0=0} = U_k(\theta), \quad k = 1, 2, \ldots, N; \tag{29}$$

$$\frac{1}{k!}\left[\frac{\partial^k R(\rho, \theta)}{\partial \rho^k}\right]_{\rho=0} = R_k(\theta), \quad k = 1, 2, \ldots, N; \tag{30}$$

$$\frac{1}{k!}[E_k(\theta; 0, \rho_0)]_{\rho_0=0} = H_k(\theta) = \frac{1}{k!}\left[\frac{\partial^k R}{\partial \rho^k}\frac{\partial f}{\partial \rho_0} + \ldots + k\frac{\partial^2 R}{\partial \rho^2}\frac{\partial^{k-1} f}{\partial \rho_0^{k-1}}\frac{\partial f}{\partial \rho_0}\right]_{\rho_0=0}$$

(the functions E_k are defined by (20)).

From (28), (29), and (30) it follows that $U_1(\theta) \equiv 1$, $R_1(\theta) \equiv 0$ and that

$$\frac{1}{k!}[E_k(\theta; 0, \rho_0)]_{\rho_0=0} = H_k(\theta) = R_k(\theta) + \ldots, \qquad k = 2, 3, \ldots, N. \tag{31}$$

The triple dots in (31) correspond to a polynomial in the functions $R_2(\theta), R_3(\theta), \ldots, R_{k-1}(\theta)$ and the functions $U_2(\theta), \ldots, U_{k-1}(\theta)$.

* See §10.1. For $\beta < 0$, $d(\rho_0) > 0$ in the stable case and $d(\rho_0) < 0$ in the unstable case.

** Logically, this term is not quite adequate, since every multiple focus has a single Lyapunov value. It is convenient, however, in that it directly identifies the running number of the focal value which is the Lyapunov value.

Let now $\rho_0 = 0$ in (20). We will successively integrate these equations as first-order linear equations, using initial conditions (21), the relations

$$f(\theta; 0, 0) \equiv 0 \text{ and } R(0, \theta) \equiv 0$$

and the notation (29), (30), (31). The first of these equations has been considered before (§10.2). Its solution is given by (28).

Multiplying the k-th equation in (20) ($k = 2, 3, \ldots, N$) by $\frac{1}{k!}$ and integrating, we find

$$\frac{1}{k!}\left[\frac{\partial^k f}{\partial \rho_0^k}\right]_{\rho_0=0} = U_k(\theta) = e^{\int\limits_0^\theta R_1(\theta)\,d\theta} \int\limits_0^\theta H_k(\theta)\, e^{-\int\limits_0^\theta R_1(\theta)\,d\theta}\, d\theta. \tag{32}$$

By (28) and (30), $R_1(\theta) = 0$. Therefore,

$$\frac{1}{k!}\left[\frac{\partial^k f}{\partial \rho_0^k}\right]_{\rho_0=0} = U_k(\theta) = \int\limits_0^\theta H_k(\theta)\, d\theta, \quad k = 2, 3, \ldots, N. \tag{33}$$

By definition, the focal values are equal to the derivatives of the function

$$d(\rho_0) = f(2\pi; 0, \rho_0) - \rho_0$$

at the point $\rho_0 = 0$. Therefore, using (28) and (33), we obtain for the focal values

$$d'(0) = 0, \quad d^{(k)}(0) = k! \int\limits_0^{2\pi} H_k(\theta)\, d\theta, \quad k = 2, 3, \ldots, N. \tag{34}$$

Let us now express the focal values in terms of the right-hand sides P and Q of system (2). To this end, we first have to derive an expression for $R_k(\theta)$. System (1) has the form

$$\frac{dx}{dt} = -\beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \psi(x, y),$$

where the functions φ and ψ are continuously differentiable to order N inclusive, and these functions together with their first derivatives vanish at the point $O(0, 0)$. As is known (see Appendix, 2), for any k, $2 \leqslant k \leqslant N$, the functions φ and ψ can be written in the form

$$\left.\begin{aligned} \varphi(x, y) &= P_2(x, y) + P_3(x, y) + \ldots + P_k(x, y) + P^*(x, y), \\ \psi(x, y) &= Q_2(x, y) + Q_3(x, y) + \ldots + Q_k(x, y) + Q^*(x, y), \end{aligned}\right\} \tag{35}$$

where $P_i(x, y)$ and $Q_i(x, y)$ are homogeneous polynomials of degree i ($i = 2, 3, \ldots, k$), and

$$P^* = \sum_{\alpha=0}^{k} x^{k-\alpha} y^\alpha P_\alpha^*(x, y), \quad Q^* = \sum_{\alpha=0}^{k} x^{k-\alpha} y^\alpha Q_\alpha^*(x, y), \tag{36}$$

where $P_\alpha^*(x, y)$ and $Q_\alpha^*(x, y)$ are continuous functions which vanish at $x = y = 0$.

Consider the function $R(\rho, \theta)$. By definition (see (3) and (4))

$$R(\rho, \theta) = \frac{\varphi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta}{\beta + \frac{\psi(\rho\cos\theta, \rho\sin\theta)}{\rho}\cos\theta - \frac{\varphi(\rho\cos\theta, \rho\sin\theta)}{\rho}\sin\theta}. \tag{37}$$

By Lemma 3, the function $R(\rho, \theta)$ has continuous partial derivatives to order N inclusive with respect to ρ. Moreover, $R(0, \theta) \equiv 0$, $\left[\frac{\partial R}{\partial \rho}\right]_{\rho=0} = 0$. Therefore we obtain the following relation, analogous to relations (35):

$$R(\rho, \theta) = R_2(\theta)\rho^2 + \ldots + R_k(\theta)\rho^k + R^*(\rho, \theta)\rho^k, \tag{38}$$

where $R_i(\theta)$, $i = 2, \ldots, k$, are computed from (30) (i.e., they are equal to $\frac{1}{i!}\left[\frac{\partial^i R(\rho, \theta)}{\partial \rho^i}\right]_{\rho=0}$), and $R^*(\rho, \theta)$ is a continuous function of ρ and θ which vanishes at $\rho = 0$.

Inserting the functions φ and ψ from (35) in (37), we obtain

$$R(\rho, \theta) = \frac{\sum_{m=2}^{k} \rho^m [P_m\cos\theta + Q_m\sin\theta] + P^*\cos\theta + Q^*\sin\theta}{\beta + \sum_{m=2}^{k} \rho^{m-1}[Q_m\cos\theta - P_m\sin\theta] + \frac{Q^*\cos\theta}{\rho} - \frac{P^*\sin\theta}{\rho}},$$

where

$$\begin{aligned} P_m &= P_m(\cos\theta, \sin\theta), & Q_m &= Q_m(\cos\theta, \sin\theta), \\ P_m^* &= P_m^*(\rho\cos\theta, \rho\sin\theta), & Q_m^* &= Q_m^*(\rho\cos\theta, \rho\sin\theta). \end{aligned} \tag{39}$$

We write $u_m(\cos\theta, \sin\theta)$ and $v_m(\cos\theta, \sin\theta)$, respectively, for the factor after ρ^m in the numerator and the factor after ρ^{m-1} in the denominator. Moreover, by (36),

$$\begin{aligned} P^*(\rho\cos\theta, \rho\sin\theta)\cos\theta + Q^*(\rho\cos\theta, \rho\sin\theta)\sin\theta &= \rho^k u^*(\rho, \cos\theta, \sin\theta), \\ Q^*(\rho\cos\theta, \rho\sin\theta)\cos\theta - P^*(\rho\cos\theta, \rho\sin\theta)\sin\theta &= \rho^k v^*(\rho, \cos\theta, \sin\theta), \end{aligned} \tag{40}$$

where $u^*(\rho, \cos\theta, \sin\theta)$ and $v^*(\rho, \cos\theta, \sin\theta)$ are continuous functions of ρ and θ which vanish at $\rho = 0$.

From (38) through (40) we find

$$R_2(\theta)\rho^2 + \ldots + R_k(\theta)\rho^k + R^*(\rho, \theta)\rho^k = \frac{\sum_{m=2}^{k} \rho^m u_m(\cos\theta, \sin\theta) + \rho^k u^*(\rho, \cos\theta, \sin\theta)}{\beta + \sum_{m=2}^{k} \rho^{m-1} v_m(\cos\theta, \sin\theta) + \rho^{k-1} v^*(\rho, \cos\theta, \sin\theta)}. \tag{41}$$

Multiplying both sides of (41) by the denominator of the right-hand side, we obtain an equality which is valid for all (sufficiently small) ρ, i.e., an identity. All the coefficients in this identity are continuous functions of ρ, and $u^*(0, \cos\theta, \sin\theta) = R^*(0, \theta) = 0$. Therefore, equating the coefficients of the corresponding powers of ρ and using the usual arguments of

continuity, we obtain the following relations:

$$\left.\begin{aligned} u_2 &= \beta R_2, \\ u_3 &= \beta R_3 + v_2 R_2, \\ &\;\;\cdot \qquad \cdot \\ u_k &= \beta R_k + v_2 R_{k-1} + \ldots + v_{k-1} R_2. \end{aligned}\right\} \tag{42}$$

It now follows that

$$R_2 = \frac{u_2}{\beta}, \qquad R_3 = \frac{u_3}{\beta} - \frac{v_2 R_2}{\beta} \tag{43}$$

and, in general,

$$R_m = \frac{u_m}{\beta} + W_m, \quad m = 3, 4, \ldots, k,$$

where

$$W_m = -\frac{R_2 v_{m-1} + R_3 v_{m-2} + \ldots + R_{m-1} v_2}{\beta}. \tag{44}$$

Evidently, W_m is expressible in terms of the functions $u_i(\cos\theta, \sin\theta)$ and $v_i(\cos\theta, \sin\theta)$ with the indices i not exceeding $m-1$.

Inserting for u_i and v_i in (43) and (44) their expressions in terms of P_i and Q_i, we obtain

$$R_m = \frac{P_m(\cos\theta, \sin\theta)\cos\theta + Q_m(\cos\theta, \sin\theta)\sin\theta}{\beta} + W_m \tag{45}$$

$(m = 2, 3, \ldots, k)$, where W_m is expressed in terms of β and in terms of the functions $P_i(\cos\theta, \sin\theta)$, $Q_i(\cos\theta, \sin\theta)$ with indices i not exceeding $m-1$.

In our case of a multiple focus (i.e., for $\alpha = 0$), the first focal value is zero. If $N \geqslant 2$, the second focal value $d''(0) = 0$ by Lemma 5. The remaining focal values $d^{(k)}(0)$ are computed from (34) using expressions (45) for $R_m(\theta)$. A comparison of the various expressions derived above will enable us to formulate a useful lemma.

(a) From (28), (29), (30), and (31) it follows that

$$H_2(\theta) = R_2(\theta).$$

(b) From (31) it follows that, for $m = 3, 4, \ldots, N$,

$$H_m(\theta) = R_m(\theta) + \Phi_m(R_2(\theta), \ldots, R_{m-1}(\theta), u_2(\theta), \ldots, u_{m-1}(\theta)),$$

where Φ_m is a polynomial in the corresponding functions.

(c) From (33) and (b) it follows that $u_2(\theta)$ is expressible in terms of $R_2(\theta)$, $u_3(\theta)$ is expressible in terms of $R_2(\theta)$ and $R_3(\theta)$, $\ldots$, $u_m(\theta)$ is expressible in terms of $R_2(\theta)$, $R_3(\theta)$, $\ldots$, $R_m(\theta)$.

(d) From (b) and (c) it follows that

$$H_m(\theta) = R_m(\theta) + \Phi_m^*(R_2(\theta), \ldots, R_{m-1}(\theta)),$$

where Φ_m^* is expressible in terms of the functions $R_2(\theta)$, $\ldots$, $R_{m-1}(\theta)$ using algebraic operations and integration.

(e) From (45) and (d) it follows that

$$H_m(\theta) = \frac{P_m(\cos\theta,\ \sin\theta)\cos\theta + Q_m(\cos\theta,\ \sin\theta)\sin\theta}{\beta} + \Phi_m^{**}$$

$(m = 2, 3, \ldots, k)$, where Φ_m^{**} is expressible, using algebraic operations and integration, in terms of the number β and the functions $P_i(\cos\theta,\ \sin\theta)$, $Q_i(\cos\theta,\ \sin\theta)$ with indices i not exceeding $m-1$.

Finally, from (34) and (d) we have the following useful result:

Lemma 6. Consider a dynamic system of class N

$$\frac{dx}{dt} = -\beta y + \varphi(x, y) = P(x, y), \quad \frac{dy}{dt} = \beta x + \psi(x, y) = Q(x, y) \tag{A}$$

where the functions φ and ψ are expressed by equations (35) *and* (36), *with k equal to one of the numbers $2, 3, \ldots, N$. Then the focal value $d^{(m)}(0)$, $m = 2, 3, \ldots, k$, may be computed from the formula*

$$d^{(m)}(0) = m! \int_0^{2\pi} \frac{P_m(\cos\theta,\ \sin\theta)\cos\theta + Q_m(\cos\theta,\ \sin\theta)\sin\theta}{\beta}\,d\theta + m! \int_0^{2\pi} \Phi_m^{**}(\theta)\,d\theta, \tag{46}$$

where the function $\Phi_m^{**}(\theta)$ is expressible in terms of the number β and functions P_i and Q_i with indices i not exceeding $m-1$.

Before proceeding with the fundamental theorem of this chapter, we will apply the above results, and in particular equation (46), to compute the Lyapunov value of the focus $O(0, 0)$ of a modified system of one particular form.

Let (A) be a dynamic system of class N (see above), and s an integer satisfying the inequalities $1 \leqslant s$, $2s + 1 \leqslant N$. Let $(\tilde{A}_s)$ be the dynamic system

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) + \lambda(x^2 + y^2)^s x = \tilde{P}(x, y), \\ \frac{dy}{dt} &= Q(x, y) + \lambda(x^2 + y^2)^s y = \tilde{Q}(x, y), \end{aligned} \tag{$\tilde{A}_s$}$$

where λ is a parameter. Let $\tilde{d}(\rho_0)$ be the analog of $d(\rho_0)$ for $(\tilde{A}_s)$. The point $O(0, 0)$ is evidently a multiple focus of $(\tilde{A}_s)$ and $\tilde{d}(0) = d(0) = 0$.

Lemma 7. If all the focal values of the focus $O(0, 0)$ of system (A), *up to $(2s+1)$-th inclusive, are zero, i.e., if*

$$d'(0) = d''(0) = \ldots = d^{(2s+1)}(0) = 0, \tag{47}$$

the focal values of the focus $O(0, 0)$ of system (A_s) up to $(2s)$-th inclusive are also zero, i.e.,

$$\tilde{d}'(0) = d''(0) = \ldots = \tilde{d}^{(2s)}(0) = 0, \tag{48}$$

and the $(2s+1)$-th focal value is

$$\tilde{d}^{(2s+1)}(0) = (2s+1)!\,\frac{\lambda}{\beta}\cdot 2\pi. \tag{49}$$

Proof. We express the right-hand sides of $(\tilde{A}_s)$ using formulas analogous to (35) in the form

$$\begin{aligned} \tilde{P}(x, y) &= -\beta y + \tilde{P}_2(x, y) + \ldots + \tilde{P}_{2s+1}(x, y) + \tilde{P}^*(x, y), \\ \tilde{Q}(x, y) &= \beta x + \tilde{Q}_2(x, y) + \ldots + \tilde{Q}_{2s+1}(x, y) + \tilde{Q}^*(x, y). \end{aligned} \tag{50}$$

Evidently,

$$\tilde{P}_i(x, y) = P_i(x, y), \quad \tilde{Q}_i(x, y) = Q_i(x, y), \tag{51}$$

if $2 \leqslant i < 2s+1$. If $i = 2s+1$, then

$$\begin{aligned} \tilde{P}_{2s+1}(x, y) &= P_{2s+1}(x, y) + \lambda(x^2+y^2)^s x, \\ \tilde{Q}_{2s+1}(x, y) &= Q_{2s+1}(x, y) + \lambda(x^2+y^2)^s y. \end{aligned} \tag{52}$$

From (51) and (46) we conclude that for $2 \leqslant i \leqslant 2s$, $\tilde{d}^{(i)}(0) = d^{(i)}(0)$, i.e., $\tilde{d}^{(i)}(0) = 0$ by (47). The first focal value is $\tilde{d}'(0) = 0$, since 0 is a multiple focus of $(\tilde{A}_s)$. Equalities (48) are thus proved.

To compute the $(2s+1)$-th focal value, we will use equations (46), (51), and (52). They directly lead to the result

$$\tilde{d}^{(2s+1)}(0) = (2s+1)! \int_0^{2\pi} \frac{\lambda(\cos^2\theta + \sin^2\theta)\cos^2\theta + \lambda(\cos^2\theta + \sin^2\theta)\sin^2\theta}{\beta}\, d\theta + d^{(2s+1)}(0).$$

Hence and using the equality $d^{(2s+1)}(0) = 0$, we find that

$$\tilde{d}^{(2s+1)}(0) = (2s+1)!\, \frac{\lambda}{\beta} \cdot 2\pi.$$

This completes the proof of the lemma.

4. The case of an analytical system

In this subsection we will consider the computation of the focal values of an analytical system (A). We assume, as before, that the multiple focus coincides with the origin and that the system has the canonical form

$$\frac{dx}{dt} = \alpha x - \beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \alpha y + \psi(x, y), \tag{A}$$

where $\beta > 0$, and φ and ψ are analytical functions.

The function $R(\rho, \theta)$ in the right-hand side of equation (4) (see §24.1) is clearly an analytical function of θ and ρ in this case in the strip

$$-\infty < \theta < +\infty, \quad |\rho| < r_0, \tag{53}$$

where r_0 is some positive number. Therefore, it can be series-expanded in powers of ρ. Since by (3) and (4), $R(0, \theta) \equiv 0$, the series expansion has the form

$$R(\rho, \theta) = R_1(\theta)\rho + R_2(\theta)\rho^2 + \ldots \tag{54}$$

Note that the function $R(\rho, \theta)$ and hence the functions $R_i(\theta)$, $i = 1, 2, \ldots$, are periodic functions of θ with period of 2π. It follows from the standard properties of analytical functions that there exists $r_1 > 0$ such that the series (54) converges for all θ, $0 \leqslant \theta \leqslant 2\pi$, and for all ρ, $|\rho| \leqslant r_1$.

By QT, Appendix, §8.3, Theorem C, the solution

$$\rho = f(\theta; 0, \rho_0) \tag{55}$$

of the equation

$$\frac{d\rho}{d\theta} = R(\rho, \theta) \tag{4}$$

satisfying the initial condition

$$f(0; 0, \rho_0) \equiv \rho_0 \tag{56}$$

is an analytical function of the arguments. Let us expand this solution in powers of the "initial value" ρ_0. Since $R(0, \theta) \equiv 0$, we see that $\rho \equiv 0$ is a solution of equation (4), so that $f(\theta; 0, 0) \equiv 0$. The expansion of $f(\theta; 0, \rho_0)$ in powers of ρ_0 therefore has the form

$$\rho = f(\theta; 0, \rho_0) = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + \ldots \tag{57}$$

There clearly exists a number $r_2 \leqslant r_1$ such that the series (57) converges for all θ, $0 \leqslant \theta \leqslant 2\pi$, and for all ρ, $|\rho| \leqslant r_2$.

By (56) and (57),

$$u_1(0) = 1, \quad u_2(0) = u_3(0) = \ldots = 0. \tag{58}$$

Inserting for ρ and R in equation (4) their expressions from (57) and (54) and equating the coefficients of the corresponding powers of ρ_0 in the right- and the left-hand sides, we obtain the following recursive differential equations for the coefficients $u_i(\theta)$ $(i = 1, 2, 3, \ldots)$:

$$\left.\begin{aligned} &u_1'(\theta) = R_1(\theta)u_1(\theta),\\ &u_2'(\theta) = R_1(\theta)u_2(\theta) + R_2(\theta)u_1^2(\theta),\\ &u_3'(\theta) = R_1(\theta)u_3(\theta) + 2R_2(\theta)u_1(\theta)u_2(\theta) + R_3(\theta)u_1^3(\theta),\\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{59}$$

Relations (58) may be considered as the initial conditions for the functions $u_i(\theta)$ satisfying differential equations (59).* Using these initial conditions and successively integrating equations (59) as linear differential equations for the corresponding functions, we obtain

$$\left.\begin{aligned} &u_1(\theta) = e^{\int_0^\theta R_1(\theta)d\theta},\\ &u_2(\theta) = e^{\int_0^\theta R_1(\theta)d\theta}\int_0^\theta R_2(\theta)u_1(\theta)\,d\theta,\\ &u_3(\theta) = e^{\int_0^\theta R_1(\theta)d\theta}\int_0^\theta [2R_2(\theta)u_2(\theta) + R_3(\theta)u_1^2(\theta)]\,d\theta,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{60}$$

By definition, the succession function is $f(\rho_0) = f(2\pi; 0, \rho_0)$ (see §24.1, (9)). Therefore, taking $\theta = 2\pi$ in (57), we obtain a series expressing the succession function

$$\rho = f(\rho_0) = u_1(2\pi)\rho_0 + u_2(2\pi)\rho_0^2 + \ldots$$

* It is readily seen that equations (59) are obtained from equations (20) if we set $\rho_0 = 0$ and introduce an appropriate notation.

Let

$$u_i(2\pi) = \alpha_i, \quad i = 1, 2, \ldots \tag{61}$$

Then

$$\rho = f(\rho_0) = \alpha_1\rho_0 + \alpha_2\rho_0^2 + \ldots \tag{62}$$

From the last expression and the equality $d(\rho_0) = f(\rho_0) - \rho_0$, we obtain for the focal values

$$d'(0) = \alpha_1 - 1 = u_1(2\pi) - 1, \quad d^{(k)}(0) = k!\,\alpha_k = k!\,u_k(2\pi) \qquad (k = 2, 3, \ldots). \tag{63}$$

Let us now derive expressions for the coefficients α_i in terms of the right-hand sides of (A). The expression for the first focal value

$$d'(0) = e^{2\pi\frac{\alpha}{\beta}} - 1$$

has been derived before (see §24.2, (25)). The successive focal values are of any interest only when $O(0, 0)$ is a multiple focus, i.e., when $\alpha = 0$. System (A) in this case has the form

$$\frac{dx}{dt} = -\beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \psi(x, y).$$

Let

$$\varphi(x, y) = P_2(x, y) + P_3(x, y) + \ldots, \qquad \psi(x, y) = Q_2(x, y) + Q_3(x, y) + \ldots, \tag{64}$$

where $P_i(x, y)$ and $Q_i(x, y)$ are homogeneous polynomials of i-th degree $(i = 2, 3, \ldots)$. By (3), (4), and (64),

$$R(\rho, \theta) = \frac{\sum\limits_{m=2}^{\infty} \rho^m u_m(\cos\theta, \sin\theta)}{\beta + \sum\limits_{m=2}^{\infty} \rho^{m-1} v_m(\cos\theta, \sin\theta)}, \tag{65}$$

where

$$u_m(\cos\theta, \sin\theta) = P_m(\cos\theta, \sin\theta)\cos\theta + Q_m(\cos\theta, \sin\theta)\sin\theta$$

and (66)

$$v_m(\cos\theta, \sin\theta) = Q_m(\cos\theta, \sin\theta)\cos\theta - P_m(\cos\theta, \sin\theta)\sin\theta$$

(compare with (39); the present treatment is in fact a repetition of the treatment of the previous subsection for systems of class N).

In virtue of the first relation in (28), we have for $\alpha = 0$

$$R_1(\theta) = \left[\frac{\partial R(\rho, \theta)}{\partial\rho}\right]_{\rho=0} = 0. \tag{67}$$

The series expansion of the function $R(\rho, \theta)$ is therefore given by

$$R(\rho, \theta) = R_2(\theta)\rho^2 + R_3(\theta)\rho^3 + \ldots \tag{68}$$

Equalities (65), (66), (68) yield, as in the previous subsection, the relations

$$\begin{aligned} u_2 &= \beta R_2, \\ u_3 &= \beta R_3 + R_2 v_2, \\ u_4 &= \beta R_4 + R_3 v_2 + R_2 v_3, \\ &\dots\dots\dots\dots \end{aligned}$$

Hence

$$\left.\begin{aligned} R_2 &= \frac{u_2}{\beta}, \\ R_3 &= \frac{u_3}{\beta} - \frac{R_2 v_2}{\beta}, \\ R_4 &= \frac{u_4}{\beta} - \frac{R_3 v_2 + R_2 v_3}{\beta}, \\ &\dots\dots\dots\dots \end{aligned}\right\} \tag{69}$$

Inserting these expressions in (60), setting $\theta = 2\pi$, and using (67) and (63), we obtain the focal values. For a multiple focus, the first focal value is zero, $d'(0) = 0$. The second focal value is also zero, $d''(0) = 2\alpha_2 = 0$, by Lemma 5. Note that this fact also emerges directly from the relation

$$d''(0) = 2\alpha_2 = 2\int_0^{2\pi} R_2(\theta)\, d\theta.$$

The integrand, as is readily seen, is an odd periodic function of period 2π. The integral therefore vanishes.

For the third focal value of a multiple focus we have

$$d'''(0) = 3!\,\alpha_3 = 6\int_0^{2\pi} [2R_2(\theta)\, u_2(\theta) + R_3(\theta)]\, d\theta.$$

Using the expressions for $R_2(\theta)$ and $R_3(\theta)$ in terms of the polynomials P_2, Q_2, P_3, Q_3 and writing these polynomials (of second and third degree, respectively) in the form

$$\left.\begin{aligned} P_2(x, y) &= a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ P_3(x, y) &= a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ Q_2(x, y) &= b_{20}x^2 + b_{11}xy + b_{02}y^2, \\ Q_3(x, y) &= b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{aligned}\right\} \tag{70}$$

we obtain after elementary, but fairly lengthy, computations the following expressions for α_3:

$$\alpha_3 = \frac{\pi}{4\beta}[3(a_{30} + b_{03}) + (a_{12} + b_{21})] - \\ - \frac{\pi}{4\beta^2}[2(a_{20}b_{20} - a_{02}b_{02}) - a_{11}(a_{02} + a_{20}) + b_{11}(b_{02} + b_{20})]. \tag{71}$$

If $\alpha_3 \neq 0$, $d'''(0) = 6\alpha_3$ is the Lyapunov value. From the results of §24.2 (see (24), (26), (27)) it follows that if $\beta > 0$, O is a stable focus for $\alpha_3 < 0$ and an unstable focus for $\alpha_3 > 0$.

If $\alpha_3 = 0$, the character of the equilibrium state (with pure imaginary characteristic numbers) can be determined by considering α_5 (if $\alpha_3 = 0$, then

by Lemma 5, $\alpha_4 = 0$ too); if $\alpha_5 = 0$, we have to consider α_7, etc. However, the computational difficulties rapidly multiply as the index increases.*

Expression (71) for α_3 has been derived assuming a system of canonical form (1). We will also give an expression for α_3 in terms of the coefficients of a system expressed in the general form

$$\frac{dx}{dt} = ax + by + P_2(x, y) + P_3(x, y) + \ldots, \qquad \frac{dy}{dt} = cx + dy + Q_2(x, y) + Q_3(x, y) + \ldots \tag{72}$$

where P_2, P_3, Q_2, Q_3 are expressed by (70). Since $O(0, 0)$ is a multiple focus, we have

$$a + d = 0, \quad \Delta = ad - bc > 0. \tag{73}$$

The characteristic numbers of the focus O are $\pm\beta i$, where

$$\beta = \sqrt{\Delta} = +\sqrt{ad - bc}. \tag{74}$$

The expression for α_3 in terms of the coefficients of system (72) has been derived in /18/. To derive this expression, the substitution

$$\xi = x, \quad \eta = -\frac{a}{\beta}x - \frac{b}{\beta}y$$

is applied to reduce (72) to the canonical form

$$\frac{d\xi}{dt} = -\beta\eta + \bar{P}_2(\xi, \eta) + \bar{P}_3(\xi, \eta) + \ldots, \qquad \frac{d\eta}{dt} = \beta\xi + \bar{Q}_2(\xi, \eta) + \bar{Q}_3(\xi, \eta) + \ldots \tag{75}$$

Expressing α_3 from (71) in terms of the coefficients of system (75) and reverting to the coefficients of the original system (72), we find

$$\begin{aligned}\alpha_3 = -\frac{\pi}{4b\beta^3}\{[&ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + ab(b_{11}^2 + a_{20}b_{11} + a_{11}b_{20}) + \\ &+ c^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2ac(b_{02}^2 - a_{20}a_{02}) - 2ab(a_{20}^2 - b_{20}b_{02}) - \\ &- b^2(2a_{20}b_{20} + b_{11}b_{20}) + (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20})] - \\ &- (a^2 + bc)[3(cb_{03} - ba_{30}) + 2a(a_{21} + b_{12}) + (ca_{12} - bb_{21})]\}\end{aligned} \tag{76}$$

(see /18/, p. 29).

Equation (71) is a particular case of (76), and can be obtained from the latter for $a = d = 0$, $b = -\beta$, $c = \beta$.

Note that (76) actually gives an expression — in terms of the coefficients of system (75) — for the focal value of system (75) and not the original system (72). This is immaterial, however, for the investigation of the

* For dynamic systems with quadratic polynomials in the right-hand sides, i.e., systems of the form

$$P(x, y) = \alpha x - \beta y + a_{20}x^2 + a_{11}xy + a_{02}y^2,$$
$$Q(x, y) = \beta x + \alpha y + b_{20}x^2 + b_{11}xy + b_{02}y^2,$$

the expressions for $\alpha_3, \alpha_5, \alpha_7$ in terms of the coefficients a_{ij} and b_{ij} have been derived in /18/.

topological structure, since systems (72) and (75) are the images of each other by a non-singular linear transformation.*

Note that the Lyapunov values were derived by Lyapunov from different considerations.

Remark concerning the case of a center in an analytical system. As we know (QT, §8.6), an equilibrium state of a dynamic system of class $N \geqslant 1$ which has pure imaginary characteristic numbers may be either a multiple focus, or a center, or a center-focus. For analytical systems, the case of a center-focus is ruled out. Indeed, if system (A) is analytical, the corresponding function $d(\rho_0)$ is also analytical. Therefore, for positive ρ_0 which are sufficiently close to zero, $d(\rho_0)$ retains a constant sign and the point $O(0, 0)$ is a multiple focus, or alternatively $d(\rho_0) \equiv 0$ and O is a center. In the former case, at least one of the focal values does not vanish. In the case of a center, all the focal values vanish, i.e.,

$$\alpha_1 = 1, \quad \alpha_2 = \alpha_3 = \ldots = 0. \tag{77}$$

Conditions (72) are clearly necessary and sufficient for point $O(0, 0)$ to be a center. A center is thus observed when an infinite number of conditions are satisfied.**

§25. CREATION OF LIMIT CYCLES FROM A MULTIPLE FOCUS

1. The fundamental theorem

Theorem 40 (theorem of the creation of limit cycles from a multiple focus). If $O(0, 0)$ is a multiple focus of multiplicity k ($k \geqslant 1$) of a dynamic system (A) of class $N \geqslant 2k+1$ or of analytical class, then

1) there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any system $(\tilde{A})$ δ_0-close to rank $2k+1$ to system (A) has at most k closed paths in $U_{\varepsilon_0}(O)$;

2) for any $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$, there exists a system $(\tilde{A})$ of class N or (respectively) of analytical class which is δ-close to rank $2k+1$ to (A) and has k closed paths in $U_\varepsilon(O)$.

Proof. 1) Let us prove the first proposition of the theorem. Without loss of generality, we shall assume that system (A) (of class N or analytical) has the canonical form

$$\frac{dx}{dt} = -\beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \psi(x, y), \tag{A}$$

where φ and ψ are functions which vanish together with their first derivatives at the point $O(0, 0)$.

Consider the function $d(\rho_0) = f(2\pi, 0, \rho_0) - \rho_0$ corresponding to system (A) (see §24.1). Let this function be defined for all ρ_0, $|\rho_0| < r_0$, where r_0 is some positive number. As we know (see §24.2, corollary from Lemma 3),

* The transformation $\xi = x$, $\eta = -\frac{a}{\beta}x - \frac{b}{\beta}y$ is non-singular, since by (73) $b \neq 0$.

** We mean here conditions each of which requires computation of one number. All these conditions are of course equivalent to the single condition $d(\rho_0) \equiv 0$, but the latter requires computation of a function, and not of numbers.

$d(\rho_0)$ is a function of class N or (respectively) an analytical function. By assumption, the point 0 is a root of multiplicity $2k+1$ of the function $d(\rho_0)$ (see §24.2, Definition 26). Consequently, there exist numbers $\varepsilon_0 > 0$ and $\sigma_0 > 0$ such that any function $\tilde{d}(\rho_0)$ defined for ρ_0, $|\rho_0| < r_0$, which is σ_0-close to rank $2k+1$ to the function $d(\rho_0)$ has at most $2k+1$ roots in the interval $(-\varepsilon_0, \varepsilon_0)$.

Consider modified systems $(\tilde{\mathrm{A}})$, which are given in canonical form and for which the point $O(0, 0)$ is a focus. Let $\tilde{d}(\rho_0)$ be the analog of $d(\rho_0)$ for $(\tilde{\mathrm{A}})$. By Theorem 3 in Appendix, 1, there exists $\delta_0 > 0$ such that if system $(\tilde{\mathrm{A}})$ is δ_0-close to rank $2k+1$ to system (A), the function $\tilde{d}(\rho_0)$ is defined for all ρ_0, $|\rho_0| < r_0$, and for these ρ_0 the functions $d(\rho_0)$ and $\tilde{d}(\rho_0)$ are σ_0-close to each other to rank $2k+1$. We will now prove that the numbers δ_0 and ε_0 satisfy the first proposition of the theorem. Suppose that this is not so, i.e., suppose that there exists a modified system $(\tilde{\mathrm{A}})$ of canonical form which is δ_0-close to rank $2k+1$ to system (A) and yet has more than k closed paths in $U_{\varepsilon_0}(O)$. Each of these paths $\tilde{L}$ crosses every ray issuing from O precisely at one point (see QT, §8.4, Lemma 1). Let ρ_1 and ρ_2 be the abscissas of the intersection points of a path $\tilde{L}$ with the rays $\theta = 0$ and $\theta = \pi$, respectively. Then ρ_1 and ρ_2 are respectively roots of the functions $\tilde{d}(\rho) = \tilde{d}_0(\rho)$ and $\tilde{d}_\pi(\rho)$, i.e., $\tilde{d}(\rho_1) = 0$, $\tilde{d}_\pi(\rho_2) = 0$. By (13) in §24.1, $\tilde{d}(-\rho_2) = -\tilde{d}_\pi(\rho_2)$. Therefore $\tilde{d}(-\rho_2) = 0$, i.e., $-\rho_2$ is also a root of the function $\tilde{d}$. Evidently ρ_1 and $-\rho_2$ are different numbers, smaller than ε_0 in absolute value. Thus, to every closed path $\tilde{L}$ of $(\tilde{\mathrm{A}})$ lying in $U_{\varepsilon_0}(O)$ correspond two roots of the function $\tilde{d}(\rho)$ from the interval $(-\varepsilon_0, \varepsilon_0)$. Hence, if system $(\tilde{\mathrm{A}})$ has in $U_{\varepsilon_0}(O)$ more than k closed paths, the function $\tilde{d}(\rho)$ has at least $2k+3$ different roots* in $(-\varepsilon_0, \varepsilon_0)$. This clearly contradicts the choice of the numbers δ_0 and ε_0. We have thus proved the first proposition of the theorem for modified systems $(\tilde{\mathrm{A}})$ given in canonical form.

Let us now consider a general modified system $(\tilde{\mathrm{A}})$ (not necessarily canonical). Suppose that the first proposition of the theorem is not satisfied. Then, for any $\varepsilon_1 > 0$ and $\delta_1 > 0$, there exists a system (A^*) δ_1-close to rank $2k+1$ to (A) which has more than k closed paths in $U_{\varepsilon_1}(O)$. If ε_1 and δ_1 are sufficiently small, there exists a linear transformation, as close as desired to the identity transformation, which transforms the system (A) to the canonical form $(\tilde{\mathrm{A}})$.** For sufficiently small ε_1 and δ_1, $(\tilde{\mathrm{A}})$ is also δ_0-close to (A) and has more than k closed paths in $U_{\varepsilon_0}(O)$, which are obtained from the closed paths of (A^*) in $U_{\varepsilon_1}(O)$ by the same linear transformation. This, however, contradicts the previous result. The first proposition of the theorem is thus completely proved.

2) We can now proceed with the proof of the second proposition. Consider a modified system of a special form

$$\begin{aligned}\frac{dx}{dt} &= \tilde{P}(x, y, \lambda_0, \lambda_1, \ldots, \lambda_{k-1}) = \\ &= P(x, y) + \lambda_0 x + \lambda_1 (x^2+y^2)x + \ldots + \lambda_{k-1}(x^2+y^2)^{k-1}x, \\ \frac{dy}{dt} &= \tilde{Q}(x, y, \lambda_0, \lambda_1, \ldots, \lambda_{k-1}) = \\ &= Q(x, y) + \lambda_0 y + \lambda_1 (x^2+y^2)y + \ldots + \lambda_{k-1}(x^2+y^2)^{k-1}y.\end{aligned} \qquad (\tilde{\mathrm{A}})$$

* Since zero is also a root of this function.

** This can be proved along the same lines as Lemma 1, §9.1. In this lemma, a similar proposition is proved for a saddle point. In the case of a focus, we should further make use of the fact that the $\begin{pmatrix}\alpha+\beta i & 0\\ 0 & \alpha-\beta i\end{pmatrix}$ is transformed to the matrix $\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$ by a linear transformation with the matrix $\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}$.

For sufficiently small λ_i, $(\tilde{A})$ is evidently arbitrarily close to (A) to any (possible) rank and belongs to the same class as (A). The point O is an equilibrium state of $(\tilde{A})$. Now, since the linear parts of the functions $\tilde{P}$ and $\tilde{Q}$ are respectively $\lambda_0 x - \beta y$ and $\beta x + \lambda_0 y$, the equilibrium state $O(0,0)$ is a focus of system (A) for $\lambda_0 \neq 0$ and a focus, a center, or a center-focus for $\lambda_0 = 0$. Let

$$\tilde{d}(\rho_0, \lambda_0, \lambda_1, \ldots, \lambda_{k-1})$$

be the analog of $d(\rho_0)$ for system $(\tilde{A})$. Clearly, $\tilde{d}$ is a continuous function of ρ_0 and of the parameters $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$. Since $(\tilde{A})$ is obtained from (A) for $\lambda_0 = \lambda_1 = \ldots = \lambda_{k-1} = 0$, we have

$$\tilde{d}(\rho_0, 0, 0, \ldots, 0) \equiv d(\rho_0). \tag{1}$$

For any $\varepsilon > 0$ and $\delta > 0$, there exist $\lambda^* > 0$ and $r^* > 0$, $r^* < r_0$,* such that for

$$|\lambda_i| < \lambda^*, \quad i = 0, 1, 2, \ldots, k-1, \tag{2}$$

1) system $(\tilde{A})$ is δ-close to rank $2k+1$ to (A);
2) the function $\tilde{d}(\rho_0, \lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ is defined for all ρ_0, $|\rho_0| < r_0$, and every root satisfying the inequality $|\rho_0| < r^*$ corresponds to a closed path of $(\tilde{A})$ entirely contained in $U_{\varepsilon_0}(O)$.

It is henceforth assumed that the numbers λ_i, $i = 0, 1, 2, \ldots, k-1$, satisfy condition (2).

We will show that for an appropriate choice of the parameters $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ system $(\tilde{A})$ has k closed paths in $U_\varepsilon(O)$.

From Maclaurin's formula, for all sufficiently small ρ_0,

$$d(\rho_0) = \frac{d^{(2k+1)}(0)}{(2k+1)!}\rho_0^{2k+1} + h(\rho_0)\rho_0^{2k+1}, \tag{3}$$

where $h(\rho_0)$ is a continuous function and $h(0) = 0$ (see proof of Theorem 5, §1.3).

By assumption, $d^{(2k+1)}(0) \neq 0$. To fix ideas, let $d^{(2k+1)}(0) > 0$. Then for all sufficiently small positive ρ_0, $d(\rho_0) > 0$. We choose one of these numbers, smaller than r^*; designating it r_1, we have

$$0 < r_1 < r^*, \quad d(r_1) = \tilde{d}(r_1, 0, 0, \ldots, 0) > 0. \tag{4}$$

Let now

$$\lambda_0 = \lambda_1 = \ldots = \lambda_{k-2} = 0, \quad \lambda_{k-1} \neq 0,$$

and consider the modified system corresponding to these parameters

$$\begin{aligned} \frac{dx}{dt} &= \tilde{P}(x, y, 0, 0, \ldots, 0, \lambda_{k-1}) = P(x, y) + \lambda_{k-1}(x^2+y^2)^{k-1}x, \\ \frac{dy}{dt} &= \tilde{Q}(x, y, 0, 0, \ldots, 0, \lambda_{k-1}) = Q(x, y) + \lambda_{k-1}(x^2+y^2)^{k-1}y \end{aligned} \tag{$\tilde{A}_1$}$$

* By definition, $r_0 > 0$ is a number such that the function $d(\rho_0)$ is defined for $|\rho_0| < r_0$.

and the corresponding function $\tilde{d}_1(\rho_0) = \tilde{d}(\rho_0, 0, 0, \ldots, 0, \lambda_{k-1})$. By Lemma 7 of the previous section,

$$\tilde{d}_1'(0) = \tilde{d}_1''(0) = \ldots = \tilde{d}_1^{(2k-2)}(0) = 0, \quad \tilde{d}_1^{(2k-1)}(0) = (2k-1)!\,\frac{\lambda_{k-1}}{\beta}\,2\pi, \tag{5}$$

and by Maclaurin's formula, for all sufficiently small ρ_0,

$$\tilde{d}_1(\rho_0) = \frac{\lambda_{k-1}}{\beta}\,\rho_0^{2k-1} + \tilde{h}_1(\rho_0)\,\rho_0^{2k-1}, \tag{6}$$

where $\tilde{h}_1(\rho_0)$ is a continuous function and $\tilde{h}_1(0) = 0$.

To fix ideas, let $\beta > 0$, and we choose λ_{k-1} so that

$$|\lambda_{k-1}| < \lambda^*, \quad \lambda_{k-1} < 0, \quad \tilde{d}_1(r_1) = \tilde{d}(r_1, 0, 0, \ldots, 0, \lambda_{k-1}) > 0. \tag{7}$$

The last of these conditions is satisfied for any sufficiently small λ_{k-1} in virtue of (4) and the continuity of $\tilde{d}(\rho_0, \lambda_0, \lambda_1, \ldots, \lambda_{k-1})$.

From $\beta > 0$, $\lambda_{k-1} < 0$ and equation (6) it follows that for all sufficiently small positive ρ_0, $\tilde{d}_1(\rho_0) < 0$. We choose one such value, smaller than r_1, and designate it r_2. Thus,

$$0 < r_2 < r_1 < r^* \tag{8}$$

and

$$\tilde{d}_1(r_1) > 0, \quad \tilde{d}_1(r_2) < 0. \tag{9}$$

Further construction is completely analogous.* Indeed, we consider a system

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) + \lambda_{k-2}(x^2 + y^2)^{k-2}x + \lambda_{k-1}(x^2 + y^2)^{k-1}x = \\ &= \tilde{P}(x, y, 0, \ldots, 0, \lambda_{k-2}, \lambda_{k-1}), \\ \frac{dy}{dt} &= Q(x, y) + \lambda_{k-2}(x^2 + y^2)^{k-2}y + \lambda_{k-1}(x^2 + y^2)^{k-1}y = \\ &= \tilde{Q}(x, y, 0, \ldots, 0, \lambda_{k-2}, \lambda_{k-1}), \end{aligned} \tag{$\tilde{A}_2$}$$

which is modified both in relation to (A) and in relation to $(\tilde{A}_1)$, with the corresponding function

$$\tilde{d}_2(\rho_0) = \tilde{d}(\rho_0, 0, 0, \ldots, 0, \lambda_{k-2}, \lambda_{k-1}).$$

From (5) and Lemma 7 of the preceding section, we have

$$\tilde{d}_2'(0) = \tilde{d}_2''(0) = \ldots = \tilde{d}_2^{(2k-4)}(0) = 0, \quad \tilde{d}_2^{(2k-3)}(0) = (2k-3)!\,\frac{\lambda_{k-2}}{\beta}\,2\pi. \tag{10}$$

Therefore, for all sufficiently small ρ_0,

$$\tilde{d}_2(\rho_0) = \frac{\lambda_{k-2}}{\beta}\,2\pi\rho_0^{2k-3} + \tilde{h}_2(\rho_0)\,\rho_0^{2k-3}, \tag{11}$$

where $\tilde{h}_2(\rho_0)$ is a continuous function and $\tilde{h}_2(0) = 0$. We choose λ_{k-2} so that

$$|\lambda_{k-2}| < \lambda^*, \quad \lambda_{k-2} > 0, \quad \tilde{d}_2(r_1) > 0, \quad \tilde{d}_2(r_2) < 0. \tag{12}$$

* Also see the proof of Lemma 1, §1.3.

By (11), $\widetilde{d}_2(\rho_0) > 0$ for all sufficiently small positive ρ_0, and there exists r_3 such that

$$0 < r_3 < r_2 \tag{13}$$

and

$$\widetilde{d}_2(r_3) > 0. \tag{14}$$

Continuing as before, we eventually obtain a system

$$\begin{aligned}\frac{dx}{dt} &= \widetilde{P}(x, y, 0, \lambda_1, \ldots, \lambda_{k-1}) = \\ &= P(x, y) + \lambda_1(x^2+y^2)x + \ldots + \lambda_{k-1}(x^2+y^2)^{k-1}x, \\ \frac{dy}{dt} &= \widetilde{Q}(x, y, 0, \lambda_1, \ldots, \lambda_{k-1}) = \\ &= Q(x, y) + \lambda_1(x^2+y^2)y + \ldots + \lambda_{k-1}(x^2+y^2)y\end{aligned} \tag{$\widetilde{A}_{k-1}$}$$

and numbers $r_1, r_2, \ldots, r_{k-1}, r_k$ such that $|\lambda_i| < \lambda^*$,

$$0 < r_k < r_{k-1} < \ldots < r_2 < r_1 < r^* \tag{15}$$

and

$$\widetilde{d}_{k-1}(r_1) > 0, \quad \widetilde{d}_{k-1}(r_2) < 0, \ \ldots, \ \widetilde{d}_{k-1}(r_k) \ \begin{cases} > 0 & \text{if } k \text{ is odd}, \\ < 0 & \text{if } k \text{ is even}. \end{cases} \tag{16}$$

Continuing to the system

$$\begin{aligned}\frac{dx}{dt} &= \widetilde{P}(x, y, \lambda_0, \lambda_1, \ldots, \lambda_{k-1}) = \widetilde{P}(x, y, 0, \lambda_1, \ldots, \lambda_{k-1}) + \lambda_0 x, \\ \frac{dy}{dt} &= \widetilde{Q}(x, y, \lambda_0, \lambda_1, \ldots, \lambda_{k-1}) = \widetilde{Q}(x, y, 0, \lambda_1, \ldots, \lambda_{k-1}) + \lambda_0 y,\end{aligned} \tag{$\widetilde{A}_k$}$$

we choose λ_0 so small that

$$|\lambda_0| < \lambda^*, \quad \widetilde{d}_k(r_1) > 0, \quad \widetilde{d}_k(r_2) < 0, \ \ldots, \ \widetilde{d}_{k-1}(r_k) \gtrless 0 \tag{17}$$

and

$$\widetilde{d}_k'(0) \ \begin{cases} < 0 & \text{if } k \text{ is odd}, \\ > 0 & \text{if } k \text{ is even}. \end{cases} \tag{18}$$

By (25), §24.2, we have for the first focal value

$$\widetilde{d}_k'(0) = e^{2\pi\frac{\lambda_0}{\beta}} - 1.$$

Therefore, to satisfy (18), a negative λ_0 should be taken if k is odd and a positive one if k is even. Clearly, if for this choice of λ_0, the number $r_{k+1} > 0$ is sufficiently small, we have

$$\widetilde{d}_k(r_{k+1}) \ \begin{cases} < 0 & \text{if } k \text{ is odd}, \\ > 0 & \text{if } k \text{ is even}. \end{cases} \tag{19}$$

We moreover assume that $r_{k+1} < r_k$. All the λ_i, $i = 0, 1, 2, \ldots, k-1$, that we have chosen are less than λ^* in absolute value. Therefore, by condition 1 imposed on λ^*, the system $(\tilde{A}_k)$ is δ-close to rank $2k+1$ to system (A). Furthermore, in virtue of (17) and (19) and the continuity of d_k, there is at least one root of the function $\tilde{d}_k(\rho_0)$ between each pair r_1 and r_2, r_2 and r_3, ..., r_k and r_{k+1}. Therefore, this function has at least k different positive roots, each smaller than r^*. By condition 2 for r^*, each of these roots corresponds to a closed path entirely contained in $U_\varepsilon(O)$. If $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$, $U_\varepsilon(O) \subset U_{\varepsilon_0}(O)$ and by the first proposition of the theorem, $(\tilde{A}_k)$ has at most k different closed paths in $U_{\varepsilon_0}(O)$. This completes the proof of the theorem.

Remark. In our proof of the first proposition of Theorem 40, ε_0 and δ_0 should be chosen so small that any system $(\tilde{A})$ δ_0-close to (A) has a single equilibrium state in $U_{\varepsilon_0}(O)$, which is moreover a focus. Then we can speak of the existence of the succession function and the function d for the system $(\tilde{A})$. In what follows, the numbers ε_0 and δ_0 are always assumed to meet this requirement. Moreover, if δ_0 is sufficiently small, the points move along the paths of $(\tilde{A})$ in $U_{\varepsilon_0}(O)$ in the same direction with increasing t (clockwise). Indeed, reducing $(\tilde{A})$ to a canonical form by a transformation close to identity, i.e., by an orientation-conserving transformation, we obtain a system

$$\frac{dx}{dt} = -\tilde{\beta}y + \tilde{\varphi}(x, y), \qquad \frac{dy}{dt} = \tilde{\beta}x + \tilde{\psi}(x, y).$$

The number $\tilde{\beta}$ is close to β and therefore has the same sign. This ensures identical directions of motion. We will assume in the following that this condition is also satisfied.

2. Bifurcations of a dynamic system in the neighborhood of a multiple focus

The following theorem strengthens the second proposition of Theorem 40. Together with Theorem 40, it plays a leading role regarding the bifurcations of a dynamic system in the neighborhood of a multiple focus.

Theorem 41. Let $O(0, 0)$ be a multiple focus of multiplicity k of a dynamic system (A) *of class $N \geqslant 2k+1$ or of analytical class, and let ε_0 and δ_0 be positive numbers defined by the first proposition of Theorem 40 and the above remark following the theorem. Then*

1) for any ε and δ, $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \delta \leqslant \delta_0$, and for any s, $1 \leqslant s < k$, there exists a system (B) *of class N (or respectively, analytical) which is δ-close to rank $2k+1$ to system* (A) *and has in $U_\varepsilon(O)$ precisely s closed paths;*

2) if system (B) *is δ_0-close to rank $2k+1$ to system* (A) *and has k limit cycles in $U_{\varepsilon_0}(O)$, all these cycles, and likewise the focus of system* (B) *lying in $U_{\varepsilon_0}(O)$, are structurally stable (simple).*

Proof. Let us prove the first proposition of the theorem. Let $1 \leqslant s < k$ (for $s = k$, this proposition coincides with the second proposition of Theorem 40). In the proof of Theorem 40, we constructed a succession of systems $(\tilde{A}_1)$, $(\tilde{A}_2)$, ..., $(\tilde{A}_{k-1})$, $(\tilde{A}_k)$. All these systems evidently can be assumed to be δ-close to rank $2k+1$ to system (A).

Consider the system $(\tilde{A}_s)$. According to our construction,

$$\tilde{d}_s'(0) = \tilde{d}_s''(0) = \ldots = \tilde{d}_s^{(2k-2s)}(0) = 0, \quad \tilde{d}_s^{(2k-2s+1)}(0) \neq 0 \tag{20}$$

and

$$\tilde{d}_s(r_1) > 0, \quad \tilde{d}_s(r_2) < 0, \ \ldots, \ \tilde{d}_s(r_{s+1}) \begin{cases} > 0 & \text{if } s \text{ is even,} \\ < 0 & \text{if } s \text{ is odd.} \end{cases} \tag{21}$$

(see proof of Theorem 40). By (21), $(\tilde{\mathrm{A}}_s)$ has at least s closed paths in $U_\varepsilon(O)$. Suppose that it has $s+1$ closed paths $L_1, L_2, \ldots, L_{s+1}$ in $U_\varepsilon(O)$. By condition (20), the point $O\,(0, 0)$ is a multiple focus of multiplicity $k-s$ for $(\tilde{\mathrm{A}}_s)$. Let V and W be two neighborhoods of O, such that $W \subset V \subset U_\varepsilon(O)$, and V lies inside all the closed paths L_i ($i = 1, 2, \ldots, s+1$). These paths are "concentric" (Figure 115). By Theorem 40 there exists a system $(\mathrm{A}^*)'$ of class N, arbitrarily close to $(\tilde{\mathrm{A}}_s)$ to rank $2k+1$, which has $k-s$ closed paths $L_{s+2}, L_{s+3}, \ldots, L_{k+1}$ in W. But then, using the same construction as in the proof of Lemma 2, §15.2, we can construct a system $(\hat{\mathrm{A}})$ of class N, arbitrarily close to rank $2k+1$ to system $(\tilde{\mathrm{A}}_s)$, which coincides with $(\tilde{\mathrm{A}}_s)$ outside V and with (A^*) inside W. The system $(\hat{\mathrm{A}})$ evidently can be regarded as δ_0-close to rank $2k+1$ to (A), and it has at least $k+1$ closed paths L_l ($i = 1, 2, \ldots, k+1$) in $U_{\varepsilon_0}(O)$. This clearly contradicts Theorem 40. Therefore, $(\tilde{\mathrm{A}}_s)$ has precisely s limit cycles in $U_\varepsilon(O)$, i.e., the first proposition of the theorem is proved.*

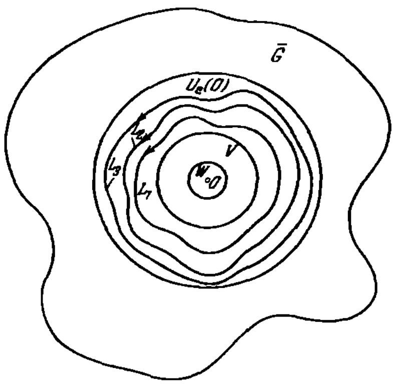

FIGURE 115

Let now (B) be δ_0-close to rank $2k+1$ to (A) and suppose that it has k limit cycles in $U_{\varepsilon_0}(O)$, at least one of which is structurally unstable. Then modifying the system only in the neighborhood of this cycle (again using the construction of Lemma 2, §15.2), we obtain a system which is close to (A) and has in $U_{\varepsilon_0}(O)$ more than k closed cycles, which is impossible. A similar contradiction is obtained if we assume that the focus of system (B) in $U_{\varepsilon_0}(O)$ is a multiple focus. The proof of the theorem is complete.

Theorems 40 and 41 lead to important conclusions regarding the possible bifurcations of a dynamic system in the neighborhood of its multiple focus of finite multiplicity. Indeed, consider a k-tuple focus $O\,(0, 0)$ of system (A) ($k \geqslant 2$). Let ε_0 and δ_0 be sufficiently small numbers (defined by Theorem 40 and the remark following the theorem), and V a neighborhood of O bounded by a cycle without contact Γ, $V \subset U_{\varepsilon_0}(O)$. We choose δ, $0 < \delta < \delta_0$, to be so small that the following condition is satisfied: if $(\tilde{\mathrm{A}})$ is δ-close to (A), the curve Γ remains a cycle without contact for $(\tilde{\mathrm{A}})$ and $(\tilde{\mathrm{A}})$ has in V a single equilibrium state $\tilde{O}$, which is a focus. By Theorems 40 and 41, system $(\tilde{\mathrm{A}})$ δ-close to rank $2k+1$ to (A) may have in V at most k limit cycles, and there exist systems $(\tilde{\mathrm{A}})$ which have in V precisely s limit cycles, where s is any number, $1 \leqslant s \leqslant k$. These limit cycles are arranged "concentrically" and enclose the focus $\tilde{O}$ inside them. Let $L_1, L_2, \ldots, L_s$ be these cycles, and suppose that L_l lies inside L_{l+1} ($i = 1, 2, \ldots, s-1$). The topological structure

* If (A) is analytical, $(\tilde{\mathrm{A}}_s)$ is also analytical.

of system $(\tilde{A})$ in V is completely determined by the stability character of the focus $\tilde{O}$, the number s of limit cycles in V, and their respective stabilities. If the stability character of the focus $\tilde{O}$ is known, it suffices to know whether each of the cycles L_i is semistable or not.* In §12, we introduced the multiplicity of a limit cycle** and established that an even-multiplicity cycle is semistable and an odd-multiplicity cycle is either stable or unstable (this follows from §12.3, (8), and §12.4, (11)). Hence, the topological structure of the dynamic system $(\tilde{A})$ in the neighborhood V is completely determined if we know

(a) the stability character of the focus $\tilde{O}$;

(b) the number s of limit cycles of $(\tilde{A})$ in V;

(c) the parity of the multiplicity of each of these cycles (whether odd or even).

Since $s \leqslant k$, system $(\tilde{A})$ clearly may have only a finite number of different topological structures in V. In other words, system (A) may undergo only a finite number of different bifurcations in a neighborhood of a focus of finite multiplicity.†

Assigning $+(-)$ to a stable (unstable) focus, and the numbers 0(1) to a cycle of even (odd) multiplicity, we obtain the following classification, corresponding to items (a), (b), (c) above:

$$+,\ 1,\ 1,\ 0,\ 1,\ 0,\ 0 \ldots 0,\ 1,\ 0,$$

where the number of ones and zeros is s. Each system $(\tilde{A})$ δ-close to rank $2k+1$ to (A) is characterized by a definite scheme of this kind. We will not consider the inverse question, namely whether or not every scheme of this kind fully characterizes a dynamic system arbitrarily close to (A). Note, however, that by Theorems 40 and 41, there exist systems arbitrarily close to (A) whose schemes contain k ones.

3. Bifurcations in the neighborhood of a multiple focus of multiplicity 1

We will consider a special case which is often encountered in applications. Let

$$\begin{aligned} \frac{dx}{dt} &= a(\lambda)x + b(\lambda)y + \varphi(x, y, \lambda) = P(x, y, \lambda), \\ \frac{dy}{dt} &= c(\lambda)x + d(\lambda)y + \psi(x, y, \lambda) = Q(x, y, \lambda) \end{aligned} \qquad (A_\lambda)$$

be a dynamic system which depends on the parameter λ. We will consider the bifurcations of this system associated with the variation of the parameter λ in the neighborhood of an equilibrium state $O(0, 0)$, when O is a multiple focus of multiplicity 1. For simplicity, we assume that the bifurcation

* If, say, the focus $\tilde{O}$ is unstable, the cycle L_1 is either unstable or semistable (unstable from inside and stable from outside). In the former case, L_2 is either stable or semistable, and in the latter case it is either unstable or semistable, etc.

** In §12.3, the multiplicity of a limit cycle is defined for analytical systems. An analogous definition for systems of class N will be found in Chapter X (§26.2, Definition 28).

† We should stress that in the case of a k-tuple focus, the closeness of systems is considered to rank $2k+1$.

value of the parameter is $\lambda = 0$. Let

$$\sigma(\lambda) = a(\lambda) + d(\lambda), \tag{22}$$

$$\Delta(\lambda) = \begin{vmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{vmatrix}. \tag{23}$$

Then

$$\sigma(0) = 0, \tag{24}$$

$$\Delta(0) > 0. \tag{25}$$

To establish the stability character of the multiple focus $O(0, 0)$ of the system (A_0), we will proceed along the same lines as in the end of §24, when dealing with system (72). We apply the transformation

$$\xi = x, \quad \eta = -\frac{a(0)}{\sqrt{\Delta(0)}}x - \frac{b(0)}{\sqrt{\Delta(0)}}y, \tag{26}$$

which reduces (A_0) to the canonical form

$$\frac{d\xi}{dt} = -\sqrt{\Delta(0)}\,\eta + \tilde{\varphi}(\xi, \eta), \quad \frac{d\eta}{dt} = \sqrt{\Delta(0)}\,\xi + \tilde{\psi}(\xi, \eta). \tag{27}$$

Since (26) is a non-singular transformation, O remains a multiple focus of multiplicity 1 for system (27) also, and its stability does not change either. The third focal value of the focus O of (27) therefore does not vanish, and is the Lyapunov or the first Lyapunov value of the focus O (see §24.4, Definition 27). We used the symbol $6\alpha_3$ for this value in §24.4, (63). Equation (76) in §24 provides an expression for α_3 in terms of the coefficients of the original system (A_0), and therefore transformation (26) need not be actually applied in practice. We have seen (§24.2, (26), (27)) that if $\alpha_3 < 0$, O is a stable focus (of system (27), and therefore of (A_0)), and if $\alpha_3 > 0$, it is an unstable focus.

Let V be a sufficiently small neighborhood of the point O bounded by a cycle without contact Γ of system (A_0) which contains no closed paths of (A_0) or equilibrium states other than O.

Let $\delta_0 > 0$ be so small that any system (A_λ) for which $|\lambda| < \delta_0$ has the following properties:

(a) the curve Γ is a cycle without contact for this system;
(b) (A_λ) has no equilibrium states, other than O, in V;
(c) the point O is a focus of system (A_λ);
(d) (A_λ) has at most one closed path in V.

The validity of the first three conditions for sufficiently small δ_0 is self-evident. Condition (d) is satisfied because O is a focus of multiplicity 1 of system (A_0) and because close systems have at most one closed path in a small neighborhood of this focus (by Theorem 40).

By condition (a), the paths of all systems (A_λ) ($|\lambda| < \delta_0$) simultaneously cross the curve Γ with increasing t, either from outside to inside or from inside to outside.

Suppose that $\sigma(\lambda)$ reverses its sign as the system passes through the bifurcation value of the parameter $\lambda = 0$, i.e., the focus O changes its stability. This condition is clearly satisfied if $\sigma'(0) \neq 0$.

Let us now consider the different possible cases.

1) $\alpha_3 < 0$. We assume that on passing through the bifurcation value of the parameter $\lambda = 0$, $\sigma(\lambda)$ changes its sign from minus to plus. If $\sigma'(0) \neq 0$, this condition is satisfied

when λ increases, for $\sigma'(0) > 0$;

when λ decreases, for $\sigma'(0) < 0$.

Since $\alpha_3 < 0$, the focus O is a stable focus of (A_0) for $\lambda = 0$. Therefore all the paths of (A_λ) enter into the cycle without contact Γ as t increases. For $\sigma(\lambda) < 0$, O is a stable focus of (A_λ). By Theorems 40 and 41, (A_λ) has at most one limit cycle in V, and if this cycle exists, it is a simple cycle, i.e., either stable or unstable. Clearly, for $\sigma(\lambda) < 0$ no such cycle exists. Indeed, if this cycle existed, it would be stable from outside and unstable from inside, i.e., it could not be simple. We have thus established that if $\alpha_3 < 0$ and $\sigma(\lambda) < 0$, (A_λ) has no limit cycles in V.

Conversely, if $\sigma(\lambda) > 0$, O is an unstable focus of (A_λ). Then, reasoning as before, we conclude that there is a single limit cycle L_λ of (A_λ) inside V, and this is a simple stable cycle. It is readily seen that if λ is sufficiently small, the cycle L_λ is arbitrarily close to O.

We thus obtain the following results. If $\alpha_3 < 0$ and $\sigma'(0) > 0$, system (A_λ) has no limit cycles in V for small negative λ and $\lambda = 0$, and O is a stable focus. As the system crosses the bifurcation value of the parameter λ (i.e., for $\lambda > 0$), the focus becomes unstable, and a stable limit cycle develops inside the neighborhood V (Figure 116).

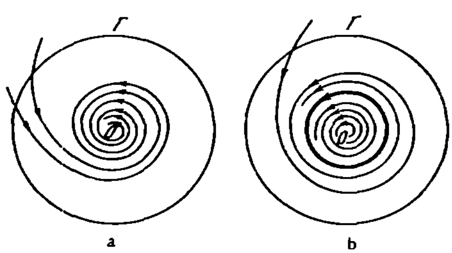

FIGURE 116. $\alpha_3 < 0$, $\sigma'(0) > 0$. a) $\lambda < 0$, stable focus; $\lambda = 0$, stable multiple focus; b) $\lambda > 0$, unstable focus, stable cycle.

If λ is varied in the opposite direction, i.e., we move from positive to negative λ, the stable limit cycle which originally existed in V would contract to the focus O and vanish for $\lambda = 0$, and the focus will change its stability accordingly.

As λ is further decreased, the focus remains stable and the topological structure of V does not change.

For $\alpha_3 < 0$, $\sigma'(0) < 0$, the stable limit cycle is created on passing from positive to negative λ, and conversely it disappears when λ increases and reaches zero.

2) $\alpha_3 > 0$. The investigation proceeds along the same lines as before.

If $\alpha_3 > 0$ and $\sigma'(0) > 0$, the point O for small negative λ is a stable focus of (A_λ) and the system has one unstable limit cycle in V. As λ increases, this cycle contracts to the point O, and at the bifurcation point $\lambda = 0$ it disappears and the focus O becomes unstable. Further increase of λ leaves

the focus O unstable and the topological structure of the system in V does not change (Figure 117).

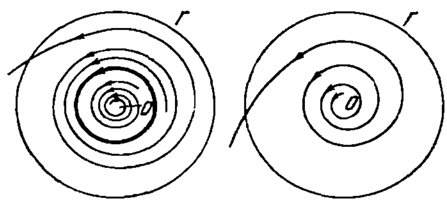

FIGURE 117. $\alpha_3 > 0$, $\sigma'(0) > 0$. a) $\lambda < 0$, stable focus, unstable cycle; b) $\lambda = 0$, multiple unstable focus: $\lambda > 0$, unstable focus.

If $\alpha_3 > 0$ and $\sigma'(0) < 0$, the unstable limit cycle is created as λ changes from negative to positive values, and conversely disappears when λ decreases to zero.

The above results can be summarized in the following table:

	$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
$\alpha_3 < 0$, $\sigma'(0) > 0$	Unstable focus, no cycle	Stable focus, no cycle	Unstable focus, stable cycle
$\alpha_3 < 0$, $\sigma'(0) < 0$	Unstable focus, stable cycle	Stable focus, no cycle	Stable focus, no cycle
$\alpha_3 > 0$, $\sigma'(0) > 0$	Stable focus, unstable cycle	Unstable focus, no cycle	Unstable focus, no cycle
$\alpha_3 > 0$, $\sigma'(0) < 0$	Unstable focus no cycle	Unstable focus, no cycle	Stable focus, unstable cycle

The above analysis shows that the change in λ brings about a change in the stability of the focus if a limit cycle is created from the focus or disappears contracting into the focus. A stable focus creates a stable cycle, and an unstable focus, an unstable cycle. Thus, a focus creates a limit cycle of the same stability, and the stability of the focus changes in the process. Conversely, when the cycle disappears (when it is "absorbed" by the focus), the focus acquires the same stability as that of the cycle before "absorption." This state of things is not limited to the case of a focus of multiplicity 1: it is observed whenever a focus creates or absorbs a cycle of definite stability (i.e., not a semistable cycle).

Example 9. Consider the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \lambda y + \beta xy + \gamma y^2 + \delta x^2 \tag{28}$$

for small values of the parameter λ.* This system has two equilibrium states, and we will consider the state $O(0, 0)$ only.

* System (28) is of importance in the theory of sustained oscillations. It was investigated by Bautin /20/.

The characteristic equation $k^2-\lambda k+1=0$ has two roots $k=\frac{\lambda}{2}\pm\sqrt{\frac{\lambda^2}{4}-1}$. Therefore, for small λ, the point $O(0,0)$ is a focus, which is stable for $\lambda<0$ and unstable for $\lambda>0$.

For this system, $\sigma(\lambda)=\lambda$, and therefore $\sigma'(\lambda)=1$. To compute α_3, we use equation (76) in §24.4, taking

$$a=0,\ b=1,\ c=-1,\ d=0,\ b_{20}=\delta,\ b_{11}=\beta,\ b_{02}=\gamma,\ \sqrt{\Delta}=1.$$

This gives $\alpha_3=\pi/4\,\beta(\gamma+\delta)$. From the inequality $\sigma'(0)>0$ and the table above we conclude that if $\beta(\gamma+\delta)<0$, the focus $O(0,0)$ is stable for $\lambda\leqslant 0$ and there are no limit cycles in its neighborhood. As we move to positive λ, the focus creates a stable limit cycle, and itself becomes unstable.

If $\beta(\gamma+\delta)>0$, the focus $O(0,0)$ is stable for $\lambda<0$ and there is an unstable limit cycle in its neighborhood. As λ increases, the cycle contracts to a point and disappears for $\lambda=0$. At this instant, the focus O becomes unstable. The topological structure in the neighborhood of O for $\lambda>0$ is the same as for $\lambda=0$.

Chapter X

CREATION OF CLOSED PATHS FROM A MULTIPLE LIMIT CYCLE

INTRODUCTION

The present chapter is entirely analogous to Chapter IX, Creation of Limit Cycles from a Multiple Focus both in regard to methods used and the results obtained. It is directly related to Chapter V, Closed Paths, and may be regarded as its continuation. As in Chapter V, the main emphasis is on the succession function $f(n)$ on a normal to the limit cycle L_0 and the function $d(n) = f(n) - n$. In Chapter V we have seen that if $d'(0) \neq 0$, the limit cycle L_0 of system (A) is structurally stable, and if $d'(0) = 0$, the limit cycle is unstable and there exist systems arbitrarily close to (A) which have at least two closed paths in any small neighborhood of L_0.

In the present chapter we consider not only the first derivative $d'(0)$, but also the value of the higher derivatives of the functions $d(n)$ at the point 0, and this leads to more refined results regarding the creation of closed paths from a multiple limit cycle.

Chapter X is divided into two sections. The first, §26, although highly significant for what follows, presents auxiliary background information. It is mainly devoted to the derivation of expressions for the derivatives of the succession functions in terms of the right-hand sides P and Q of the dynamic system. Moreover, the multiplicity of a limit cycle is defined in §26 (Definition 28, §26.2). A limit cycle L_0 is said to have multiplicity r (or to be an r-tuple limit cycle) if

$$d'(0) = d''(0) = \ldots d^{(r-1)}(0) = 0, \qquad d^{(r)}(0) \neq 0.$$

The fundamental theorems concerning the creation of closed paths from a multiple limit cycle are presented in §27 (Theorems 42 and 43). Theorem 42 is analogous to Theorem 40 on the creation of closed paths from a multiple focus. It amounts to the following: if L_0 is a multiple limit cycle of multiplicity k of a dynamic system (A), systems sufficiently close to (A) can have at most k closed paths in a sufficiently small neighborhood of L_0. On the other hand, there exist systems as close as desired to (A) with precisely k closed paths in any small neighborhood of L_0. In distinction from the case of a k-tuple focus, when closeness to rank $2k+1$ is postulated, here we are dealing with closeness to rank k. Theorem 43 shows that if L_0 is a k-tuple limit cycle, and s is an integer, $1 \leqslant s \leqslant k$, there exist systems arbitrarily close to (A) which have precisely s limit cycles in any small

neighborhood of L_0. It follows from Theorems 42 and 43 that a dynamic system may undergo only a finite number of different bifurcations in the neighborhood of a limit cycle of finite multiplicity. These different bifurcations can be readily classified.

§26. EXPRESSIONS FOR THE DERIVATIVES OF THE SUCCESSION FUNCTION. MULTIPLICITY OF A LIMIT CYCLE

1. Expressions for the derivatives of succession functions

Let

$$\frac{dx}{dt}=P(x,\ y),\qquad \frac{dy}{dt}=Q(x,\ y) \tag{A}$$

be a dynamic system of class $N\geqslant 1$ or an analytical system, L_0 a closed path of (A),

$$x=\varphi(t),\ y=\psi(t) \tag{1}$$

the motion corresponding to this path, $\tau>0$ the period of the functions φ and ψ.

Consider the neighborhood Ω of the path L_0 described in Chapter V (§13.1) with the curvilinear coordinates s and n, defined by the relations

$$x=\bar{\varphi}(s,\ n),\qquad y=\bar{\psi}(s,\ n), \tag{2}$$

where

$$\bar{\varphi}(s,\ n)=\varphi(s)+n\cdot\psi'(s),\qquad \bar{\psi}(s,\ n)=\psi(s)-n\cdot\varphi'(s). \tag{3}$$

The functions $\bar{\varphi}$ and $\bar{\psi}$ are considered in the strip

$$-\infty<s<+\infty,\qquad -\delta<n<\delta, \tag{4}$$

where δ is a sufficiently small positive number.

The properties of mapping (2) and of functions (3) are described in Chapter V, §13.1.

Changing over to the variables s and n in system (A), using relations (2), we obtain the system

$$\frac{ds}{dt}=\frac{P(\bar{\varphi},\ \bar{\psi})\cdot\bar{\psi}'_n-Q(\bar{\varphi},\ \bar{\psi})\cdot\bar{\varphi}'_n}{\Delta(s,\ n)},\qquad \frac{dn}{dt}=\frac{Q(\bar{\varphi},\ \bar{\psi})\cdot\bar{\varphi}'_s-P(\bar{\varphi},\ \bar{\psi})\cdot\bar{\psi}'_s}{\Delta(s,\ n)}, \tag{5}$$

where

$$\Delta(s,\ n)=\begin{vmatrix}\bar{\varphi}'_s & \bar{\varphi}'_n\\ \bar{\psi}'_s & \bar{\psi}'_n\end{vmatrix} \tag{6}$$

(see §13.2, (8)). As in §13, we will reduce system (5) to a single differential equation

$$\frac{dn}{ds}=\frac{Q(\overline{\varphi},\overline{\psi})\cdot\overline{\varphi}'_s-P(\overline{\varphi},\overline{\psi})\cdot\overline{\psi}'_s}{P(\overline{\varphi},\overline{\psi})\cdot\overline{\psi}'_n-Q(\overline{\varphi},\overline{\psi})\cdot\overline{\varphi}'_n}=R(s,\ n), \tag{7}$$

which is obtained when the second equation in (5) is divided through by the first equation. This division is permissible by §13.2, (11). We will reiterate here the properties of the function $R(s, n)$ and of equation (7).

$R(s, n)$ is defined in the strip (4), where it is a continuous function, periodic in s with a period of τ. If (A) is a dynamic system of class N (analytical system), $R(s, n)$ is also a function of class N (analytical function). Moreover,

$$R(s,\ 0)\equiv 0. \tag{8}$$

Therefore, $n\equiv 0$ is a solution of equation (7). This solution evidently corresponds to the closed path L_0 of (A).

Let

$$n=f(s;\ s_0,\ n_0) \tag{9}$$

be a solution of (7) satisfying the initial condition

$$f(s_0;\ s_0,\ n_0)=n_0. \tag{10}$$

Then

$$f(s_0+\tau;\ s_0,\ n_0)$$

is a succession function on the arc without contact $s = s_0$ which is a normal to the path L_0. In what follows, we shall take for simplicity $s_0 = 0$ (this evidently does not restrict the generality of our analysis); the corresponding succession function will be designated $f(n_0)$ and the normal $s = 0$ will be designated l. Thus,

$$f(n_0)=f(\tau;\ 0,\ n_0) \tag{11}$$

is the succession function on the arc without contact l.

Together with the succession function $f(n_0)$, we consider the function

$$d(n_0)=f(n_0)-n_0. \tag{12}$$

In Chapter V we computed the first derivative of this function at $n_0 = 0$,

$$d'(0)=e^{\int_0^\tau [P'_x(\varphi(s),\ \psi(s))+Q'_y(\varphi(s),\ \psi(s))]ds}-1. \tag{13}$$

Our immediate aim is to derive expressions for the higher derivatives, i.e.,

$$d''(0),\ d'''(0),\ \ldots,$$

or, equivalently, for the derivatives of the succession function

$$f''(0),\ f'''(0),\ \ldots$$

The derivation of these expressions is analogous to the derivation of the expressions for the focal values in §24.3.

Since the right-hand side $R(s, n)$ of the differential equation (7) is a function of class N, the solution

$$n = f(s;\ 0,\ n_0)$$

of this equation has continuous partial derivatives with respect to n_0 to order N inclusive.

As in §24.2, Lemma 4, we can show that these partial derivatives, treated as functions of s (i.e., for constant n_0), satisfy the following system of differential equations:*

$$\left.\begin{aligned}
&\frac{d}{ds}\left(\frac{\partial f}{\partial n_0}\right) = \frac{\partial R(s,\ f(s;\ 0,\ n_0))}{\partial n}\,\frac{\partial f(s;\ 0,\ n_0)}{\partial n_0},\\
&\frac{d}{ds}\left(\frac{\partial^2 f}{\partial n_0^2}\right) = \frac{\partial R(s,\ f)}{\partial n}\,\frac{\partial^2 f}{\partial n_0^2} + \frac{\partial^2 R(s,\ f)}{\partial n^2}\left(\frac{\partial f}{\partial n_0}\right)^2 =\\
&\qquad\qquad = \frac{\partial R(s,\ f)}{\partial n}\,\frac{\partial^2 f}{\partial n_0^2} + E_2(s,\ n_0),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{d}{ds}\left(\frac{\partial^N f}{\partial n_0^N}\right) = \frac{\partial R(s,\ f)}{\partial n}\,\frac{\partial^N f}{\partial n_0^N} + \frac{\partial^N R}{\partial n^N}\left(\frac{\partial f}{\partial n_0}\right)^N + \dots + N\frac{\partial^2 R}{\partial n^2}\,\frac{\partial^{N-1} f}{\partial n_0^{N-1}}\,\frac{\partial f}{\partial n_0} =\\
&\qquad\qquad = \frac{\partial R(s,\ f)}{\partial n}\,\frac{\partial^N f}{\partial n_0^N} + E_n(s,\ n_0)\ .
\end{aligned}\right\} \tag{14}$$

By (10), $f(0;\ 0,\ n_0) \equiv n_0$. Therefore,

$$\left[\frac{\partial f}{\partial n_0}\right]_{s=0} = 1,\quad \left[\frac{\partial^2 f}{\partial n_0^2}\right]_{s=0} = 0,\ \dots,\ \left[\frac{\partial^N f}{\partial n_0^N}\right]_{s=0} = 0. \tag{15}$$

Equalities (15) provide the initial conditions for equations (14).

Proceeding as in §24.3, we introduce the notation

$$\frac{1}{k!}\left[\frac{\partial^k f(s;\ 0,\ n_0)}{\partial n_0^k}\right]_{n_0=0} = u_k(s) \qquad (k = 1,\ 2,\ \dots,\ N), \tag{16}$$

$$\frac{1}{k!}\left[\frac{\partial^k R(s,\ n)}{\partial n^k}\right]_{n=0} = R_k(s) \qquad (k = 1,\ 2,\ \dots,\ N), \tag{17}$$

$$\frac{1}{k!}[E_k(s,\ n_0)]_{n_0=0} = H_k(s) =$$
$$= \frac{1}{k!}\left[\frac{\partial^k R}{\partial n^k}\left(\frac{\partial f}{\partial n_0}\right)^k + \dots + k\frac{\partial^2 R}{\partial n^2}\,\frac{\partial^{k-1} f}{\partial n_0^{k-1}}\,\frac{\partial f}{\partial n_0}\right]_{n_0=0} \tag{18}$$

($k = 2,\ \dots,\ N$, the functions E_k are defined by equations (14)).

Since $n \equiv 0$ is a solution of equation (7), we have $f(s;\ 0,\ 0) \equiv 0$,

$$\left[\frac{\partial^k R(s,\ f(s;\ 0,\ n_0))}{\partial n^k}\right]_{n_0=0} = \left[\frac{\partial^k R(s,\ n)}{\partial n^k}\right]_{n=0} \qquad (k = 1,\ 2,\ \dots,\ N). \tag{19}$$

Therefore,

$$H_k(s) = R_k(s)\left[\frac{\partial f}{\partial n_0}\right]^k_{n_0=0} + \dots \tag{20}$$

($k = 2,\ 3,\ \dots,\ N$; the missing terms in (20) correspond to a polynomial in the functions $R_2(s),\ R_3(s),\ \dots,\ R_{k-1}(s)$ and the functions $u_1(s),\ u_2(s),\ \dots,\ u_{k-1}(s)$).

* The expressions for $E_3,\ E_4,\ \dots,\ E_N$ are not given in explicit form; see §24.2, footnote to Lemma 4.

Let $n = 0$ in (14). We will integrate them successively using initial conditions (15) and the relations

$$f(s;\ 0,\ 0) \equiv 0 \quad \text{and} \quad R(s,\ 0) \equiv 0$$

together with (16)–(20). The first of these equations has been considered in Chapter V (§13.3). Its solution is

$$\left[\frac{\partial f(s;\ 0,\ n_0)}{\partial n_0}\right]_{n_0=0} = u_1(s) = e^{\int_0^s R_1(s)ds} \tag{21}$$

(see §13.3, (25)).

Multiplying the k-th equation in (14) by $\frac{1}{k!}$ and integrating, we find

$$\frac{1}{k!}\left[\frac{\partial^k f(s;\ 0,\ n_0)}{\partial n_0^k}\right]_{n_0=0} = u_k(s) = e^{\int_0^s R_1(s)ds}\int_0^s H_k(s)\, e^{-\int_0^s R_1(s)ds} ds. \tag{22}$$

From (11), (21), and (22) we obtain the following expressions for the derivatives of the succession function at the point $n_0 = 0$:

$$f'(0) = e^{\int_0^\tau R_1(s)ds}, \tag{23}$$

$$d^{(k)}(0) = f^{(k)}(0) = k!\, e^{\int_0^\tau R_1(s)ds}\int_0^\tau H_k(s)\, e^{-\int_0^s R_1(s)ds} ds. \tag{24}$$

Let us now express the functions $R_k(s)$ and $H_k(s)$ in terms of the right-hand sides $P(x, y)$ and $Q(x, y)$ of system (A). The expression

$$R_1(s) = \left[\frac{\partial R(s,\ n)}{\partial n}\right]_{n=0}$$

was computed in Chapter V. Indeed, differentiating the function $R(s, n)$ defined by (7) with respect to n,

$$R(s,\ n) = \frac{Q(\overline{\varphi},\ \overline{\psi})\overline{\varphi}'_s - P(\overline{\varphi},\ \overline{\psi})\cdot\overline{\psi}'_s}{P(\overline{\varphi},\ \overline{\psi})\overline{\psi}'_n - Q(\overline{\varphi},\ \overline{\psi})\overline{\varphi}'_n},$$

we obtained after some manipulations

$$R_1(s) = \left[\frac{\partial R(s,\ n)}{\partial n}\right]_{n=0} = P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s)) - \\ - \frac{d}{ds}[\ln(\varphi'(s)^2 + \psi'(s)^2)] \tag{25}$$

(see §13.3, (28)).

Let us now compute the derivative $\frac{\partial^k R(s,\ n)}{\partial n^k}$, where $2 \leqslant k \leqslant N$. We will write out in explicit form only those terms which contain k-th order derivatives of P and Q. Differentiating (7) k times with respect to n and

using the symbolic power notation,* we obtain, as is readily verified,

$$\frac{\partial^k R(s,n)}{\partial n^k} = \frac{\left[\left(\bar{\varphi}'_n \frac{\partial}{\partial x} + \bar{\psi}'_n \frac{\partial}{\partial y}\right)^k Q(\bar{\varphi},\bar{\psi})\right]\bar{\varphi}'_s - \left[\left(\bar{\varphi}'_n \frac{\partial}{\partial x} + \bar{\psi}'_n \frac{\partial}{\partial y}\right)^k P(\bar{\varphi},\bar{\psi})\right]\bar{\psi}'_s}{P(\bar{\varphi},\bar{\psi})\bar{\psi}'_n - Q(\bar{\varphi},\bar{\psi})\bar{\varphi}'_n} -$$

$$- \frac{Q(\bar{\varphi},\bar{\psi})\bar{\varphi}'_s - P(\bar{\varphi},\bar{\psi})\bar{\psi}'_s}{[P(\bar{\varphi},\bar{\psi})\bar{\psi}'_n - Q(\bar{\varphi},\bar{\psi})\bar{\varphi}'_n]^2}\left\{\left[\left(\bar{\varphi}'_n \frac{\partial}{\partial x} + \bar{\psi}'_n \frac{\partial}{\partial y}\right)^k P(\bar{\varphi},\bar{\psi})\right]\bar{\psi}'_n - \right.$$

$$\left. - \left[\left(\bar{\varphi}'_n \frac{\partial}{\partial x} + \bar{\psi}'_n \frac{\partial}{\partial y}\right)^k Q\right]\bar{\varphi}'_n + \right\} \ldots, \tag{26}$$

where the triple dots correspond to terms which contain no k-th order derivatives of P and Q (these terms contain the functions P and Q and their derivatives to order $\leqslant k-1$). Let $n = 0$ in (26). By (3),

$$\bar{\varphi}(s,0) = \varphi(s), \qquad \bar{\psi}(s,0) = \psi(s),$$
$$\bar{\varphi}'_s(s,0) = \varphi'(s), \qquad \bar{\psi}'_s(s,0) = \psi'(s),$$
$$\bar{\varphi}'_s(s,n) = \psi'(s), \qquad \bar{\psi}'_n(s,n) = -\varphi'(s).$$

Moreover, since φ and ψ are solutions of system (A), we have

$$\varphi'(s) = P(\varphi(s),\ \psi(s)),\ \psi'(s) = Q(\varphi(s),\ \psi(s)).$$

Therefore, for $n = 0$, the numerator of the fraction before the braces in (26) vanishes, and we find

$$\left[\frac{\partial^k R(s,n)}{\partial n^k}\right]_{n=0} = W_k(s) + \ldots, \tag{27}$$

where

$$W_k(s) = \frac{\left[\left(\psi'(s)\frac{\partial}{\partial x} - \varphi'(s)\frac{\partial}{\partial y}\right)^k Q(\varphi(s),\ \psi(s))\right]\varphi'(s)}{-\varphi'(s)^2 - \psi'(s)^2} -$$

$$- \frac{\left[\left(\psi'(s)\frac{\partial}{\partial x} - \varphi'(s)\frac{\partial}{\partial y}\right)^k P(\varphi(s)\psi(s))\right]\psi'(s)}{-\varphi'(s)^2 - \psi'(s)^2}, \tag{28}$$

and the triple dots represent terms containing the functions P and Q and their derivatives to order $k-1$.

By (16)–(22) and (25)–(27),

$$H_k(s) = \frac{1}{k!} W_k(s)[u_1(s)]^k + \Phi_k(s), \tag{29}$$

where $\Phi_k(s)$ is expressed — with the aid of algebraic operations and integration — in terms of the functions P and Q and their derivatives to order $(k-1)$ and in terms of the functions $\varphi(s)$, $\psi(s)$, $\varphi'(s)$, $\psi'(s)$.

* If $f(x,y)$ is a function of two variables, the symbolic binomial power $\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)^k$ is used as an abbreviated notation for the operator defined by the equality

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)^k f = u^k \frac{\partial^k f}{\partial x^k} + ku^{k-1}v\frac{\partial^k f}{\partial x^{k-1}\partial y} +$$
$$+ \frac{k(k-1)}{2!}u^{k-2}v^2\frac{\partial^k f}{\partial x^{k-2}\partial y^2} + \ldots + v^k\frac{\partial^k f}{\partial y^k}.$$

We can now compute the derivatives of the succession function $f(n_0)$ at the point $n_0 = 0$. From (23) and (25) it follows (seeing that φ and ψ, and therefore their derivatives, are periodic functions) that

$$f'(0) = e^{\int_0^\tau [P'_x(\varphi(s),\,\psi(s)) + Q'_y(\varphi(s),\,\psi(s))]\,ds} \tag{30}$$

(This expression was originally derived in Chapter V, §13.3, (30).)

From (21), (23), (24), and (29) we have

$$d^{(k)}(0) = f^{(k)}(0) =$$

$$= f'(0) \int_0^\tau W_k(s)\, e^{(k-1)\int_0^s R_1(s)\,ds}\, ds + k!\, f'(0) \int_0^\tau \Phi_k(s)\, e^{-\int_0^s R_1(s)}\, ds, \tag{31}$$

where $k = 2, 3, \ldots, N$.

This expression is analogous to (46), §24.3 and it plays an important role in the proof of the fundamental theorem of this section. , and $W_k(s)$ are expressed by (30), (25), and (28), respectively. We should again emphasize that the second term in the right-hand side of (31) does not contain derivatives of P and Q of higher than $(k-1)$-th order, and the values of these derivatives are taken at points of the curve $x = \varphi(x)$, $y = \psi(s)$.

2. Multiplicity of a limit cycle

In Chapter V we defined the multiplicity of a limit cycle for the case of analytical systems (see §12.3). We will now define this concept for systems of class N. Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system of class $N \geqslant 1$, L_0 a closed path of this system, $x = \varphi(t)$, $y = \psi(t)$ the motion corresponding to this path, $\tau > 0$ the period of the functions φ and ψ.

We recall that the path L_0 is a limit cycle if it is an isolated closed path. A limit cycle L_0 is said to be multiple (see §13.3, Definition 18) if

$$J = \int_0^\tau [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\, ds = 0, \tag{32}$$

otherwise the limit cycle is simple.

As we have repeatedly mentioned, if (A) is a dynamic system of class N, the function $f(n_0)$ and hence the function $d(n_0)$, for sufficiently small n_0, have derivatives to order N inclusive; in particular, the numbers $d'(0), d''(0), \ldots$ $\ldots, d^{(N)}(0)$ exist. We have seen in §12.4 that if at least one of these numbers does not vanish, the closed path L_0 is isolated, i.e., it is a limit cycle (stable, unstable, or semistable).

Definition 28. A closed path L_0 of a dynamic system (A) *of class N is said to be a limit cycle of multiplicity r (or an r-tuple limit cycle) if*

$$d'(0) = d''(0) = \ldots = d^{(r-1)}(0) = 0, \quad d^{(r)}(0) \neq 0 \tag{33}$$

(r is a natural number, $r \leqslant N$).

This concept of multiplicity clearly coincides for analytical systems with the definition introduced in §12.3. As in §12.3, it involves a certain arc without contact l, and we have to prove that the definition is independent of the particular choice of this arc.* However, we will not give this proof here, since it will emerge as a direct consequence of the fundamental theorem of this section.

It follows from Definition 18 (§13.3) and Definition 28 that any simple limit cycle is a limit cycle of multiplicity 1 and vice versa. With regard to multiple cycles, the situation is as follows: if (A) is an analytical system and L_0 is a multiple limit cycle of (A), it has a definite multiplicity r, where r may be any natural number greater than 1.** If (A) is a system of class $N \geq 2$ and L_0 is a multiple limit cycle of (A), it generally has a definite multiplicity r, $2 \leqslant r \leqslant N$. In some cases, however, it may turn out that for the cycle L_0

$$d'(0) = d''(0) = \ldots = d^{(N-1)}(0) = d^{(N)}(0) = 0. \tag{34}$$

If (A) is not a system of class $N+1$, Definition 28 becomes meaningless and the multiple limit cycle L_0 has no definite multiplicity.

If a closed path L_0 is a limit cycle of multiplicity r, i.e., if relations (33) are satisfied, Maclaurin's formula gives

$$d(n) = \frac{d^{(r)}(\theta n)}{r!} n^r.$$

Hence, using the results of §12.2, we conclude that a limit cycle of even multiplicity is semistable, and a limit cycle of odd multiplicity is stable or unstable. For odd r, if $d^{(r)}(0) < 0$, the cycle is stable, and if $d^{(r)}(0) > 0$, the cycle is unstable.

Let us consider a special modified system, for which a lemma analogous to Lemma 7 of §24.3 will be proved.

Let (A) be a dynamic system of class N, L_0 a closed path of (A), $x = \varphi(t)$, $y = \psi(t)$ the motion corresponding to this path (φ and ψ are periodic functions of period $\tau > 0$). By Lemma 1, §15.1 and Remark 1 to that lemma, there exists a function $F(x, y)$ of class $N+1$ in region $\bar{G}$ where system (A) is defined, such that for all s, $-\infty < s < +\infty$, we have

$$F(\varphi(s), \psi(s)) \equiv 0 \tag{35}$$

and

$$[F'_x(\varphi(s), \psi(s))]^2 + [F'_y(\varphi(s), \psi(s))]^2 \neq 0. \tag{36}$$

From (35) and (36) it follows that for all s we also have

$$F'_x(\varphi(s), \psi(s))\psi'(s) - F'_y(\varphi(s), \psi(s))\varphi'(s) \neq 0. \tag{37}$$

Indeed, suppose that condition (37) does not hold true. Then, for some $s = s_0$,

$$F'_x(\varphi(s_0), \psi(s_0))\psi'(s_0) - F'_y(\varphi(s_0), \psi(s_0))\varphi'(s_0) = 0.$$

* See footnote on p. 108.

** If (A) is an analytical system, L_0 is a closed path of (A), and all the derivatives vanish, $d^{(k)}(0) = 0$ ($k = 1, 2, \ldots$), we have $d(n_0) \equiv 0$ and all the paths close to L_0 are closed paths, i.e., L_0 is no longer a limit cycle. See §12.3.

On the other hand, differentiating identity (35) and inserting $s = s_0$, we get

$$F'_x(\varphi(s_0), \psi(s_0))\,\varphi'(s_0) + F'_y(\varphi(s_0), \psi(s_0))\,\psi'(s_0) = 0$$

Consider the last two relations as linear homogeneous equations for $F'_x(\varphi(s_0), \psi(s_0))$ and $F'_y(\varphi(s_0), \psi(s_0))$. It follows from (36) that these equations have a nonzero solution. But then the determinant of the system vanishes, $[\varphi'(s_0)]^2 + [\psi'(s_0)]^2 = 0$, i.e., $\varphi'(s_0) = P(\varphi(s_0), \psi(s_0)) = 0$ and $\psi'(s_0) = Q(\varphi(s_0), \psi(s_0)) = 0$. This is impossible, since the point $(\varphi(s_0)\,\psi(s_0))$ lies on a closed path L_0 and is therefore not an equilibrium state. Relation (37) is thus proved.

Together with system (A), consider a modified system of particular form

$$\frac{dx}{dt} = \tilde{P}(x, y) = P(x, y) + \lambda F^m F'_x, \quad \frac{dy}{dt} = \tilde{Q}(x, y) = Q(x, y) + \lambda F^m F'_y, \qquad (\tilde{\mathrm{A}}_\lambda)$$

where λ is a parameter, and m is a natural number. Clearly, $(\tilde{\mathrm{A}}_\lambda)$ is also a system of class N, and if λ is sufficiently small, $(\tilde{\mathrm{A}}_\lambda)$ is as close as desired to system (A).

It follows from (35) that a closed path L_0 of system (A) is also a path of system $(\tilde{\mathrm{A}}_\lambda)$. Indeed, since L_0 is a path of (A), we have

$$\varphi'(t) = P(\varphi(t), \psi(t)), \quad \psi'(t) = Q(\varphi(t), \psi(t)).$$

From these equalities and from (35) it follows that

$$\varphi'(t) = \tilde{P}(\varphi(t), \psi(t)), \quad \psi'(t) = \tilde{Q}(\varphi(t), \psi(t)),$$

which implies that L_0 is a path of system $(\tilde{\mathrm{A}}_\lambda)$.

Let

$$\tilde{f} = \tilde{f}(n_0, \lambda) \qquad (38)$$

be the succession function constructed for system $(\tilde{\mathrm{A}}_\lambda)$ on the same arc without contact l and for the same choice of the parameter as the succession function f_0 for system (A) (for sufficiently small λ, the function $\tilde{f}$ is defined; see §4, Lemmas 1, 2, and 11).

Clearly,

$$\tilde{f}(n_0, 0) \equiv f(n_0). \qquad (39)$$

Moreover, since L_0 is also a path of system $(\tilde{\mathrm{A}}_\lambda)$, we have

$$\tilde{f}(0) = \tilde{f}(0, \lambda) = 0. \qquad (40)$$

Together with the function $\tilde{f}(n_0)$, we also introduce $\tilde{d}(n_0) = \tilde{d}(n_0, \lambda)$, analogous to the function $d(n_0)$ of (A),

$$\tilde{d}(n_0) = \tilde{d}(n_0, \lambda) = \tilde{f}(n_0, \lambda) - n_0 = \tilde{f}(n_0) - n_0. \qquad (41)$$

In topics related to the creation of limit cycles from a closed path L_0, the system $(\tilde{\mathrm{A}}_\lambda)$ plays precisely the same role as system $(\tilde{\mathrm{A}}_s)$ (§24.3) in connection with the creation of limit cycles from a focus.

Lemma 1. Let for a closed path L_0 of a dynamic system (A) *of class* $N > 1$

$$d'(0) = d''(0) = \ldots = d^{(r)}(0) = 0, \qquad (42)$$

where $1 < r \leqslant N$, *and let* m *be an integer,*

$$1 < m \leqslant r, \tag{43}$$

and λ *a nonzero real number. Then the closed path* L_0 *is a multiple limit cycle of multiplicity* m *for the system* $(\widetilde{A}_\lambda)$, *i.e.,*

$$\widetilde{d}'(0) = \widetilde{d}''(0) = \ldots = \widetilde{d}^{(m-1)}(0) = 0, \tag{44}$$

and

$$\widetilde{d}^{(m)}(0) \neq 0. \tag{45}$$

Also

$$\widetilde{d}^{(m)}(0) = \lambda \cdot m!\,[\varphi'(0)^2 + \psi'(0)^2]^{m-1} \times \int_0^\tau \frac{[F'_x(\varphi(s), \psi(s))\,\psi'(s) - F'_y(\varphi(s), \psi(s))\,\varphi'(s)]^{m+1}}{[\varphi'(s)^2 + \psi'(s)^2]^m} \times e^{(m-1)\int_0^s [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\,ds}\,ds. \tag{46}$$

Proof. Since L_0 is a path of system $(\widetilde{A}_\lambda)$, the derivatives $\widetilde{d}^{(i)}(0)$, $i = 1, 2, \ldots, m$, may be found using (30) and (31) (for the derivatives $d^{(i)}(0)$). In these equations, $P(x, y)$ and $Q(x, y)$ and their derivatives should be replaced with $\widetilde{P}(x, y)$ and $\widetilde{Q}(x, y)$ and their derivatives (the functions φ, ψ, φ', ψ' clearly remain as before).

We recall that in computations using (30) and (31), the values of the functions $\widetilde{P}$ and $\widetilde{Q}$ and their derivatives should be taken for $x = \varphi(s)$, $y = \psi(s)$, $0 \leqslant s \leqslant \tau$.

From the relations

$$\begin{gathered} \widetilde{P}(x, y) = P(x, y) + \lambda F^m F'_x, \\ \widetilde{Q}(x, y) = Q(x, y) + \lambda F^m F'_y, \\ F(\varphi(s), \psi(s)) \equiv 0 \text{ and } m > 1, \end{gathered} \tag{47}$$

it follows, as will be seen from elementary calculations, that at the points $(\varphi(s), \psi(s))$ the values of the functions $\widetilde{P}$ and $\widetilde{Q}$ and their partial derivatives to order $m-1$ inclusive are respectively equal to the values of the functions P and Q and their derivatives to order $m-1$, i.e.,

$$\left.\begin{gathered} \widetilde{P}(\varphi(s), \psi(s)) = P(\varphi(s), \psi(s)), \qquad \widetilde{Q}(\varphi(s), \psi(s)) = Q(\varphi(s), \psi(s)), \\ \left[\frac{\partial^l \widetilde{P}(x, y)}{\partial x^i\, \partial y^{l-i}}\right]_{\substack{x=\varphi(s)\\ y=\psi(s)}} = \left[\frac{\partial^l P(x, y)}{\partial x^i\, \partial y^{l-i}}\right]_{\substack{x=\varphi(s)\\ y=\psi(s)}}, \\ \left[\frac{\partial^l \widetilde{Q}(x, y)}{\partial x^i\, \partial y^{l-i}}\right]_{\substack{x=\varphi(s)\\ y=\psi(s)}} = \left[\frac{\partial^l Q(x, y)}{\partial x^i\, \partial y^{l-i}}\right]_{\substack{x=\varphi(s)\\ y=\psi(s)}} \\ (l = 1, 2, \ldots, m-1;\ 0 \leqslant i \leqslant l). \end{gathered}\right\} \tag{48}$$

Let us now compute the m-th order partial derivatives of the function $\widetilde{P}$ and $\widetilde{Q}$ at the points $(\varphi(s), \psi(s))$. In virtue of the relation $F(\varphi(s), \psi(s)) \equiv 0$, it

suffices to consider only those terms which result from the differentiation of the functions P, Q, and F^m (all the other terms produced by differentiation contain the function $F(x, y)$ as a multiplicative factor and therefore vanish for $x = \varphi(s)$, $y = \psi(s)$). Simple manipulations thus lead to the relations

$$\left.\begin{aligned} \left[\frac{\partial^m \widetilde{P}(x, y)}{\partial x^i \partial y^{m-i}}\right]_{\substack{x=\varphi(s)\\ y=\psi(s)}} &= \left[\frac{\partial^m P(x, y)}{\partial x^i \partial y^{m-i}}\right]_{\substack{x=\varphi(s)\\ y=\psi(s)}} + \\ &\quad + \lambda\cdot m!\,[(F'_x(x, y))^{i+1}(F'_y(x, y))^{m-i}]_{\substack{x=\varphi(s),\\ y=\psi(s)}} \\ \left[\frac{\partial^m \widetilde{Q}(x, y)}{\partial x^i \partial y^{m-i}}\right]_{\substack{x=\varphi(s)\\ y=\psi(s)}} &= \left[\frac{\partial^m Q(x, y)}{\partial x^i \partial y^{m-i}}\right]_{\substack{x=\varphi(s)\\ y=\psi(s)}} + \lambda\cdot m!\,[(F'_x)^i (F'_y)^{m-i+1}]_{\substack{x=\varphi(s)\\ y=\psi(s)}} \\ &(i = 0, 1, 2, \ldots, m). \end{aligned}\right\} \quad (49)$$

Let $\widetilde{R}_i(s)$, $\widetilde{H}_i(s)$, etc., be the functions of $(\widetilde{A}_\lambda)$ which are analogous to the functions $R_i(s)$, $H_i(s)$, etc., of (A). Relation (25), the analogous relation for $\widetilde{R}_1(s)$, and equations (48) lead to the equality

$$\widetilde{R}_1(s) = R_1(s). \quad (50)$$

Thus, using (23), we see that

$$\widetilde{f}'(0) = f'(0), \quad (51)$$

and therefore

$$\widetilde{d}'(0) = d'(0). \quad (52)$$

Furthermore, using (21), (28), (29), (31), the analogous relations for $(\widetilde{A}_\lambda)$, and also equations (48), (50), and (51), we conclude that

$$\widetilde{W}_k(s) = W_k(s) \quad (53)$$

for $k \leqslant m-1$;

$$\widetilde{\Phi}_k(s) = \Phi_k(s) \quad (54)$$

for $k \leqslant m$;

$$\widetilde{H}_k(s) = H_k(s) \quad (55)$$

for $k \leqslant m-1$, and finally,

$$\widetilde{d}^{(k)}(0) = d^{(k)}(0) \quad (56)$$

for $k = 2, 3, \ldots, m-1$.

From (42), (43), (52), and (56) it follows that

$$\widetilde{d}'(0) = \widetilde{d}''(0) = \ldots = \widetilde{d}^{(m-1)}(0) = 0.$$

We have thus proved relation (44).

Let us now compute $\widetilde{d}^{(m)}(0)$. To this end, we will use equation (31), replacing f, W, R, etc., with $\widetilde{f}, \widetilde{W}, \widetilde{R}$, etc. By (50), (51), and (54),

$$\widetilde{d}^{(m)}(0) = f'(0)\int_0^\tau \widetilde{W}_m(s)\, e^{(k-1)\int_0^s R_1(s)\,ds} ds + m!f'(0)\int_0^\tau \Phi_k(s)\, e^{-\int_0^s R_1(s)\,ds} ds. \quad (57)$$

The expression for $\widetilde{W}_m(s)$ can be found using (28). Replacing P and Q with $\widetilde{P}$ and $\widetilde{Q}$ and using (49), we obtain

$$\widetilde{W}_m(s) = W_m(s) + \frac{\lambda\cdot m!\,[F'_x(\varphi(s), \psi(s))\psi'(s) - F'_y(\varphi(s), \psi(s))\varphi'(s)]^{m+1}}{\varphi'(s)^2 + \psi'(s)^2}. \quad (58)$$

From (31) and (58) we obtain

$$\tilde{d}^{(m)}(0) = d^{(m)}(0) + f'(0) \int_0^{\tau} \frac{\lambda \cdot m!\,[F'_x\psi' - F'_y\varphi']^{m-1}}{\varphi'(s)^2 + \psi'(s)^2}\, e^{(m-1)\int_0^s R_1(s)\,ds}\, ds.$$

By the conditions of the lemma, $d^{(m)}(0) = 0$. Furthermore, $d'(0) = f'(0) - 1 = 0$, i.e., $f'(0) = 1$. Inserting these numerical values and expression (25) in the last relation, we finally obtain

$$\tilde{d}^{(m)}(0) = \lambda \cdot m!\,[\varphi'(0)^2 + \psi'(0)^2]^{m-1} \times$$
$$\times \int_0^{\tau} \frac{[F'_x(\varphi(s), \psi(s))\,\psi'(s) - F'_y(\varphi(s), \psi(s))\,\varphi'(s)]^{m+1}}{[\varphi'(s)^2 + \psi'(s)^2]^m} \times$$
$$\times e^{(m-1)\int_0^s [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\,ds}\, ds,$$

i.e., equality (46) holds true.

Inequality (45) follows directly from (46) and inequality (37). This completes the proof of the lemma.

Remark. We have assumed in the lemma that $r \geqslant 2$ (and thus $N \geqslant 2$) and, imposing conditions (42), considered the modified system $(\tilde{A}_\lambda)$ with $m \geqslant 2$. Under these conditions, the closed path L_0 of system $(\tilde{A}_\lambda)$ is a multiple limit cycle of multiplicity m. The case $m = 1$, i.e., when the modified system $(\tilde{A}_\lambda)$ has the form

$$\frac{dx}{dt} = P + \lambda F F'_x, \qquad \frac{dy}{dt} = Q + \lambda F F'_y, \qquad (\tilde{A}_{\lambda_1})$$

has been considered in connection with the proof of Theorem 19 (§15.2). We have seen there that if (A) is a dynamic system of class $N \geqslant 1$, L_0 a closed path of (A) for which $d'(0) = 0$ ($d''(0)$ need not be equal to zero, or even need not exist altogether), and $\lambda \neq 0$, L_0 is a simple limit cycle of $(\tilde{A}_{\lambda_1})$ and the corresponding derivative is expressed in the form

$$\tilde{d}'_{\lambda_1}(0) = e^{\lambda \int_0^{\tau} [(F'_x(\varphi(s), \psi(s)))^2 + (F'_y(\varphi(s), \psi(s)))^2]\,ds} - 1. \qquad (59)$$

§27. CREATION OF LIMIT CYCLES FROM A MULTIPLE LIMIT CYCLE

1. The fundamental theorem

In Chapter V we considered a multiple limit cycle and showed that it may "create" closed paths (§15.2, Theorem 19). In this section we will elucidate the number of paths that may be "created" in the neighborhood of a multiple limit cycle on passing to sufficiently close systems. We will in fact prove the following theorem, analogous to Theorem 40 of §25.1.

Theorem 42 (theorem of the creation of limit cycles from a multiple limit cycle). If (A) *is a dynamic system of class* $N > 1$ *or an analytical system, and* L_0 *is a multiple limit cycle of multiplicity* k $(2 \leqslant k \leqslant N)$, *then*

1) there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any system $(\tilde{A})$ *δ_0-close to rank k to* (A) *has at most k closed paths in $U_{\varepsilon_0}(L_0)$;*

2) for any positive $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$, there exists a system $(\tilde{A})$ *of class N (of analytical class, respectively) which is δ-close to rank k to* (A) *and has k closed paths in $U_{\varepsilon_0}(L_0)$.**

Proof. 1) Let us prove the first proposition of the theorem. As in the previous section, consider an arc without contact l, which is a normal to the path L_0 (see §26.1), and the succession function $f(n_0)$ of system (A) on the arc l, together with the function $d(n_0) = f(n_0) - n_0$. Let these functions be defined for all n_0, $|n_0| \leqslant n^*$, where n^* is some positive number. As we have noted in §26, $d(n_0)$ is a function of class N.

Since L_0 is a limit cycle of multiplicity k of system (A), we see that

$$d'(0) = d''(0) = \ldots = d^{(k-1)}(0) = 0, \quad d^{(k)}(0) \neq 0, \tag{1}$$

i.e., the number 0 is a root of multiplicity k of the function $d(n_0)$. Therefore (see Chapter I, §1.3) there exist positive numbers $\eta \leqslant n^*$ and σ such that any function $\tilde{d}(n_0)$ defined for all n_0, $|n_0| \leqslant n^*$, and σ-close to $d(n_0)$ to rank k may have at most k roots on the segment $[-\eta, \eta]$.

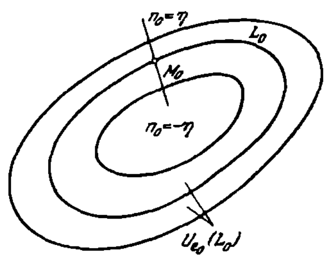

FIGURE 118

A sufficiently small positive number is taken for ε_0, so that all the points of the normal l lying in $U_{\varepsilon_0}(L_0)$ correspond to the values of the parameter n_0 less than η in magnitude (Figure 118). δ_0 is also taken so small that the following condition is satisfied: if system $(\tilde{A})$ is δ_0-close to rank k to (A), the succession function $\tilde{f}(n_0)$, and hence the function $\tilde{d}(n_0)$, are defined for $(\tilde{A})$ on the arc l for all n_0, $|n_0| \leqslant n^*$, and for $|n_0| \leqslant n^*$ the function $\tilde{d}(n_0)$ is σ-close to $d(n_0)$ to rank k. This δ_0 exists by Theorem 3, Appendix, 1. The numbers ε_0 and δ_0 chosen in this way evidently satisfy the first proposition of the theorem. The first proposition is thus proved.

2) The second proposition will first be proved for a dynamic system (A) of class N. Consider a modified system of a particular form

$$\begin{aligned} \frac{dx}{dt} &= \tilde{P}(x, y, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) = P(x, y) + \lambda_1 F F'_x + \lambda_2 F^2 F'_x + \ldots + \lambda_{k-1} F^{k-1} F'_x, \\ \frac{dy}{dt} &= \tilde{Q}(x, y, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) = Q(x, y) + \lambda_1 F F'_y + \lambda_2 F^2 F'_y + \ldots + \lambda_{k-1} F^{k-1} F'_y, \end{aligned} \tag{$\tilde{A}$}$$

* The closed paths of Theorem 42 are clearly isolated, i.e., they are limit cycles.

where λ_i are parameters, and $F(x, y)$ is a function of class $N+1$ defined in $\tilde{G}$ and satisfying conditions (35) and (36) of the previous section (§26.2). By (35), the function $F(x, y)$ vanishes at the points of the limit cycle L_0 of system (A) and therefore L_0 is also a path of system $(\tilde{\text{A}})$. Evidently, $(\tilde{\text{A}})$ is also a system of class N.

For sufficiently small λ_i, $(\tilde{\text{A}})$ is arbitrarily close to (A) to any (possible) rank, and by §4, Lemmas 1, 2, and 11, the function

$$\tilde{d}(n_0, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}),$$

analogous to the function $d(n_0)$ for (A), is defined for $(\tilde{\text{A}})$ on the arc without contact l for all n_0, $|n_0| \leqslant n^*$.* Clearly, $\tilde{d}$ is a continuous function of n_0 and of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$. Since (A) is obtained from $(\tilde{\text{A}})$ for $\lambda_1 = \lambda_2 = \ldots = \lambda_{k-1} = 0$, we see that

$$\tilde{d}(n_0, 0, 0, \ldots, 0) = d(n_0). \tag{2}$$

For any $\varepsilon > 0$ and $\delta > 0$, there exist $\lambda^* > 0$ and $\hat{n} \leqslant n^*$ such that if

$$|\lambda_i| < \lambda^*, \quad i = 1, 2, \ldots, k-1, \tag{3}$$

then

(a) system $(\tilde{\text{A}})$ is δ-close to rank k to system (A);

(b) the function $\tilde{d}(n_0, \lambda_1, \lambda_2, \ldots, \lambda_{k-1})$ is defined for all n_0, $|n_0| \leqslant n^*$, and any root of this function satisfying the inequality $|n_0| < \hat{n}$ corresponds to a closed path of $(\tilde{\text{A}})$ completely contained in $U_\varepsilon(L_0)$.

Let ε and δ be fixed positive numbers, $\varepsilon < \varepsilon_0$, $\delta < \delta_0$. Let further, λ^* and $\hat{n}$ be the numbers corresponding to ε and δ (i.e., such that if (3) is satisfied, conditions (a) and (b) hold true). We will show that by an appropriate choice of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$, system $(\tilde{\text{A}})$ can be made δ-close to rank k to (A) and will have k closed paths in $U_\varepsilon(L_0)$.

We will assume that the numbers λ_i henceforth satisfy condition (3).

From Maclaurin's formula we conclude that, in virtue of relations (1), for all sufficiently small n_0,

$$d(n_0) = \frac{d^{(k)}(0)}{k!} n_0^k + h(n_0) n_0^k, \tag{4}$$

where $h(n_0)$ is a continuous function and $h(0) = 0$ (see proof of Theorem 5, §1.3).

By assumption, $d^{(k)}(0) \neq 0$. To fix ideas, let $d^{(k)}(0) > 0$. Then for all sufficiently small positive n_0, $d(n_0) > 0$. We choose one of these numbers, smaller than $\hat{n}$, and denote it by n_1. Thus,

$$0 < n_1 < \hat{n}, \quad d(n_1) = \tilde{d}(n_1, 0, \ldots, 0) > 0. \tag{5}$$

Now suppose that

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{k-2} = 0, \quad \lambda_{k-1} \neq 0,$$

* We recall that $n^* > 0$ is a number such that for $|n_0| \leqslant n^*$ the function $d(n_0)$ is defined.

and consider the modified system corresponding to these values of the parameters,

$$\begin{aligned}\frac{dx}{dt} &= \widetilde{P}(x, y, 0, 0, \ldots, 0, \lambda_{k-1}) = P(x, y) + \lambda_{k-1}F^{k-1}F'_x,\\ \frac{dy}{dt} &= \widetilde{Q}(x, y, 0, 0, \ldots, 0, \lambda_{k-1}) = Q(x, y) + \lambda_{k-1}F^{k-1}F'_y,\end{aligned} \qquad (\widetilde{A}_1)$$

and the corresponding function $\widetilde{d}_1(n_0) = \widetilde{d}(n_0, 0, 0, \ldots, 0, \lambda_{k-1})$. By (1) and Lemma 1 of the previous section,

$$\widetilde{d}_1'(0) = \widetilde{d}_1''(0) = \ldots = \widetilde{d}_1^{(k-2)}(0) = 0, \qquad (6)$$

and $\widetilde{d}_1^{(k-1)}(0)$ is expressed by equality (46) of the previous section, which gives

$$\widetilde{d}_1^{(k-1)}(0) = C_1\lambda_{k-1},$$

where C_1 is a nonzero constant (C_1 is the factor before λ in (46) for $m = k-1$. Its explicit expression will not be needed here, however).

As before, Maclaurin's formula and relations (6) give for all sufficiently small n_0

$$\widetilde{d}_1(n_0) = \frac{C_1\lambda_{k-1}}{(k-1)!}n_0^{k-1} + \widetilde{h}_1(n_0)n_0^{k-1}, \qquad (7)$$

where $\widetilde{h}_1(n_0)$ is a continuous function and $\widetilde{h}_1(0) = 0$.

Let $C_1 > 0$ and choose λ_{k-1} so that

$$|\lambda_{k-1}| < \lambda^*, \quad \lambda_{k-1} < 0, \qquad \widetilde{d}_1(n_1) = \widetilde{d}(n_1, 0, 0, \ldots, 0, \lambda_{k-1}) > 0. \qquad (8)$$

The last of these conditions is satisfied for any sufficiently small λ_{k-1} in virtue of equality (5) and the continuity of the function $\widetilde{d}(n_0, \lambda_1, \lambda_2, \ldots, \lambda_{k-1})$.

The inequalities $C_1 > 0$, $\lambda_{k-1} < 0$ and equality (7) show that for sufficiently small positive n_0, $\widetilde{d}_1(n_0) < 0$. Choose one of these numbers, smaller than n_1, and denote it by n_2. Thus,

$$0 < n_2 < n_1 < \hat{n} \qquad (9)$$

and

$$\widetilde{d}_1(n_1) > 0, \qquad \widetilde{d}_1(n_2) < 0. \qquad (10)$$

Further construction proceeds along the same lines as above (compare with proof of the second proposition of Theorem 40, § 25.1). After the $(k-1)$-th step,* we end up with the system

$$\begin{aligned}\frac{dx}{dt} &= \widetilde{P}(x, y, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) =\\ &= P(x, y) + \lambda_1 FF'_x + \lambda_2 F^2F'_x + \ldots + \lambda_{k-1}F^{k-1}F'_x,\\ \frac{dy}{dt} &= \widetilde{Q}(x, y, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}) =\\ &= Q(x, y) + \lambda_1 FF'_y + \lambda_2 F^2F'_y + \ldots + \lambda_{k-1}F^{k-1}F'_y,\end{aligned} \qquad (\widetilde{A})$$

* In our construction of the systems $(\widetilde{A}_1)$, $(\widetilde{A}_2)$, ..., $(\widetilde{A}_{k-2})$ and the numbers $n_1, n_2, \ldots, n_{k-1}$, we always use Lemma 1 of the previous section. However, in the last, $(k-1)$-th step, the lemma itself is replaced by the remark to Lemma 1 and formula (59) of the last section is used instead of formula (46).

and the numbers $n_1, n_2, \ldots, n_k$ are such that $|\lambda_i| < \lambda^*$,

$$0 < n_k < n_{k-1} < \ldots < n_1 < \hat{n} \tag{11}$$

and

$$\tilde{d}(n_1) > 0,\ \tilde{d}(n_2) < 0,\ \ldots,\ \tilde{d}(n_k) \left\{ \begin{array}{l} > 0 \text{ if } k \text{ is odd,} \\ < 0 \text{ if } k \text{ is even.} \end{array} \right. \tag{12}$$

From inequalities (12) and the continuity of the function $\tilde{d}(n_0)$ it follows that at least one root of the function $\tilde{d}(n_0)$ falls between each pair of numbers n_1 and n_2, n_2 and $n_3, \ldots, n_{k-1}$ and n_k. These roots correspond to closed paths of system $(\tilde{\mathrm{A}})$. By (3) and (11), these closed paths lie in $U_\varepsilon(L_0)$. Moreover, the path L_0 of (A) is itself a path of $(\tilde{\mathrm{A}})$. Thus, at least k paths of $(\tilde{\mathrm{A}})$ exist inside the neighborhood $U_\varepsilon(L_0)$. Since $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$, the first proposition of the theorem indicates that $U_\varepsilon(L_0)$ may contain at most k paths of $(\tilde{\mathrm{A}})$, i.e., it contains precisely k paths. This proves the second proposition of the theorem for systems of class N.

3) Let us now consider the analytical case. Our proof is inapplicable to this case, since, in general, no analytical function $F(x, y)$ exists in the entire $\bar{G}$ which satisfies conditions (35) and (36) of the previous section.* We will therefore proceed along the same lines as in the proof of Theorem 19 (§15.2).

Let (A) be an analytical system, and L_0 a multiple limit cycle of multiplicity k of (A), ε_0 and δ_0 the numbers introduced in the first proposition of the theorem. Take any positive numbers $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. As before, we construct a system $(\tilde{\mathrm{A}})$ of class $N \geqslant k$ and numbers $n_1, n_2, \ldots, n_k$ such that the following conditions are satisfied:

(a) System $(\tilde{\mathrm{A}})$ is $\delta/2$ close to rank k to system (A).

(b) Relations (11) and (12) are satisfied, and $\hat{n}$ is a number with the following properties: any path $\tilde{L}$ of system $(\tilde{\mathrm{A}})$ crossing the arc without contact l at point $\tilde{M}_1$ corresponding to a value n_0, $|n_0| < \hat{n}$, of the parameter crosses the arc l with increasing t at another point $\tilde{M}_2$, so that the arc $\tilde{M}_1\tilde{M}_2$ of the path $\tilde{L}$ is entirely contained in $U_{\varepsilon/2}(L_0)$.

By equation (59) of the preceding section, $\tilde{d}'(0) \neq 0$, i.e., L_0 is a simple, and hence structurally stable, limit cycle of $(\tilde{\mathrm{A}})$.

Let η be a number, $0 < \eta < \frac{\delta}{2}$, and (A^*) an analytical system η-close to rank k to system $(\tilde{\mathrm{A}})$ (e.g., a system whose right-hand sides are polynomials adequately approximating to the right-hand sides of $(\tilde{\mathrm{A}})$). Clearly, if η is sufficiently small, the following conditions are satisfied:

(a*) The function $d^*(n_0)$ corresponding to (A^*) is defined for all n_0, $|n_0| < \hat{n}$, and

$$d^*(n_1) > 0,\ d^*(n_2) < 0,\ \ldots,\ d^*(n_k) \left\{ \begin{array}{l} > 0 \text{ if } k \text{ is odd,} \\ < 0 \text{ if } k \text{ is even.} \end{array} \right. \tag{13}$$

(b*) Any path L^* of system (A^*) crossing the arc without contact l at point M_1^* corresponding to the value n_0, $|n_0| < \hat{n}$, of the parameter crosses

* We have seen in Chapter V (§15.2) that such a function may be constructed in the neighborhood of the path L_0. This is not sufficient, however, since $(\tilde{\mathrm{A}})$ should be defined in the entire $\bar{G}$.

the arc l again with increasing t at point M_2^*, so that the arc $M_1^*M_2^*$ of the path L^* is entirely contained in $U_\varepsilon(L_0)$.*

(c*) There exists a closed path L_0^* of (A*) which lies entirely in $U_\varepsilon(L_0)$ and crosses the arc without contact l at point M_0^* corresponding to the value n_0^* of the parameter, where $n_0^* < n_1$ (condition (c*) is satisfied for sufficiently small η because L_0 is a structurally stable limit cycle of ($\tilde{A}$)).

From (a*), (b*), and (c*) it clearly follows that (A*) is an analytical system, δ-close to rank k to system (A), which has at least k closed paths in $U_\varepsilon(L_0)$, i.e., it satisfies the second proposition of the theorem. This completes the proof of the theorem.

Remark. If (A) is a system of class N, the condition – in the second proposition of Theorem 42 – that system ($\tilde{A}$) is δ-close to rank k to (A) may be replaced by requirement of δ-closeness of rank N. If (A) is an analytical system, we can find an analytical system ($\tilde{A}$) satisfying the second proposition of the theorem and yet close to (A) to rank m, where m is an arbitrary natural number. The validity of this remark follows directly from the proof of Theorem 42.

2. Supplements

On the creation of limit cycles from a focus of finite multiplicity. Theorem 42 and the remark following the theorem enable us to strengthen the second proposition of Theorem 41. Indeed, the following theorem obtains:

Theorem 41'. Let $O(0, 0)$ be a multiple focus of multiplicity k of a dynamic system (A) *of class $N \geqslant 2k+1$ (or analytical), and let ε_0 and δ_0 be sufficiently small positive numbers (introduced in the first proposition of Theorem 40 and the remark to that theorem). If system* (B) *is δ_0-close to rank $2k+1$ to system* (A), *the sum of the multiplicities of the focus and the limit cycles of* (B) *lying in $U_{\varepsilon_0}(O)$ is at most k.*

Proof. Let the sum of the multiplicities of the focus and the limit cycles of (B) lying in $U_{\varepsilon_0}(O)$ be $k^* > k$. Then, using the construction of Lemma 2, §15.2, and Theorems 40 and 42, together with the remark to Theorem 42, we can modify (B) in the neighborhood of each of these limit cycles and the focus, to obtain a system (B*) which is arbitrarily close to rank $2k+1$ to system (B) and has $k^* > k$ closed paths in $U_{\varepsilon_0}(O)$. The existence of this system (B*), however, contradicts the condition that $O(0, 0)$ is a focus of multiplicity k. If only analytical systems are considered, the limit cycles of (B*) lying in $U_{\varepsilon_0}(O)$ should further be made structurally stable (see §15.2, Lemma 2) and (B*) should be approximated with an analytical system. The theorem is thus proved. It is clear that the second proposition of Theorem 41 follows from Theorem 41'.

Bifurcations of a dynamic system in the neighborhood of a limit cycle of finite multiplicity. We will first consider a proposition which is analogous to Theorem 41 and strengthens the second proposition of Theorem 42. This proposition, together with Theorem 42, plays a fundamental role in the entire topic of bifurcations of dynamic systems in the neighborhood of a limit cycle of finite multiplicity.

* For a small η, condition (b*) is satisfied because of condition (b) and Lemma 11, §4.2.

Theorem 43. Let (A) *be a dynamic system of class* $N > 1$ *or an analytical system,* L_0 *a multiple limit cycle of multiplicity* $k\,(2 \leqslant k \leqslant N)$ *of* (A), *and* ε_0 *and* δ_0 *sufficiently small positive numbers (introduced in the first proposition of Theorem 42).* Then*

1) for any ε *and* δ, $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \delta \leqslant \delta_0$, *and for any* s, $1 \leqslant s \leqslant k$, *there exists a system* (B) *of class* N *(or an analytical system) which is* δ*-close to rank* k *to* (A) *and has precisely* s *closed paths in* $U_\varepsilon(L_0)$*;*

2) if (B) *is* δ_0*-close to rank* k *to* (A), *the sum of the multiplicities of all the limit cycles of* (B) *lying in* $U_{\varepsilon_0}(L_0)$ *is at most* k.

The proof of the first proposition is analogous to the proof of the corresponding proposition of Theorem 41. The proof of the second proposition is analogous to the proof of Theorem 41'. The reader will be able to reconstruct the detailed proof without difficulty.

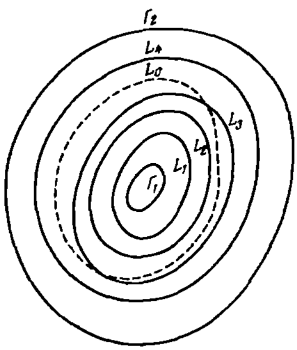

FIGURE 119

The investigation of the bifurcations of a dynamic system in the neighborhood of a limit cycle of finite multiplicity is analogous to the investigation performed at the end of §25.2 for a multiple focus. Theorems 42 and 43 play a leading role in the entire treatment. Let L_0 be a k-tuple limit cycle of system (A) $(k > 2)$, V a sufficiently small neighborhood of this cycle bounded by cycles without contact Γ_1 and Γ_2, δ a sufficiently small positive number. By Theorems 42 and 43, system $(\tilde{\mathrm{A}})$ δ-close to rank k to (A) may have at most k closed paths in V. Moreover, there exist systems $(\tilde{\mathrm{A}})$ which have precisely s closed paths in V, where s is any number, $1 \leqslant s \leqslant k$. These closed paths, naturally, are limit cycles and are arranged "concentrically" (Figure 119). As in the case of a multiple focus (§25.2), the topological structure of $(\tilde{\mathrm{A}})$ in V is entirely determined by the number s of the limit cycles lying in V and their stability characteristics. Let these cycles be $L_1, L_2, \ldots, L_s$, and we assume that L_i lies inside L_{i+1} $(i = 1, 2, \ldots, s-1)$. Suppose that the behavior of the paths of (A) in relation to the cycles without contact Γ_1 and Γ_2 is known.** The paths of any system $(\tilde{\mathrm{A}})$ sufficiently close to (A) behave in relation to Γ_1 and Γ_2 just like the paths of (A). Therefore the topological structure of $(\tilde{\mathrm{A}})$ in V is completely determined if we know:

(a) the number s of limit cycles of $(\tilde{\mathrm{A}})$ in V;

(b) whether each of these cycles is of even or odd multiplicity.

Hence it follows, as for a multiple focus, that in the neighborhood of a limit cycle of finite multiplicity a dynamic system (A) may only have a finite number of different bifurcations. We will not try to describe these bifurcations, since the situation is precisely the same as for a focus (§25.2).

Example 10. Consider the system

$$\frac{dx}{dt} = -y + x\,(x^2 + y^2 - 1)^k, \qquad \frac{dy}{dt} = x + y\,(x^2 + y^2 - 1)^k, \qquad (\mathrm{B}_k)$$

where k is a natural number.

* It is further assumed that all systems δ_0-close to (A) have no equilibrium states in $U_{\varepsilon_0}(L_0)$.

** In other words, it is known whether the paths crossing the cycle Γ_i $(i = 1, 2)$ enter into V or leave V as t increases.

Direct computations show that this system has a single equilibrium state $O(0,0)$, which is an unstable focus for even k and a stable focus for odd k. It is readily verified that the circles

$$x^2+y^2=C^2 \tag{14}$$

for $C\neq 1$ are cycles without contact of system (B_k), and the circle

$$x^2+y^2=1 \tag{15}$$

is a path of the system. Hence it follows that this circle is the only closed path of the system, i.e., it is a limit cycle. We will use the symbol L_0 to designate the circle (15). All the paths of (B_k), except the focus O and the limit cycle L_0, are spirals. Since

$$x\dot{x}+y\dot{y}=\frac{1}{2}\frac{d\rho^2}{dt}=\rho^2(\rho^2-1)^k,$$

the infinity is absolutely stable. These data uniquely determine the topological structure of the dynamic system (B_k). For an even k, the limit cycle L_0 is semistable, and for an odd k it is unstable. The path configuration is shown schematically in Figure 120 (even k) and in Figure 121 (odd k).

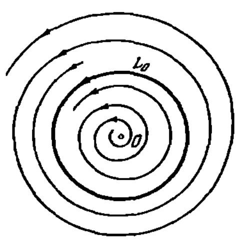

FIGURE 120. For even k, the focus is unstable and the limit cycle is semistable.

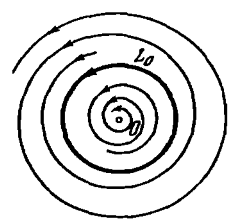

FIGURE 121. For odd k, the focus is stable and the limit cycle is unstable.

We will now show that the path L_0 is a cycle of multiplicity k for (B_k). This follows directly from Lemma 1, §26.2. For the starting system (A) of the lemma we take the system

$$\frac{dx}{dy}=-y, \qquad \frac{dy}{dt}=x \tag{16}$$

and set

$$F(x,y)=x^2+y^2-1.$$

The numbers m and λ are assigned the values k and $1/2$. Then system $(\tilde{A}_\lambda)$ introduced in Lemma 1, §26.2, coincides with (B_k). The paths of system (16) are the equilibrium state $O(0,0)$ (a center) and the concentric circles $x=C\cos t, y=C\sin t$, including the circle L_0. Therefore, for

system (16), the succession function is $f(\rho_0) \equiv \rho_0$, and $d(\rho_0) \equiv 0$. But then conditions (42), §26.2, are satisfied for any r. The following conditions are additionally satisfied on the path L_0:

$$F(x, y) \equiv 0, \qquad (F'_x)^2 + (F'_y)^2 \neq 0$$

(conditions (35) and (36), §26). Thus all the conditions of Lemma 1, §26.2 are satisfied, i.e., circle L_0 is a limit cycle of multiplicity k of (B_k).

The case $k = 2$ was considered in detail in Chapter VIII (§22, Example 8). In that example we established the exact changes in the topological structure of the system as its field rotated and elucidated the fate of the limit cycle L_0.

Chapter XI

CREATION OF LIMIT CYCLES FROM THE LOOP OF A SADDLE-POINT SEPARATRIX

INTRODUCTION

Consider a dynamic system (D) with a simple (structurally stable) saddle point $O(x_0, y_0)$ and a path L_0 which goes to the saddle point O both for $t \to -\infty$ and for $t \to +\infty$. This path is both an α-separatrix and an ω-separatrix of the saddle point O, and we say that it forms a loop. In Chapter IV (Theorem 16, § 11.2) it is shown that a separatrix forming a loop is a structurally unstable path and there exist modified systems arbitrarily close to (D) and such that the separatrix loop disappears on moving to these systems.

In the present chapter we consider the creation of closed paths from a separatrix loop upon moving to close systems.

The first of the two sections, § 28, presents auxiliary background material. The succession function on an arc without contact crossing the separatrix loop is considered in § 28.1, and some properties of this function are established. *All the principal results of this chapter are derived in what follows using this succession function.* The behavior of the saddle point and its separatrices on moving to close systems constitutes the subject of § 28.2.

The principal results of the chapter are contained in § 29.

To fix ideas, suppose that two separatrices of the saddle point O which do not belong to L_0 lie inside the loop formed by the separatrix L_0. We first prove (Theorem 44, § 29.1) that if the parameter

$$\sigma_0(x_0, y_0) = P'_x(x_0, y_0) + Q'_y(x_0, y_0)$$

is positive (negative), the loop L_0 is unstable (stable) from inside, i.e., all the paths passing through points interior to the loop which are sufficiently close to the loop go to this loop for $t \to -\infty$ $(t \to +\infty)$.

We further consider the creation of a closed path from a separatrix loop (Theorems 45 and 46, § 29.2). It is established that if the separatrix loop is stable or unstable (in particular, if $\sigma_0 \neq 0$), there exist modified systems arbitrarily close to the original system such that the loop disappears on passing to any of these close systems, and yet at least one closed path is created in any arbitrarily small neighborhood of the loop (Figure 122).

In § 29.3 it is proved that if $\sigma_0 \neq 0$, a separatrix loop will create at most one closed path in a sufficiently small neighborhood of itself. For the creation of the closed path it is necessary (but not sufficient) that the

separatrix loop disappear. If a closed path is created on moving to a close system, this closed path is a limit cycle of the same stability as the stability of the disappearing separatrix loop (Theorems 47 and 48).

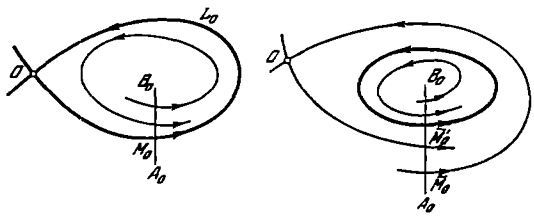

FIGURE 122

The case $\sigma_0 = 0$ is considered in §29.4. In this case, the uniqueness theorem does not apply, i.e., there exist modified systems as close as we desire to (D) with at least two closed paths in any arbitrarily small neighborhood of the loop (Theorem 50).

In addition to the above topics, §29 also discusses the conditions when the disappearance of a loop of necessity leads, or conversely does not lead, to the creation of a closed path in its neighborhood. Theorems 45 and 49 provide a comprehensive answer to this question for the case $\sigma_0 \neq 0$.

In Chapter XI we are dealing with analytical dynamic systems only. However, all the results remain valid for systems of class n $(n \geqslant 1)$, and the proof is completely analogous to that for analytical systems.

Also note that the closeness of dynamic systems in Chapter XI is to be understood, as always, in the sense of closeness in some fixed closed region $\overline{G}$.

§28. AUXILIARY MATERIAL

We will present a number of lemmas which later on are actively used in the proof of the principal proposition of this chapter. Some of these lemmas are contained in QT, but they are nevertheless reproduced here, sometimes without proof.

1. Correspondence function and succession function

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{D}$$

be a dynamic system, l_1 and l_2 two arcs without contact of the system which have no common points, and

$$x = g_1(u), \quad y = h_1(u), \quad a \leqslant u \leqslant b$$

and

$$x = g_2(\bar{u}), \quad y = h_2(\bar{u}), \quad \bar{a} \leqslant \bar{u} \leqslant \bar{b}$$

are the parametric equations of the arcs l_1 and l_2, respectively. Let g_1, h_1, g_2, h_2 be functions of class 2. A point on the arc l_1 (l_2) corresponding to the value u ($\bar{u}$) of the parameter will be designated $M(u)$ (or $\bar{M}(\bar{u})$).

Suppose that every path L of system (D) which for $t = t_0$ passes through the point $M(u)$ of the arc l_1 ($a \leqslant u \leqslant b$) will pass through the point $\bar{M}(\bar{u})$ of the arc l_2 for some other $\bar{t} > t_0$. Moreover, suppose that for $t_0 < t < \bar{t}$ the path L has no common points either with l_1 or with l_2. The parameters $\bar{t}$ and $\bar{u}$ are functions of u. We will designate them as $\chi(u)$ and $\omega(u)$ respectively:

$$\bar{t} = \chi(u), \quad \bar{u} = \omega(u).$$

The functions χ and ω are defined for all u, $a \leqslant u \leqslant b$, and, as is shown in QT, § 3.6, Remark 2 to Lemma 9, they are functions of class 1. The function $\omega(u)$ is called the correspondence function between arcs l_1 and l_2. It is readily seen that $\omega(u)$ is a monotonic function.

Let, as always,

$$x = \varphi(t;\ t_0,\ x_0,\ y_0), \quad y = \psi(t;\ t_0,\ x_0,\ y_0) \tag{1}$$

be the solution of (D) which passes through the point (x_0, y_0) for $t = t_0$. Let

$$\varphi(t;\ t_0,\ g_1(u),\ h_1(u)) = \Phi(t,\ u), \quad \psi(t;\ t_0,\ g_1(u),\ h_1(u)) = \Psi(t,\ u). \tag{2}$$

Then

$$\Phi(t_0,\ u) \equiv g_1(u), \quad \Psi(t_0,\ u) \equiv h_1(u). \tag{3}$$

On the other hand, in view of the above assumptions,

$$\Phi(\chi(u),\ u) \equiv g_2(\omega(u)), \quad \Psi(\chi(u),\ u) \equiv h_2(\omega(u)). \tag{4}$$

Since l_1 and l_2 are arcs without contact, each of the determinants

$$\Delta_1(u) = \begin{vmatrix} \Phi'_t(t_0,\ u) & \Psi'_t(t_0,\ u) \\ g'_1(u) & h'_1(u) \end{vmatrix} \quad \text{and} \quad \Delta_2(u) = \begin{vmatrix} \Phi'_t(\chi(u),\ u) & \Psi'_t(\chi(u),\ u) \\ g'_2(\omega(u)) & h'_2(\omega(u)) \end{vmatrix}$$

retains the same sign for all u, $a \leqslant u \leqslant b$. Without loss of generality, we may assume that both these determinants are positive. Then $\omega(u)$ is a monotonically increasing function (Figure 123).

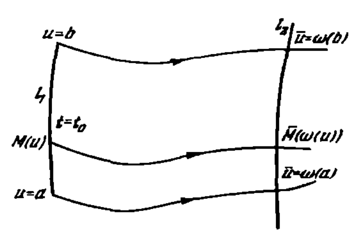

FIGURE 123

Also note that

$$\Phi_t'(t, u) = P(\Phi(t, u), \Psi(t, u)), \quad \Psi_t'(t, u) = Q(\Phi(t, u), \Psi(t, u)). \tag{5}$$

This follows from (2) and from the fact that (1) is a solution of system (D).

Lemma 1. The functions

$$x = \varphi(t; t_0, x_0, y_0), \quad y = \psi(t; t_0, x_0, y_0) \tag{1}$$

satisfy the partial differential equations

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\varphi}{\partial x_0} P(x_0, y_0) + \frac{\partial\varphi}{\partial y_0} Q(x_0, y_0), \quad \frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial x_0} P(x_0, y_0) + \frac{\partial\psi}{\partial y_0} Q(x_0, y_0). \tag{6}$$

Proof. From the definition of φ and ψ, if

$$x = \varphi(t; t_0, x_0, y_0), \quad y = \psi(t; t_0, x_0, y_0),$$

then

$$x_0 = \varphi(t_0; t, x, y), \quad y_0 = \psi(t_0; t, x, y). \tag{7}$$

Hence

$$\frac{\partial x_0}{\partial t_0} = P(x_0, y_0), \quad \frac{\partial y_0}{\partial t_0} = Q(x_0, y_0). \tag{8}$$

By (1) and (7),

$$x \equiv \varphi(t; t_0, \varphi(t_0; t, x, y), \psi(t_0; t, x, y)),$$
$$y \equiv \psi(t; t_0, \varphi(t_0; t, x, y), \psi(t_0; t, x, y)).$$

Differentiating the last identities with respect to t_0 and using (8), we find

$$\begin{aligned} &\frac{\partial\varphi}{\partial t_0} + \frac{\partial\varphi}{\partial x_0} P(x_0, y_0) + \frac{\partial\varphi}{\partial y_0} Q(x_0, y_0) = 0, \\ &\frac{\partial\psi}{\partial t_0} + \frac{\partial\psi}{\partial x_0} P(x_0, y_0) + \frac{\partial\psi}{\partial y_0} Q(x_0, y_0) = 0. \end{aligned} \tag{9}$$

As we know,

$$\varphi(t; t_0, x_0, y_0) = \varphi(t - t_0, 0, x_0, y_0), \quad \psi(t; t_0, x_0, y_0) = \psi(t - t_0, 0, x_0, y_0).$$

Therefore

$$\frac{\partial\varphi}{\partial t_0} = -\frac{\partial\varphi}{\partial t}, \quad \frac{\partial\psi}{\partial t_0} = -\frac{\partial\psi}{\partial t}.$$

Equations (6) follow from the last relations and from (9). Q.E.D.

Lemma 2. Let

$$J(t; t_0, x_0, y_0) = J = \begin{vmatrix} \frac{\partial\varphi}{\partial x_0} & \frac{\partial\varphi}{\partial y_0} \\ \frac{\partial\psi}{\partial x_0} & \frac{\partial\psi}{\partial y_0} \end{vmatrix}, \tag{10}$$

where $\varphi = \varphi(t; t_0, x_0, y_0)$, $\psi = \psi(t; t_0, x_0, y_0)$. *Then*

$$J = e^{\int_{t_0}^{t} [P_x'(\varphi, \psi) + Q_y'(\varphi, \psi)]\, dt} \tag{11}$$

Proof. Let us evaluate $\frac{\partial J}{\partial t}$.

Differentiating the determinant,

$$\frac{\partial J}{\partial t}=\begin{vmatrix}\frac{\partial}{\partial t}\frac{\partial\varphi}{\partial x_0} & \frac{\partial\varphi}{\partial y_0}\\ \frac{\partial}{\partial t}\frac{\partial\psi}{\partial x_0} & \frac{\partial\psi}{\partial y_0}\end{vmatrix}+\begin{vmatrix}\frac{\partial\varphi}{\partial x_0} & \frac{\partial}{\partial t}\frac{\partial\varphi}{\partial y_0}\\ \frac{\partial\psi}{\partial x_0} & \frac{\partial}{\partial t}\frac{\partial\psi}{\partial y_0}\end{vmatrix}=\begin{vmatrix}\frac{\partial P(\varphi,\psi)}{\partial x_0} & \frac{\partial\varphi}{\partial y_0}\\ \frac{\partial Q(\varphi,\psi)}{\partial x_0} & \frac{\partial\psi}{\partial y_0}\end{vmatrix}+\begin{vmatrix}\frac{\partial\varphi}{\partial x_0} & \frac{\partial P(\varphi,\psi)}{\partial y_0}\\ \frac{\partial\psi}{\partial x_0} & \frac{\partial Q(\varphi,\psi)}{\partial y_0}\end{vmatrix}=$$

$$=\begin{vmatrix}\frac{\partial P}{\partial x}\frac{\partial\varphi}{\partial x_0}+\frac{\partial P}{\partial y}\frac{\partial\psi}{\partial x_0} & \frac{\partial\varphi}{\partial y_0}\\ \frac{\partial Q}{\partial x}\frac{\partial\varphi}{\partial x_0}+\frac{\partial Q}{\partial y}\frac{\partial\psi}{\partial x_0} & \frac{\partial\psi}{\partial y_0}\end{vmatrix}+\begin{vmatrix}\frac{\partial\varphi}{\partial x_0} & \frac{\partial P}{\partial x}\frac{\partial\varphi}{\partial y_0}+\frac{\partial P}{\partial y}\frac{\partial\psi}{\partial y_0}\\ \frac{\partial\psi}{\partial x_0} & \frac{\partial Q}{\partial x}\frac{\partial\varphi}{\partial y_0}+\frac{\partial Q}{\partial y}\frac{\partial\psi}{\partial y_0}\end{vmatrix},$$

we obtain after simple manipulations

$$\frac{\partial J}{\partial t}=[P'_x(\varphi,\psi)+Q'_y(\varphi,\psi)]J. \tag{12}$$

We choose fixed t_0, x_0, y_0, i.e., the last relation is considered as an ordinary differential equation. Since

$$\varphi(t_0;\,t_0,\,x_0,\,y_0)\equiv x_0,\qquad \psi(t_0;\,t_0,\,x_0,\,y_0)\equiv y_0,$$

we have

$$\left[\frac{\partial\varphi}{\partial x_0}\right]_{t=t_0}=1,\qquad \left[\frac{\partial\varphi}{\partial y_0}\right]_{t=t_0}=0,$$
$$\left[\frac{\partial\psi}{\partial x_0}\right]_{t=t_0}=0,\qquad \left[\frac{\partial\psi}{\partial y_0}\right]_{t=t_0}=1.$$

Therefore

$$\left[\frac{\partial J}{\partial t}\right]_{t=t_0}=1. \tag{13}$$

Integration of equation (12) with the initial condition (13) gives (11). Q.E.D.

Let $\Delta(t,u)$ denote the Jacobian

$$\frac{D(\Phi,\Psi)}{D(t,u)}=\begin{vmatrix}\Phi'_t(t,u) & \Phi'_u(t,u)\\ \Psi'_t(t,u) & \Psi'_u(t,u)\end{vmatrix}.$$

Lemma 3.

$$\Delta(t,u)=\frac{D(\Phi,\Psi)}{D(t,u)}=J(t;\,t_0,\,g_1(u),\,h_1(u))\begin{vmatrix}P(g_1(u),\,h_1(u)) & Q(g_1(u),\,h_1(u))\\ g'_1(u) & h'_1(u)\end{vmatrix}, \tag{14}$$

where $J(t;\,t_0,\,g_1(u),\,h_1(u))$ *is defined by* (10).

Proof. By (2) and (6),

$$\Delta(t,u)=\begin{vmatrix}\frac{\partial\varphi}{\partial x_0}P(g_1(u),\,h_1(u))+\frac{\partial\varphi}{\partial y_0}Q(g_1(u),\,h_1(u)) & \frac{\partial\varphi}{\partial x_0}g'_1(u)+\frac{\partial\varphi}{\partial y_0}h'_1(u)\\ \frac{\partial\psi}{\partial x_0}P(g_1(u),\,h_1(u))+\frac{\partial\psi}{\partial y_0}Q(g_1(u),\,h_1(u)) & \frac{\partial\psi}{\partial x_0}g'_1(u)+\frac{\partial\psi}{\partial y_0}h'_1(u)\end{vmatrix}, \tag{15}$$

where the values of the derivatives $\frac{\partial\varphi}{\partial x_0}$, $\frac{\partial\varphi}{\partial y_0}$, $\frac{\partial\psi}{\partial x_0}$, $\frac{\partial\psi}{\partial y_0}$ are taken at the point $(t;\,t_0,\,g_1(u),\,h_1(u))$. The determinant (15) is equal to the product of the determinants $J(t;\,t_0,\,g_1(u),\,h_1(u))$ and $\begin{vmatrix}P(g_1,h_1) & Q(g_1,h_1)\\ g'_1(u) & h'_1(u)\end{vmatrix}$, which proves the lemma.

The above lemmas enable us to derive an expression for the derivative of the correspondence function. This expression will prove to be of considerable importance in what follows, and we will now proceed with its derivation.

We have already introduced the determinants

$$\Delta_1(u)=\begin{vmatrix}\Phi'_t(t_0,u) & \Psi'_t(t_0,u)\\ g'_1(u) & h'_1(u)\end{vmatrix}\quad\text{and}\quad \Delta_2(u)=\begin{vmatrix}\Phi'_t(\chi(u),u) & \Psi'_t(\chi(u),u)\\ g'_2(\omega(u)) & h'_2(\omega(u))\end{vmatrix}.$$

As we have seen, the two determinants can be taken positive without loss of generality. From (2), (4), and (5) we have

$$\Delta_1(u)=\begin{vmatrix}P(g_1(u),h_1(u)) & g'_1(u)\\ Q(g_1(u),h_1(u)) & h'_1(u)\end{vmatrix}, \tag{16}$$

$$\Delta_2(u)=\begin{vmatrix}P(g_2(\omega(u)),h_2(\omega(u))) & g'_2(\omega(u))\\ Q(g_2(\omega(u)),h_2(\omega(u))) & h'_2(\omega(u))\end{vmatrix}. \tag{17}$$

Lemma 4. The derivative of the correspondence function is expressible in the form

$$\omega'(u)=\frac{\Delta_1(u)}{\Delta_2(u)}e^{\int_{t_0}^{\chi(u)}[P'_x(\varphi,\psi)+Q'_y(\varphi,\psi)]\,dt} \tag{18}$$

where

$$\varphi=\varphi(t;t_0,g_1(u),h_1(u))=\Phi(t,u),$$
$$\psi=\psi(t;t_0,g_1(u),h_1(u))=\Psi(t,u).$$

Proof. By (4),

$$\Phi(\chi(u),u)\equiv g_2(\omega(u)),\qquad \Psi(\chi(u),u)\equiv h_2(\omega(u)).$$

Differentiation of these identities with respect to u gives

$$\Phi'_t(\chi(u),u)\chi'(u)+\Phi'_u(\chi(u),u)\equiv g'_2(\omega(u))\,\omega'(u),$$
$$\Psi'_t(\chi(u),u)\chi'(u)+\Psi'_u(\chi(u),u)\equiv h'_2(\omega(u))\,\omega'(u).$$

The last relations can be considered as a linear system in $\chi'(u)$, $\omega'(u)$. Its determinant is evidently equal to $\Delta_2(u)$, so that we are dealing with a Cramers system. Its solution is

$$\omega'(u)=\frac{\begin{vmatrix}\Phi'_t(\chi(u),u) & \Phi'_u(\chi(u),u)\\ \Psi'_t(\chi(u),u) & \Psi'_u(\chi(u),u)\end{vmatrix}}{\Delta_2(u)}. \tag{19}$$

The determinant in the last fraction is $\Delta(\chi(u),u)$. By (14)

$$\Delta(\chi(u),u)=J(\chi(u);t_0,g_1(u),h_1(u))\,\Delta_1(u).$$

Hence, using (11) and (19), we obtain (18). Q.E.D.

Remark. The derivative $\omega'(u)$ evidently does not depend on the choice of motion (2) along the path L passing through the point $(g(u),h(u))$ (i.e., it is independent of the initial time t_0).

Let us now consider the succession function. Let the succession function

$$\bar{u} = f(u) \tag{20}$$

be defined in some interval of the arc without contact l described by the parametric equations

$$x = g(u), \quad y = h(u)$$

(g and h are functions of class 2).

Let M_0 be the point of l corresponding to the value u_0 of the parameter u, L_0 the path through M_0, and

$$x = \varphi(t), \quad y = \psi(t)$$

the motion along this path in which the point M_0 corresponds to the time t_0. Let the succession function $f(u)$ be defined at u_0 and let

$$\bar{u}_0 = f(u_0) \neq u_0. \tag{21}$$

According to the definition of the succession function, this means that the path L_0 crosses the arc without contact l again for some $T > t_0$ at the point $\bar{M}_0(\bar{u}_0)$ and that for $t_0 < t < T$ it has no common points with l (Figure 124). By (21), M_0 and $\bar{M}_0$ are two different points and the path L_0 is therefore not closed.

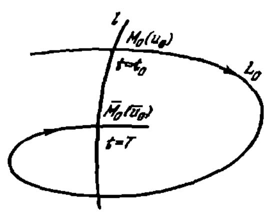

FIGURE 124

Let x_0, y_0 be the coordinates of M_0, and $\bar{x}_0, \bar{y}_0$ the coordinates of $\bar{M}_0$. Evidently,

$$\begin{aligned} \varphi'(t_0) &= P(x_0, y_0), & \psi'(t_0) &= Q(x_0, y_0), \\ \varphi'(T) &= P(\bar{x}_0, \bar{y}_0), & \psi'(T) &= Q(\bar{x}_0, \bar{y}_0). \end{aligned} \tag{22}$$

Let further

$$\Delta_0 = \begin{vmatrix} \varphi'(t_0) & \psi'(t_0) \\ g'(u_0) & h'(u_0) \end{vmatrix}, \quad \bar{\Delta} = \begin{vmatrix} \varphi'(T) & \psi'(T) \\ g'(u_0) & h'(u_0) \end{vmatrix} \tag{23}$$

We write

$$\Delta(x, y, u) = \begin{vmatrix} P(x, y) & Q(x, y) \\ g'(u) & h'(u) \end{vmatrix}. \tag{24}$$

By (22) – (24)

$$\Delta_0 = \Delta(x_0, y_0, u_0), \quad \bar{\Delta} = \Delta(\bar{x}_0, \bar{y}_0, \bar{u}_0). \tag{25}$$

Lemma 5. The derivative of the succession function at the point $M_0(u_0)$ of the arc without contact l through which passes the non-closed path L_0 is expressible in the form

$$f'(u_0)=\frac{\Delta(x_0,\ y_0,\ u_0)}{\Delta(\bar{x}_0,\ \bar{y}_0,\ \bar{u}_0)}\,e^{\int\limits_{t_0}^{T}[P'_x(\varphi(t),\ \psi(t))+Q'_y(\varphi(t),\ \psi(t))]\,dt} \tag{26}$$

Proof. Since by assumption M_0 and $\bar{M}_0$ are two different points, the succession function $\bar{u}=f(u)$ in a sufficiently small neighborhood of the point M_0 may be treated as a correspondence function between two arcs l' and l'' without common points, which are segments of the arc l containing the points M_0 and $\bar{M}_0$, respectively.

Then, by Lemma 4 (equality (18)),

$$f'(u_0)=\frac{\Delta_0}{\bar{\Delta}}\,e^{\int\limits_{t_0}^{T}[P'_x(\varphi,\ \psi)+Q'_y(\varphi,\ \psi)]\,dt}$$

Hence, using (25), we obtain (26). Q.E.D.

Remark 1. The expression for the derivative of the succession function at a point through which passes a closed path is evidently obtained from (26) if we take $\bar{u}_0=u_0$, $\bar{x}_0=x_0$, $\bar{y}_0=y_0$. In this case $\Delta(\bar{x}_0,\bar{y}_0,\bar{u}_0)=\Delta(x_0,\ y_0,\ u_0)$ and $f'(u_0)=e^{\int\limits_{t_0}^{T}[P'_x(\varphi,\ \psi)+Q'_y(\varphi,\ \psi)]\,dt}$ is the expression previously derived in Chapter V (§13.3, (30)).

Remark 2. Alongside with the arc without contact l on which the succession function (20) is defined, let us consider an arc without contact l^* defined by the parametric equations

$$x=g^*(v),\qquad y=h^*(v)$$

(g^* and h^* are functions of class 2). Let the path L_0 which for $t=t_0$ crosses the arc l at the point $M_0(u_0)$ cross the arc l^* for $t=t_0^*>t_0$ (or $t_0^*<t_0$) at some point $N_0(v_0)$, so that for t lying between t_0 and t_0^* the path L_0 has no common points either with l or with l^* (Figure 125). We moreover assume that M_0 and N_0 are interior points of l and l^*, respectively.

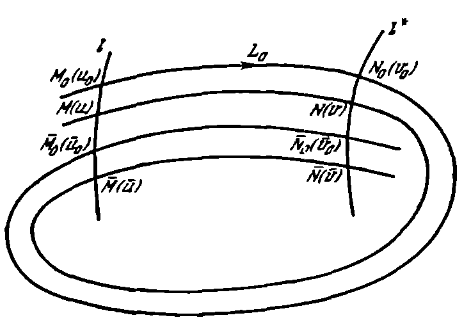

FIGURE 125

It is readily seen that a correspondence function between the arcs l and l^* is defined on the arc l near the point u_0,

$$v=\omega(u),$$

and a succession function is defined on the arc l^* in the neighborhood of the point v_0,

$$\bar{v}=f^*(v),$$

so that

$$\bar{v} = f^*(v) = f^*(\omega(u)) = \omega(f(u)) = \omega(\bar{u}). \tag{27}$$

Differentiation of (27) with respect to v gives

$$\frac{d\bar{v}}{dv} = f^{*\prime}(v) = \omega'(\bar{u})\, f'(u)\, \frac{du}{dv} = \frac{\omega'(\bar{u})}{\omega'(u)} f'(u). \tag{28}$$

Equation (28) establishes a relation between the derivatives of the succession function on different arcs without contact. This expression will be used in what follows. Note that if L_0 is a closed path, then $\bar{u}_0 = u_0$, $\bar{v}_0 = v_0$ and by (28)

$$f^{*\prime}(v_0) = f'(u).$$

This indicates that the value of the derivative of the succession function in this case is independent of the particular choice of the arc without contact and is also evidently independent of the particular choice of the parameter on the arc without contact.

Lemma 6. Let $\bar{u} = f(u)$ be the succession function on the arc without contact l defined for all u, $a \leqslant u \leqslant b$, and let for all these values of the parameter

$$f'(u) < 1 \qquad (f'(u) > 1). \tag{29}$$

Then there exists at most one closed path crossing the segment of l corresponding to the above values of u, and if such a closed path does exist, it will be a stable (correspondingly unstable) structurally stable limit cycle.

Proof. Suppose that there are two points on the arc l, $M_1(u_1)$ and $M_2(u_2)$, through which pass closed paths, where u_1 and u_2 belong to the segment $[a, b]$. Then $f(u_1) = u_1$, $f(u_2) = u_2$,

$$f(u_1) - f(u_2) = u_1 - u_2.$$

From Lagrange's formula, $f(u_1) - f(u_2) = f'(\hat{u})(u_1 - u_2)$, where $\hat{u}$ is a value of the parameter from the interval (u_1, u_2) and hence from the segment $[a, b]$. But then $f'(\hat{u})(u_1 - u_2) = u_1 - u_2$, i.e., $f'(\hat{u}) = 1$, at variance with (29). The first proposition of the lemma is thus proved. The fact that a closed path for which $f'(u) < 1$ $(f'(u) > 1)$ is a stable (unstable) structurally stable limit cycle was established in Chapter V (§ 12.4, and also § 14, Theorem 18). This completes the proof of the lemma.

We will give without proof two further lemmas which deal with (D) and other close systems.

Let l_1 and l_2 be two arcs without contact of system (D) without common points, and let a correspondence function $\bar{u} = \omega(u)$ between the arcs l_1 and l_2 be defined on the arc l_1 for all values of the parameter u, $a \leqslant u \leqslant b$, so that the values $\bar{a} = \omega(a)$ and $\bar{b} = \omega(b)$ of the correspondence function represent interior points of the arc l_2.

Lemma 7. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $(\tilde{D})$ δ-close to (D), l_1 and l_2 are arcs without contact and there exists a correspondence function between these arcs

$$\bar{u} = \tilde{\omega}(u),$$

which is defined on the segment $[a, b]$ *and is* ε*-close to the function* $\omega(u)$ *on this segment.**

A similar proposition applies to the succession function. Indeed, let a succession function $\bar{u} = f(u)$ be defined for system (D) on the arc l for all values of the parameter u on this arc, $a \leqslant u \leqslant b$, such that the values $\bar{a} = f(a)$ and $\bar{b} = f(b)$ represent interior points of l.

L e m m a 8. *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every system* $(\tilde{D})$ δ*-close to* (D), l *is an arc without contact and on this arc exists the succession function*

$$\bar{u} = \tilde{f}(u),$$

which is defined on the segment $[a, b]$ *and is* ε*-close to the function* $f(u)$ *on this segment.*

Lemma 7 follows almost directly from Lemma 2, §4.1, and from Theorem 4, §1.1, if we remember that the correspondence function $\tilde{\omega}(u)$, together with some function $\tilde{\chi}(u)$, satisfy the equations

$$\tilde{\Phi}(\tilde{\chi}(u), u) \equiv g_2(\tilde{\omega}(u)), \quad \tilde{\Psi}(\tilde{\chi}(u), u) \equiv h_2(\tilde{\omega}(u)),$$

which are analogous to equations (4). Lemma 8 may be considered as a particular case of Lemma 7.

2. Some properties of a saddle point and its separatrices

Simple equilibrium states classified as saddle points are considered in detail in QT, Chapter IV, §7.3, and also in Chapter IV of the present book (§9). We will now give those properties of saddle points and separatrices which will be needed in what follows.

Let $O(x_0, y_0)$ be a saddle point of the dynamic system (D), which is an interior point of $\bar{G}$.

L e m m a 9. *(a) There exist* $\varepsilon_0 > 0$ *and* $\delta_0 > 0$ *such that any system* $(\tilde{D})$ δ_0*-close in* $\bar{G}$ *to* (D) *has a single equilibrium state* $\tilde{O}$ *in* $U_{\varepsilon_0}(O)$, *which is moreover a saddle point.*

(b) For every ε, $0 < \varepsilon < \varepsilon_0$, *there exists* δ, $0 < \delta < \delta_0$, *such that if* $(\tilde{D})$ *is* δ*-close to* (D), *it has a saddle point* $\tilde{O}$ *lying in* $U_\varepsilon(O)$.

P r o o f. The validity of Lemma 9 follows directly from the definition of a simple equilibrium state (see §7.3, Definition 15, and also §2.1, Definition 5) and from the fact that $\Delta < 0$ for a saddle point.

R e m a r k. Consider a dynamic system whose right-hand sides are functions of the parameter μ, i.e., a system

$$\frac{dx}{dt} = P(x, y, \mu), \quad \frac{dy}{dt} = Q(x, y, \mu), \qquad (\tilde{D}_\mu)$$

where $(\tilde{D}_{\mu_0})$ is identified with the original system (D).

Then there exist $\varepsilon_0 > 0$ and $\alpha > 0$ such that if $|\mu - \mu_0| < \alpha$, $(\tilde{D}_\mu)$ has in $U_{\varepsilon_0}(O)$ a single equilibrium state, which is a saddle point, and its coordinates $x_0(\mu)$ and $y_0(\mu)$ are continuous functions of μ; in particular

$$\lim_{\mu \to \mu_0} x_0(\mu) = x_0, \quad \lim_{\mu \to \mu_0} y_0(\mu) = y_0.$$

* We recall that c l o s e n e s s is to be understood always as closeness at least to rank 1. See §3.1, Definition 6, and §1.1, Definition 1.

The validity of the last relations, i.e., the continuity of the functions $x_0(\mu)$ and $y_0(\mu)$ at the point μ_0, follows directly from Lemma 9. Continuity for all the other values of μ, close to μ_0, follows from the fact that each of these μ may be identified with μ_0.

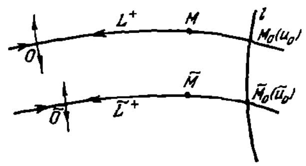

FIGURE 126

The next proposition is contained in the remark to Lemma 3, § 9.2, and it is given here without proof. Let O be a saddle point of (D), L^+ its ω-separatrix, l an arc without contact crossing the separatrix L^+ at a single point M_0, which does not coincide with either end point of the arc l, without crossing the second ω-separatrix of the saddle point (Figure 126). Let, furthermore, $x = g(u)$, $y = h(u)$ be the parametric equations of the arc l, the point M_0 corresponding to the value u_0 of the parameter u.

By Lemma 9, any modified system ($\tilde{\text{D}}$) sufficiently close to (D) has a single saddle point $\tilde{O}$ sufficiently close to O in some fixed neighborhood $U_{\varepsilon_0}(O)$.

Lemma 10. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if* ($\tilde{\text{D}}$) *is δ-close to* (D), *then:*

(a) One of the ω-separatrices of the saddle point $\tilde{O}$ (which we denote $\tilde{L}^+$) crosses the arc l at a single point $\tilde{M}_0$, corresponding to the value $\tilde{u}_0$ of the parameter, such that $\tilde{M}_0 \in U_\varepsilon(M_0)$, and the second ω-separatrix of the saddle point $\tilde{O}$ does not cross the arc l.

(b) If motion is defined on the separatrices L^+ and $\tilde{L}^+$ so that the points M_0 and $\tilde{M}_0$ correspond to the same time $t = t_0$, then for any $t > t_0$, the point $\tilde{M}(t)$ of the separatrix $\tilde{L}^+$ corresponding to the time t lies in an ε-neighborhood of the point $M(t)$ of the separatrix L^+ corresponding to the same time.

A similar proposition is true for the α-separatrices of the saddle point O.

Remark. As in the remark to Lemma 9, let (D_μ) be a system whose right-hand sides are continuous functions of μ and which coincides with the original system (D) for $\mu = \mu_0$. Then by Lemma 10, there exists $\alpha > 0$ such that if $|\mu - \mu_0| < \alpha$, ($\tilde{\text{D}}_\mu$) has a single saddle point $O(\mu)$ in $U_{\varepsilon_0}(O)$ one of whose ω-separatrices $\tilde{L}^+_\mu$ crosses the arc l at a single point $\tilde{M}_0(\mu)$, and the other separatrix has no common point with l. Moreover, the value $\tilde{u}_0(\mu)$ of the parameter u corresponding to the point $\tilde{M}_0(\mu)$ is a continuous function of μ. A similar proposition applies to the case when the right-hand sides of the system are continuous functions of several parameters.

Still another useful proposition can be derived from an analysis of the behavior of paths in the neighborhood of a saddle point (QT, § 7.3). This proposition is formulated in the form of a lemma, without proof.

Consider a circle C centered at the saddle point O; there are no equilibrium states of system (D), except O, either inside the area enclosed by the circle or on the circle. Let L_1^+ and L_2^- be ω- and α-separatrices of the saddle point O. Suppose that each of these separatrices has points lying outside C, and let M_1 and M_2 be the last common points of these separatrices with C (so that the segments OM_1 and OM_2 of the semipaths L_1^+ and L_2^- contain no points of C; Figure 127). The segments OM_1 and OM_2 of the separatrices L_1^+ and L_2^- divide the circular area enclosed by the circle C into two (curvilinear) sectors, one of

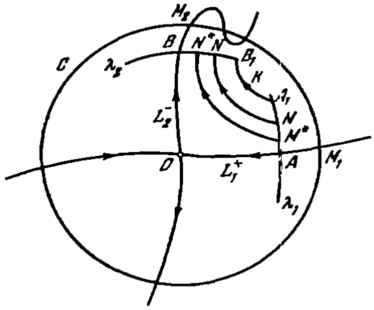

FIGURE 127

which contains the other two separatrices of the saddle point O. Let the other sector be designated K. Let A and B be two points on the segments OM_1 and OM_2 of the separatrices L_1^+ and L_2^- which do not coincide with M_1 and M_2, and λ_1, λ_2 arcs without contact passing through A and B which have no common points.

Lemma 11. There exist segments AA_1 and BB_1 of the arcs without contact λ_1 and λ_2 which are entirely contained (with the exception of the end points A and B) in the sector K and have the following property: every path L of (D) *which for $t = t_1$ passes through a point M of the arc AA_1 other than A, will cross for $t_2 > t_1$ the arc λ_2 at some point N other than B; moreover,*

(a) all the points of the path corresponding to t, $t_1 < t < t_2$, lie inside the sector K;

(b) a path passing through the point A_1 of the arc AA_1 crosses the arc BB_1 at the point B_1;

(c) the point N goes to B when M goes to A;

(d) for any $T > 0$, there exists a point M^* of the arc AA_1 such that for any path which for $t = t_1$ crosses the segment M^*A of the arc AA_1 we have the inequality $t_2 - t_1 > T$.

Retaining the notation of the previous lemma, let us consider a modified system $(\tilde{D})$.

From the theorems of the continuous dependence of the solution on the initial conditions and on the right-hand side and from the previous lemmas, it follows that there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ with the following properties: if $(\tilde{D})$ is δ_0-close to (D), then

1) $U_{\varepsilon_0}(O)$ contains one and only one equilibrium state, the saddle point $\tilde{O}$;

2) λ_1 and λ_2 are arcs without contact of $(\tilde{D})$;

3) there exist separatrices $\tilde{L}_1^+$ and $\tilde{L}_2^-$ of the saddle point $\tilde{O}$ of $(\tilde{D})$ which cross the arcs λ_1 and λ_2 at the points $\tilde{A}$ and $\tilde{B}$, respectively, the point $\tilde{A}$ lying on λ_1 in the same direction from A_1 as the point A, and the point $\tilde{B}$ lying on λ_2 in the same direction from B_1 as the point B;

4) the path of $(\tilde{D})$ passing through A_1 crosses the arc λ_2 at some point $\tilde{B}_1$ which lies on λ_2 in the same direction from $\tilde{B}$ as the point B_1.

The following two lemmas are now self-evident.

Lemma 12. For any $\varepsilon > 0$ $(\varepsilon < \varepsilon_0)$ there exists $\delta > 0$ $(\delta < \delta_0)$ such that if $(\tilde{D})$ is δ-close to (D), *then*

(a) $\tilde{A} \in U_\varepsilon(A)$, $\tilde{B} \in U_\varepsilon(B)$, $\tilde{B}_1 \in U_\varepsilon(B_1)$;

(b) the path of $(\tilde{D})$ which for $t = t_1$ passes through some point M of the arc $\tilde{A}A_1$ will cross the arc λ_2 at the point $\tilde{N}$ for $t = \tilde{t}_2 > t_1$;

(c) the point $\tilde{N}$ goes to $\tilde{B}$ when M goes to $\tilde{A}$.

Lemma 13. For a fixed $T > 0$, let M^* be a point of the arc λ_1 satisfying condition (d) of Lemma 11. Then there exists $\delta > 0$ such that if $(\tilde{D})$ is δ-close to (D), *the point $\tilde{A}$ lies on the arc λ_1 in the same direction from*

M^* as the point A, and for every path of $(\tilde{D})$ crossing the segment $M^*\tilde{A}$ of the arc λ_1 we have the inequality $\tilde{t}_2-t_1>T$.

In what follows, we will often require the parameter

$$\sigma(x_0, y_0)=P'_x(x_0, y_0)+Q'_y(x_0, y_0), \tag{30}$$

where x_0, y_0 are the coordinates of the saddle point O of (D). We will show that it is invariant under a transformation of coordinates.

We perform in G a transformation of coordinates defined by the equalities

$$\xi=f(x, y), \quad \eta=g(x, y) \tag{31}$$

or the equivalent equalities

$$x=\varphi(\xi, \eta), \quad y=\psi(\xi, \eta), \tag{32}$$

where f, g, φ, and ψ are functions of class 2.

In the new coordinates, (D) takes the form

$$\frac{d\xi}{dt}=P^*(\xi, \eta), \quad \frac{d\eta}{dt}=Q^*(\xi, \eta), \tag{D*}$$

where

$$\begin{aligned} P^*(\xi, \eta)&=f'_x(\varphi, \psi)P(\varphi, \psi)+f'_y(\varphi, \psi)Q(\varphi, \psi), \\ Q^*(\xi, \eta)&=g'_x(\varphi, \psi)P(\varphi, \psi)+g'_y(\varphi, \psi)Q(\varphi, \psi). \end{aligned} \tag{33}$$

The new coordinates of the saddle point O in this case are

$$\xi_0=f(x_0, y_0), \quad \eta_0=g(x_0, y_0).$$

Lemma 14. $\sigma(x_0, y_0)=P'_x(x_0, y_0)+Q'_y(x_0, y_0)$ *is invariant under a transformation of coordinates, i. e.,*

$$\sigma^*(\xi_0, \eta_0)=P^{*\prime}_{\xi}(\xi_0, \eta_0)+Q^{*\prime}_{\eta}(\xi_0, \eta_0)=\sigma(x_0, y_0). \tag{34}$$

Proof. Differentiating the first equality in (33) with respect to ξ and the second with respect to η, adding them and inserting for ξ and η their values ξ_0 and η_0, respectively, we obtain, using the relation $P(x_0, y_0)=Q(x_0, y_0)=0$,

$$\begin{aligned} \sigma^*(\xi_0, \eta_0)=\{&f'_x(\varphi, \psi)[P'_x(\varphi, \psi)\varphi'_\xi+P'_y(\varphi, \psi)\psi'_\xi]+ \\ &+f'_y(\varphi, \psi)[Q'_x(\varphi, \psi)\varphi'_\xi+Q'_y(\varphi, \psi)\psi'_\xi]+g'_x(\varphi, \psi)[P'_x(\varphi, \psi)\varphi'_\eta+P'_y(\varphi, \psi)\psi'_\eta]+ \\ &+g'_y(\varphi, \psi)[Q'_x(\varphi, \psi)\varphi'_\eta+Q'_y(\varphi, \psi)\psi'_\eta]\}_{\substack{\xi=\xi_0 \\ \eta=\eta_0}}. \end{aligned}$$

Equality (34) follows directly from the last relation in virtue of the identities

$$x\equiv\varphi(f(x, y), g(x, y)), \quad y\equiv\psi(f(x, y), g(x, y))$$

which give, when differentiated with respect to x and y,

$$\begin{aligned} \frac{\partial\varphi}{\partial\xi}f'_x+\frac{\partial\varphi}{\partial\eta}g'_x\equiv 1, \quad \frac{\partial\varphi}{\partial\xi}f'_y+\frac{\partial\varphi}{\partial\eta}g'_y\equiv 0, \\ \frac{\partial\psi}{\partial\xi}f'_x+\frac{\partial\psi}{\partial\eta}g'_x\equiv 0, \quad \frac{\partial\psi}{\partial\xi}f'_y+\frac{\partial\psi}{\partial\eta}g'_y\equiv 1. \end{aligned}$$

§29. CREATION OF LIMIT CYCLES FROM THE SEPARATRIX LOOP OF A SIMPLE SADDLE POINT

1. Some properties of the separatrix loop

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{D}$$

be a dynamic system, $O(x_0, y_0)$ a simple equilibrium state of this system, which is a saddle point. Suppose that one of the α-separatrices L_0 of the saddle point O is also an ω-separatrix, i.e., it forms a whole path which goes to O both for $t \to -\infty$ and for $t \to +\infty$. In this case we say that the separatrix L_0 *forms a loop*.* Let C_0 denote the simple closed curve comprising the path L_0 and the point O. The curve C_0 will be called a *loop of the separatrix* L_0 or simply a *loop*.

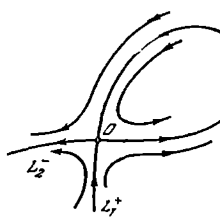

FIGURE 128

Besides the path L_0, the saddle point O also has two other separatrices L_1^+ and L_2^- (either different or coincident) which both lie either inside the curve C_0 or outside this curve. We will henceforth assume, without loss of generality, that the separatrices L_1^+ and L_2^- lies outside the curve C_0 (Figure 128). The case when they lie inside the curve C_0 is entirely analogous.

Lemma 1. A separatrix forming a loop may have at most one common point with an arc without contact.

Proof. Suppose a separatrix L_0 forming a loop has two common points A and B with some arc without contact l (Figure 129). Then the saddle point O should evidently lie both inside the closed curve formed by the segments AB of the path L_0 and the arc without contact l and outside this curve, which is clearly impossible. This proves the lemma.

Let $x = \varphi_0(t)$, $y = \psi_0(t)$ be a solution corresponding to the separatrix L_0, M_0 and M_1 the points of the separatrix corresponding to the times t_0 and t_1. We may take $t_0 < t_1$. Arcs without contact l_0 and l_1, without common points, are drawn through the points M_0 and M_1 (Figure 130).

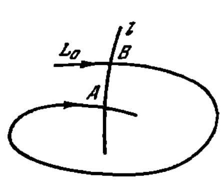

FIGURE 129

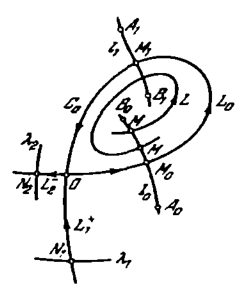

FIGURE 130

* By Theorem 23 (§18.2), a separatrix forming a loop may exist only in a structurally unstable system.

Let M_0B_0 be a segment of the arc l_0 which is entirely enclosed by the curve C_0, with the exception of its end point M_0; let M_0A_0 be a segment of the arc l_0 which lies entirely outside the curve C_0, with the exception of the point M_0. Furthermore, let M_1B_1 and M_1A_1 be the analogous segments of the arc l_1 (Figure 130).

L e m m a 2. *For any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: every path L which for $t = t_0$ passes through a point M_0 of the arc M_0B_0 which lies in $U_\delta(M_0)$ and does not coincide with M will again cross the segment M_0B_0 of the arc l_0 at some point $\bar{M}$ for $T > t_0$, without leaving the ε-neighborhood of the loop C_0 during the time $t_0 \leqslant t \leqslant T$ ($\bar{M}$ may differ from M, as in Figure 130, or coincide with M).*

The proof of Lemma 2 follows directly from Lemma 11 of the previous section, if we take into consideration the properties of paths crossing two arcs without contact (QT, §3.4, Lemma 5).

R e m a r k. Let $\Delta > 0$ be some number. The number δ introduced in Lemma 2 can be made sufficiently small, so that for some t_1', $t_0 < t_1' < T$, $|t_1' - t_1| < \Delta$, the path L will cross the segment M_1B_1 of the arc l_1.

In the next lemma, we consider paths which cross the segment M_0A_0 of the arc l_0 lying outside the loop C_0.

L e m m a 3. *There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ with the following property: if the path L passes through the point M of the arc M_0A_0 which lies in $U_{\delta_0}(M_0)$ and does not coincide with M_0, the path L will leave the ε_0-neighborhood of the path C_0 both with increasing and with decreasing t.*

P r o o f. Consider the ω- and α-separatrices L_1^+ and L_2^- of the saddle point O which lie outside the curve C_0. Let N_1 and N_2 be two points on these separatrices, and λ_1 and λ_2 arcs without contact passing through these respective points (Figure 130). By Lemma 11 of the previous section, there exists $\delta_0 > 0$ with the following property: if the path L passes through the point M of the arc M_0A_0 which lies in $U_{\delta_0}(M_0)$ and does not coincide with M_0, this path will cross the arc λ_2 with increasing t and the arc λ_1 with decreasing t. As $\varepsilon_0 > 0$ we choose a number such that the neighborhood $U_{\varepsilon_0}(C_0)$ does not intersect with the arcs without contact λ_1 and λ_2. The numbers δ_0 and ε_0 evidently satisfy the proposition of the lemma.

Let us now return to the case of the arc M_0B_0 lying i n s i d e the curve C_0. By Lemma 2, every path which for $t = t_0$ crosses this arc at a point M sufficiently close to M_0, will again cross this arc at a point $\bar{M}$ ("successor" of M) for $T > t_0$; when M goes to M_0, $\bar{M}$ also goes to M_0. Since the arcs M_0B_0 and M_1B_1 may be identified with the arcs BB_1 and AA_1, respectively, of Lemma 11 of the previous section, this lemma together with Lemma 2 directly lead to the following useful proposition.

L e m m a 4. *When the point M on the arc M_0B_0 (Figure 130) goes to the point M_0 on a separatrix, the time T corresponding to the point $\bar{M}$ goes to $+\infty$.*

We will say that the path L goes to the loop C_0 for $t \to +\infty$ ($t \to -\infty$) if its ω-limit (correspondingly, α-limit) set coincides with the loop C_0.

The proof of the following lemma, which uses Lemma 2, is self-evident and is thus omitted.

L e m m a 5. *If among the paths crossing the arc M_0B_0 at points sufficiently close to M_0 there are no closed paths, then either all these paths go to a loop for $t \to +\infty$ or they all go to a loop for $t \to -\infty$.*

Definition 29. A loop C_0 is said to be stable (unstable) if all the paths crossing the arc M_0B_0 at points sufficiently close to M_0 (and yet different from M_0) go to the loop C_0 for $t \to +\infty$ (correspondingly, for $t \to -\infty$).

By Lemma 2, a succession function is defined on some segment M_0B of the arc M_0B_0 (the point M_0 excepted).

Let

$$x = g_0(u), \qquad y = h_0(u) \tag{1}$$

be the parametric equations of the arc l_0. As in the previous section, we assume that g_0 and h_0 are functions of class 2.*

Suppose that the points M_0, A_0, B_0, B correspond to the values u_0, a_0, b_0, b of the parameter, where $a_0 < u_0 < b < b_0$.

Let

$$\bar{u} = f(u) \tag{2}$$

be the succession function on the arc M_0B. In virtue of the above assumption, $f(u)$ is defined for all u, $u_0 < u \leqslant b$ (we should stress that we are dealing with a succession function in the direction of increasing t, i.e., each "succeeding" point corresponds to a later time than the time of the "preceding" point).

In what follows, we will also consider the function

$$d(u) = f(u) - u. \tag{3}$$

By Lemma 2,

$$\lim_{u \to u_0} d(u) = 0. \tag{4}$$

If a closed path L^* passes through the point $M^*(u^*)$ of the arc M_0B, we have $d(u^*) = 0$.

If there exists u_1, $u_0 < u_1 \leqslant b$, such that for all u, $u_0 < u \leqslant u_1$, $d(u) \neq 0$, i.e., if there are no closed paths crossing the arc M_0B_0 at the points u, $u_0 < u \leqslant u_1$, then

$$\begin{aligned} &\text{the loop is stable if } d(u) < 0, \\ &\text{the loop is unstable if } d(u) > 0 \end{aligned} \tag{5}$$

(for $u_0 < u \leqslant u_1$).

In what follows we will consider the limit of the derivative of the succession function $f'(u)$ for $u \to u_0$.

Let

$$x = \varphi_M(t), \qquad y = \psi_M(t) \tag{6}$$

be the solution corresponding to the path L passing through the point $M(u)$ of the arc M_0B. As before, we assume that the solution (6) passes through the point M for $t = t_0$ and through the successor $\bar{M}(\bar{u})$ of M for $t = T > t_0$.

The coordinates of the points M_0, M, $\bar{M}$ and the saddle point O are (ξ_0, η_0), (ξ, η), $(\bar{\xi}, \bar{\eta})$, and (x_0, y_0), respectively. Let, as before,

$$\sigma_0 = \sigma(x_0, y_0) = P'_x(x_0, y_0) + Q'_y(x_0, y_0). \tag{7}$$

* If, in particular, l_0 is a segment of the normal to the path L_0 at the point M_0, the functions g_0 and h_0 can be taken in the form

$$g_0(u) = \varphi_0(t_0) - \psi'_0(t_0)\,u, \qquad h_0(u) = \psi_0(t_0) + \varphi'_0(t_0)\,u.$$

Lemma 6. Let the point M on the arc M_0B go to the point M_0, i.e., $u \to u_0$. Then

$$f'(u) \to +\infty \text{ for } \sigma(x_0, y_0) = P'_x(x_0, y_0) + Q'_y(x_0, y_0) > 0, \tag{8}$$
$$f'(u) \to 0 \text{ for } \sigma(x_0, y_0) < 0. \tag{9}$$

Proof. First note that $\lim_{u \to u_0} f'(u)$ is independent of the particular choice of the arc without contact l_0. This follows directly from equation (28) of the previous section. In virtue of this equation, if l_0^* is an arc without contact analogous to l, v and $f^*(v)$ are the parameter and the succession function on this arc, so that the separatrix L_0 crosses l^* at the point v_0, and if $\omega(u)$ is the correspondence function between arcs without contact l and l^*, we have

$$f^{*\prime}(v) = \frac{\omega'(\bar{u})}{\omega'(u)} f'(u).$$

If $u \to u_0$, then $v \to v_0$, $\bar{u} \to u_0$, and since $\omega'(u_0) \neq 0$ in virtue of §28.1, (18), we have

$$\lim_{v \to v_0} f^{*\prime}(v) = \lim_{u \to u_0} f'(u).$$

First consider the case when

$$\sigma_0 = \sigma(x_0, y_0) > 0.$$

Let C be a circle centered at O with a sufficiently small radius so that at any point (x, y) inside C we have

$$\sigma(x, y) = P'_x(x, y) + Q'_y(x, y) > \frac{\sigma_0}{2}. \tag{10}$$

The points M_0 and M_1 and the arcs l_0 and l_1 are chosen so that these arcs as well as the segments of the separatrix L_0 corresponding to $t \leqslant t_0$ and $t \geqslant t_1$ lie inside the circle C (Figure 131).

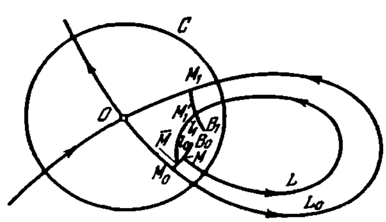

FIGURE 131

By Lemma 5 of the previous section, we have

$$f'(u) = \frac{\Delta(\xi, \eta, u)}{\Delta(\bar{\xi}, \bar{\eta}, \bar{u})} e^{\int_{t_0}^{T} \sigma(\varphi_M(t), \psi_M(t))\, dt} \tag{11}$$

where

$$\Delta(\xi, \eta, u) = \begin{vmatrix} P(\xi, \eta) & Q(\xi, \eta) \\ g_0'(u) & h_0'(u) \end{vmatrix},$$

and

$$\Delta(\bar{\xi}, \bar{\eta}, \bar{u}) = \begin{vmatrix} P(\bar{\xi}, \bar{\eta}) & Q(\bar{\xi}, \bar{\eta}) \\ g_0'(\bar{u}) & h_0'(\bar{u}) \end{vmatrix}.$$

When $u \to u_0$, both determinants $\Delta(\xi, \eta, u)$ and $\Delta(\bar{\xi}, \bar{\eta}, \bar{u})$ go to the same non-zero limit $\Delta(\xi_0, \eta_0, u_0)$, so that

$$\lim_{u \to u_0} \frac{\Delta(\xi, \eta, u)}{\Delta(\bar{\xi}, \bar{\eta}, \bar{u})} = 1. \tag{12}$$

We should therefore find only the limit

$$\lim_{M \to M_0} \int_{t_0}^{T} \sigma(\varphi_M(t), \psi_M(t))\, dt.$$

Clearly,

$$\int_{t_0}^{T} \sigma(\varphi_M(t), \psi_M(t))\, dt = \int_{t_0}^{t_1'} \sigma(\varphi_M, \psi_M)\, dt + \int_{t_1'}^{T} \sigma(\varphi_M, \psi_M)\, dt, \tag{13}$$

where t_1' is the value of t corresponding to the intersection point M_1' of the path L with the arc without contact l_1 (Figure 131 ; we recall that the path L passes through the point M for $t = t_0$).

When $M \to M_0$, we have $\int_{t_0}^{t_1'} \sigma(\varphi_M, \psi_M)\, dt \to \int_{t_0}^{t_1} \sigma(\varphi_0(t), \psi_0(t))\, dt$, where $x = \varphi_0(t)$, $y = \psi_0(t)$ is the solution corresponding to the separatrix L_0 which passes through the point M_0 for $t = t_0$. The first integral on the right in (13) thus goes to a finite limit.

Let us consider the second integral, i.e.,

$$\int_{t_1'}^{T} \sigma(\varphi_M(t), \psi_M(t))\, dt.$$

If the point M is sufficiently close to M_0, then M_1' is arbitrarily close to M_1 and the segment $M_1'\bar{M}$ of the path L is entirely enclosed inside the circle C (see §28.2, Lemma 11). But then, in virtue of inequality (10), the integrand in the last integral is greater than $\frac{\sigma_0}{2}$ for any $t \in [t_1', T]$, i.e.,

$$\int_{t_1'}^{T} \sigma(\varphi_M(t), \psi_M(t))\, dt > \frac{\sigma_0}{2}(T - t_1').$$

For $M \to M_0$, $t_1' \to t_1$ and $T \to +\infty$ (by Lemma 4). The last integral, and hence $\int_{t_0}^{T} \sigma(\varphi_M, \psi_M)\, dt$, go to $+\infty$ for $M \to M_0$. This result and equalities (11) and (12) show that $\lim_{u \to u_0} f'(u) = +\infty$. We have thus proved (8). It can be

similarly shown that if $\sigma_0 < 0$, then

$$\lim_{M \to M_0} \int_{t_0}^{T} \sigma(\varphi_M(t), \psi_M(t))\,dt = -\infty, \quad \text{т. е.} \quad f'(u) \to 0.$$

This completes the proof of the lemma.

Remark. We have assumed so far that the separatrices L_1^+ and L_2^- of the saddle point O which do not coincide with L_0 lie outside the loop C_0 and that the segment M_0B of the arc without contact l_0 on which the succession function $f(u)$ is defined corresponds to $u > u_0$. If the separatrices L_1^+ and L_2^- lie inside the loop C_0, the loop may only be an α- or ω-limit continuum for the paths of the system from outside, whereas in all other respects the situation does not change. In particular, all the previous lemmas of this section remain valid. The assumption that the separatrices L_1^+ and L_2^- lie outside the loop is thus of no consequence and has been introduced for convenience only.

Let us now see what happens if the direction along the arc l_0 is reversed (e.g., by defining a new parameter $u^* = -u$). First note that the succession function $f(u)$ on the arc l_0 (or, more precisely, on the segment of the arc adjoining the point M_0, where it is defined) is a monotonically increasing function and its derivative $f'(u)$ is positive regardless of the particular direction on the arc l_0 which is chosen as the positive direction. This follows immediately from geometrical considerations (or, alternatively, from Lemma 5 of the previous section and the remark to this lemma). Our Lemma 6 thus remains in force for any choice of the parameter on the arc without contact l_0.

Conversely, conditions (5) derived in the preceding are true only when the succession function $f(u)$ is defined for $u > u_0$. If, on the other hand, it is defined for u smaller than u_0 (e.g., for $u_0 > u \geqslant u_1$) and there are no closed paths crossing the arc l_0 at points u, $u_0 < u \leqslant u_1$, we clearly see that

$$\begin{array}{l} \text{the loop is stable for } d(u) > 0, \\ \text{the loop is unstable for } d(u) < 0 \end{array} \tag{14}$$

(for $u_0 > u \geqslant u_1$).

We are now in a position to derive a sufficient condition of stability (instability) of a separatrix loop.

Theorem 44. Let $O(x_0, y_0)$ be a saddle point of the dynamic system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{D}$$

and L_0 its separatrix which together with the saddle point O forms the loop C_0. Then, if $\sigma_0 = P'_x(x_0, y_0) + Q'_y(x_0, y_0) > 0$, the loop is unstable, and if $\sigma_0 < 0$, the loop is stable.

Proof. Let us first consider the case when the succession function is defined for u close to u_0, but greater than u_0. Let $\sigma_0 > 0$. In virtue of (4),

$$\lim_{u \to u_0} d(u) = 0.$$

On the other hand, in virtue of Lemma 6, if $\sigma_0 > 0$, we have

$$\lim_{u \to u_0} f'(u) = +\infty.$$

Therefore, there exists u_1 such that for all u, $u_0 < u < u_1$,

$$f'(u) > 1 \quad \text{and} \quad d'(u) = f'(u) - 1 > 0. \tag{15}$$

It follows from (4) and (15) that for all u, $u_0 < u < u_1$, $d(u) > 0$, i.e., the loop is unstable in virtue of (5).

Let now $\sigma_0 < 0$. Then, by Lemma 6

$$\lim_{u \to u_0} f'(u) = 0,$$

i.e., there exists u_1 such that for all u, $u_0 < u < u_1$,

$$f'(u) < 1 \text{ and } d'(u) < 0. \tag{16}$$

By (4) and (16) we see that for all u, $u_0 < u < u_1$, $d(u) < 0$, i.e., the loop is stable in virtue of (5).

If the succession function is defined for $u < u_0$, the proof proceeds along the same lines. In this case, relations (4) and (15) show that for $u_0 > u > u_1$, $d(u) < 0$, and then by (14) the loop is unstable. By (4) and (16) we see that $d(u) > 0$, i.e., the loop is stable by (14). This completes the proof of the theorem.

Remark 1. It follows from Theorem 44 that if the four separatrices of the saddle point O form two loops (which lie one outside the other or one inside the other, Figures 132 and 133), then for $\sigma_0 > 0$ both loops are unstable, and for $\sigma_0 < 0$ both loops are stable.

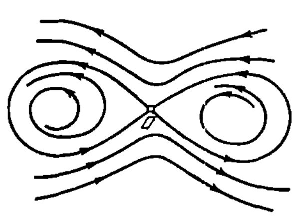

FIGURE 132

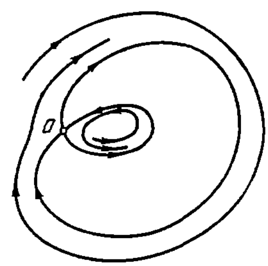

FIGURE 133

Remark 2. Theorem 44 is concerned with the case $\sigma_0 = P'_x(x_0, y_0) + Q'_y(x_0, y_0) \neq 0$. We will now show that if $\sigma_0 = 0$, there may be cases when an arbitrarily small neighborhood of the loop contains closed paths, as well as cases when the loop is stable or unstable.

Example 11. Consider the system

$$\frac{dx}{dt} = 2y = P_1(x, y), \quad \frac{dy}{dt} = 12x - 3x^2 = Q_1(x, y), \tag{D$_1$}$$

which is investigated in QT, §1.14, Example VIII. It can be checked that this system has a general integral

$$x^3 - 6x^2 + y^2 = C. \tag{17}$$

The shape of the curves (17) is readily established from their explicit equation

$$y=\pm\sqrt{6x^2-x^3+C}$$

considered together with the auxiliary curves

$$z=6x^2-x^3+C,$$

which can be constructed without difficulty. Curves (17) are shown in Figure 134. $C=0$ corresponds to the curve with a loop. For $C>0$, curve (17) comprises a single branch located outside the loop. For $0>C>-32$, the curve consists of two branches. One of these branches lies to the left of the y axis, and the other branch is an oval enclosed inside the loop. For $C=-32$, the curve comprises a branch lying in the left half-plane and the point $O_1(4,0)$. Finally for $C<-32$, the curve has a single branch in the left half-plane. Each of the curves (17) is either a path of system (D_1) (for $C>0$ and $C<-32$), or consists of two (for $0>C\geqslant-32$) or four (for $C=0$) paths. System (D_1) has two equilibrium states: $O(0,0)$ and $O_1(4,0)$. The first of these equilibrium states is a saddle point, with two of its separatrices forming the loop L_0, and the second is a center.

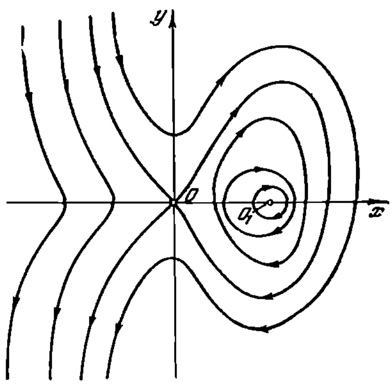

FIGURE 134

Here $\sigma_0(0,0)=P'_{1x}(0,0)+Q'_{1y}(0,0)=0$, i.e., we are dealing with the case $\sigma_0=0$, and any neighborhood of the loop contains closed paths.

Together with system (D_1), let us now consider the system

$$\begin{aligned}\frac{dx}{dt}&=2y-\mu(x^3-6x^2+y^2)(12x-3x^2)=P_2(x,y),\\ \frac{dy}{dt}&=12x-3x^2+\mu(x^3-6x^2+y^2)\,2y=Q_2(x,y),\end{aligned}\qquad (D_2)$$

where μ is a small (in absolute magnitude) number.

Clearly,

$$P_2=P_1-\mu fQ_1,\quad Q_2=Q_1+\mu fP_1,\qquad (18)$$

where

$$f=f(x,y)=x^3-6x^2+y^2.\qquad (19)$$

It is directly verified that the four paths of the system (D_1) making the curve

$$x^3-6x^2+y^2=0\qquad (20)$$

are also paths of system (D_2). Moreover, (D_2) has the same equilibrium states as (D_1), i.e., the points $O(0,0)$ and $O_1(4,0)$, and the point O is a saddle point of (D_2), whereas O_1 for small $\mu\neq0$ is a structurally stable

focus, which is stable for $\mu > 0$ and unstable for $\mu < 0$. Thus L_0 is a separatrix of (D_2) forming a loop. The vector field of (D_2) is obtained by rotating the vector field of (D_1) through an angle $\tan^{-1} \mu/(x, y)$(this is found by direct computation; see also QT, § 1.14, remark preceding Example VII). But then all closed paths of (D_1) are cycles without contact for (D_2). Hence it clearly follows that inside the loop L_0 (D_2) has no closed paths, i.e., all the paths of (D_2) lying inside the separatrix loop either wind onto the loop for $t \to -\infty$ and onto the focus O_1 for $t \to +\infty$, or, conversely, wind onto the focus O_1 for $t \to -\infty$ and onto the separatrix loop for $t \to +\infty$. Since for $\mu > 0$ ($\mu < 0$), the focus O_1 is stable (unstable), the separatrix loop L_0 is

stable for $\mu < 0$,
unstable for $\mu > 0$.

Now $\sigma_0 = P'_{2x}(0, 0) + Q'_{2y}(0, 0)$ vanishes for any μ. Our example thus proves that for $\sigma_0 = 0$, any of the three alternatives mentioned in Example 2 may be observed.

2. Theorems of the creation of a closed path from a separatrix loop

Assuming, as before, that system (D) has a saddle point O whose separatrix L_0 forms a loop, we will consider, alongside with (D), a modified system

$$\frac{dx}{dt} = \tilde{P}(x, y), \quad \frac{dy}{dt} = \tilde{Q}(x, y). \tag{$\tilde{D}$}$$

Retaining the same notation as in the previous subsection, we write M_0 and M_1 for points of the separatrix L_0 corresponding to $t = t_0$ and $t = t_1$, $t_0 < t_1$, l_0 and l_1 for the arcs without contact passing through these points, and u for the parameter on the arc l_0 (Figure 130). Let L_0^+ be the positive half of the path L_0 containing the point M_0 (and hence also the point M_1; L_0^+ is an ω-separatrix of O, and its points correspond to $t \geqslant t_0^*$, where $t_0^* < t_0$), and L_0^- the negative half of the path L_0 containing M_0 and M_1 (the points of L_0^- correspond to $t \leqslant t_1^*$, where $t_1^* > t_1$).

By Lemmas 9 and 10, § 28, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if the modified system ($\tilde{D}$) is δ_0-close to (D), then

(a) $U_{\varepsilon_0}(O)$ contains a single equilibrium state of ($\tilde{D}$), the saddle point $\tilde{O}$;

(b) there exist an ω-separatrix $\tilde{L}_0^+$ and an α-separatrix $\tilde{L}_0'^-$ of the saddle point $\tilde{O}$ crossing the arc l_0 at the points $\tilde{M}_0$ and $\tilde{M}_0'$, respectively, which lie inside the arc l_0;

(c) the ω- and the α-separatrices $\tilde{L}_0^+$ and $\tilde{L}_0'^-$ cross the arc l_1 at points $\tilde{M}_1$ and $\tilde{M}_1'$, respectively, which lie inside the arc l_1 (Figure 135).

Moreover, by Lemmas 9 and 10, § 28, for any positive $\varepsilon < \varepsilon_0$, there exists $\delta < \delta_0$ such that for every system ($\tilde{D}$) which is δ-close to (D), the saddle point $\tilde{O}$ is contained in $U_\varepsilon(O)$, and each point of the semipath $\tilde{L}_0^+$ ($\tilde{L}_0'^-$) lies in the ε-neighborhood of the point of the semipath L_0^+ (L_0^-) corresponding to the same time. The points $\tilde{M}_0$ and $\tilde{M}_0'$ lie in $U_\varepsilon(M_0)$, and the points $\tilde{M}_1$ and $\tilde{M}_1'$ in $U_\varepsilon(M_1)$.

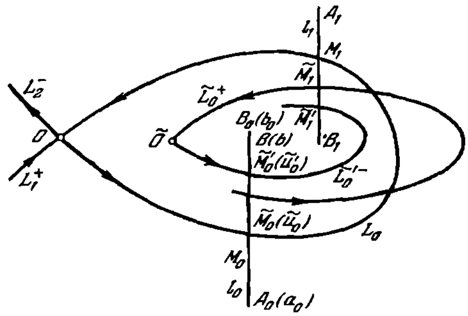

FIGURE 135

Let $\widetilde{u}_0$ and $\widetilde{u}_0'$ be the values of the parameter u corresponding to the points $\widetilde{M}_0$ and $\widetilde{M}_0'$ $(a < \widetilde{u}_0 < b,\ a < \widetilde{u}_0' < b)$. The following cases are possible for different modified systems:

1) $\widetilde{M}_0$ and $\widetilde{M}_0'$ are two different points, i.e., $\widetilde{u}_0' \neq \widetilde{u}_0$.

2) $\widetilde{M}_0$ and $\widetilde{M}_0'$ are the same point, i.e., $\widetilde{u}_0' = u_0$.

In case 1, Lemma 1 shows that $(\widetilde{D})$ has no separatrices which form a loop whose semipaths are $\widetilde{L}_0^+$ and $\widetilde{L}_0'^-$. We will say in this case that on moving from (D) to $(\widetilde{D})$, the loop is broken. There evidently always exist modified systems as close as desired to (D) and such that the loop is broken when moving to these systems. An appropriate example is the system

$$\frac{dx}{dt} = P - \mu Q, \quad \frac{dy}{dt} = Q + \mu P, \tag{D*}$$

where $\mu \neq 0$ is a sufficiently small (in absolute magnitude) number. By the lemma of § 11.1, if $\mu > 0$, we have $\widetilde{u}_0' > u_0$, $\widetilde{u}_0 < u_0$ for system (D*), considered as $(\widetilde{D})$, i.e.,

$$\widetilde{u}_0 < \widetilde{u}_0',$$

and if $\mu < 0$, then $\widetilde{u}_0' < u_0$, $\widetilde{u}_0 > u_0$, i.e.,

$$\widetilde{u}_0 > \widetilde{u}_0'. \tag{21}$$

In case 2, when the points $\widetilde{M}_0$ and $\widetilde{M}_0'$ coincide, $(\widetilde{D})$ has a separatrix $\widetilde{L}_0$ which forms a loop, $\widetilde{L}_0^+$ and $\widetilde{L}_0'^-$ are its semipaths, and in this case the semipath $\widetilde{L}_0'^-$ may be simply designated $\widetilde{L}_0^-$.

Let, as before, the succession function be defined for (D) everywhere along the arc M_0B, with the exception of the point M_0.

Lemma 7. There exists $\delta_0 > 0$ such that if $(\widetilde{D})$ is δ_0-close to (D), l_0 is an arc without contact for $(\widetilde{D})$, and every path of $(\widetilde{D})$ which for $t = t_0$ crosses the segment $\widetilde{M}_0B$ of the arc l_0 will cross for some $T > t_0$ the segment $\widetilde{M}_0'B_0$ of this arc.

The proof of this lemma follows almost immediately from Lemmas 7 and 9–12, § 28, and it is therefore omitted. Note, however, that the cases $\widetilde{u}_0' > \widetilde{u}_0$, $\widetilde{u}_0' = \widetilde{u}_0$, and $\widetilde{u}_0' < \widetilde{u}_0$ should be treated separately (Figures 135, 136, and 137, respectively). The proof proceeds along the same lines in all the three cases.

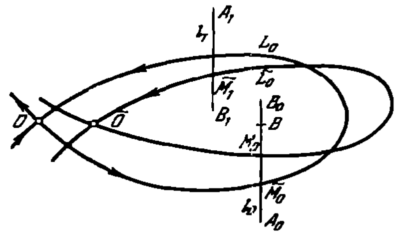

FIGURE 136

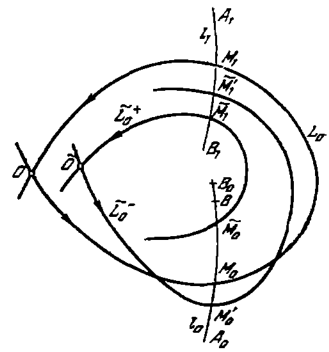

FIGURE 137

Lemma 8. There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ which satisfy the following conditions: if $(\tilde{D})$ is δ_0-close to (D), and $\tilde{u}_0' < \tilde{u}_0$ $(\tilde{u}_0' > \tilde{u}_0)$, all the paths of $(\tilde{D})$ crossing the segment $\tilde{M}_0\tilde{M}_0'$ of the arc l_0 will leave the ε_0-neighborhood of the loop L_0 as t increases (decreases).

Proof. The validity of Lemma 8 follows directly from the properties of the paths of system $(\tilde{D})$ which cross the segment $\tilde{M}_0\tilde{M}_0'$ of the arc without contact l_0 (see Figures 135 and 137), if we use Lemma 3 of this section and also Lemmas 9 and 10 of §28.2.

We can now prove one of the fundamental theorems, the so-called theorem of creation of a closed path from a separatrix loop. It establishes sufficient conditions for the appearance of a closed path when a separatrix loop disappears (breaks).

Let O be a saddle point of a dynamic system (D), L_0 its separatrix which forms a loop. Here, l_0, $\tilde{L}_0^+$, $\tilde{L}_0'^-$, $\tilde{u}_0$, $\tilde{u}_0'$, A_0, B_0, a_0, b_0, b, etc., have the same meaning as before; moreover, $a_0 < b_0$ and the point B_0 lies inside the loop (Figure 135 or 137).

Theorem 45. Let the loop formed by the separatrix L_0 of the saddle point O be stable (unstable). Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $(\tilde{D})$ is δ-close to (D) and if $\tilde{u}_0 < \tilde{u}_0'$ $(\tilde{u}_0 > \tilde{u}_0')$, then in the ε-neighborhood of the loop there exists at least one closed path $\tilde{L}^*$ of $(\tilde{D})$ which crosses the arc without contact l_0 at the point M^* (u^*), where

$$\tilde{u}_0 < u^* < b \qquad (\tilde{u}_0' < u^* < b).$$

A similar proposition applies if the separatrices of the saddle point O other than L_0 lie inside the loop.

Proof. Consider the case of a stable separatrix loop. Let $\varepsilon > 0$ be given. On the segment M_0B of the arc l_0 we select a point N_1 (n_1) sufficiently close to M_0, so that the following conditions are satisfied:

(a) the path L_N of (D) passing through N_1 goes to the loop for $t \to +\infty$;

(b) the segment of the path L_N between the point N_1 and the "succeeding" intersection point N_2 (u_2) of the path L_N with the arc l_0, together with the

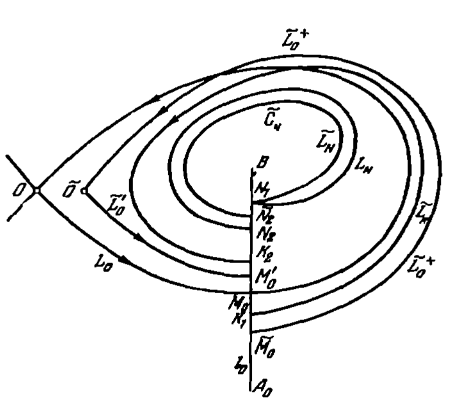

FIGURE 138

segment N_1N_2 of the arc l_0, form a simple closed curve C_N which is entirely contained in $U_{\varepsilon/3}(L_0)$. The region limited by the loop of the separatrix L_0 and the curve C_N enclosed inside this loop is also contained in $U_{\varepsilon/3}$ (Figure 138).

Since the separatrix loop is stable, it is clear that if point N_1 is sufficiently close to M_0, both conditions are satisfied. It is also clear that $n_2 < n_1$.

Let us now take $\delta_1 > 0$ so small that the following conditions are satisfied:

(a) $\delta_1 < \delta_0$, where δ_0 is the number defined by Lemma 7;

(b) if $(\tilde{D})$ is δ_1-close to (D), the path $\tilde{L}_N$ of $(\tilde{D})$ which passes through the point N_1 crosses again the arc l_0 at the point $\tilde{N}_2(\tilde{n}_2)$, so that $\tilde{n}_2 < n_1$, and the simple closed curve $\tilde{C}_N$, analogous to the curve C_N, is entirely enclosed inside the loop of the separatrix L_0, delimiting together with the loop a region which is entirely contained in $U_{\varepsilon/3}(L)$.

It is clear that if δ_1 is sufficiently small, both conditions are satisfied. In particular, condition (b) is satisfied since for sufficiently small δ_1 the curve $\tilde{C}_N$ may be obtained from the curve C_N by an arbitrarily small translation. Let us further choose $\delta_2 > 0$ which satisfied the following condition: if $(\tilde{D})$ is δ_2-close to (D), the simple closed curve $\tilde{C}$ which consists of the segment $\tilde{O}\tilde{M}_0$ of the separatrix $\tilde{L}_0^+$, the segment $\tilde{O}\tilde{M}_0'$ of the separatrix $\tilde{L}_0'^-$, and the segment $\tilde{M}_0\tilde{M}_0'$ of the arc l_0, is entirely contained in $U_{\varepsilon/3}(L_0)$ and encloses the curve $\tilde{C}_N$; the region delimited by the curves $\tilde{C}$ and $\tilde{C}_N$ is also contained in $U_{\varepsilon/3}(L_0)$.

The existence of such δ_2 follows from Lemmas 9 and 10 of § 28.2, and also from the fact that, for sufficiently small δ_2, the curve $\tilde{C}$ may be obtained from the loop of the separatrix L_0 by an arbitrarily small translation.

We will now show that the number $\delta = \min\{\delta_1, \delta_2\}$ satisfies the proposition of the lemma. Let $(\tilde{D})$ be δ-close to (D) and let $\tilde{u}_0 < \tilde{u}_0'$. Let, further, $K_1(k_1)$ be a point on the segment $\tilde{M}_0B$ of the arc l_0, sufficiently close to the point $\tilde{M}_0$. Then the path $\tilde{L}_k$ through this point crosses again the arc l_0 at a point $K_2(\tilde{k}_2)$, such that the following conditions are satisfied:

(a) $\tilde{k}_2 > k_1$;

(b) the closed curve $\tilde{C}_k$ consisting of the segment K_1K_2 of the path $\tilde{L}_k$ and the segment K_2K_1 of the path l_0 encloses the curve $\tilde{C}_N$ and together with $\tilde{C}_N$ delimits a region which is entirely contained in $U_\varepsilon(L_0)$.

Let $\tilde{f}(u)$ be the succession function on the arc without contact l_0 corresponding to $(\tilde{D})$. By Lemma 7, this function is defined on the segment K_1N_1. Also

$$\tilde{f}(k_1) = \tilde{k}_2 > k_1$$

and

$$\tilde{f}(n_1) = \tilde{n}_2 < n_1.$$

Hence it follows that there exists a number u^*, $k_1 < u^* < n_1$, such that

$$\tilde{f}(u^*) = u^*. \tag{22}$$

The point M^* corresponding to this value of the parameter lies between K_1 and N_1, and the path L^* of system $(\tilde{D})$ passing through this point is closed (in virtue of (22)). Moreover, L^* lies inside the region delimited by the simple closed curves $\tilde{C}_N$ and $\tilde{C}_K$, and it is therefore contained in $U_\varepsilon(L_0)$. The theorem is thus proved for the case of a loop which is stable from inside. An analogous proof can be given for the other cases.

We have seen before that if the separatrix L_0 of the saddle point O of the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{D}$$

forms a loop, the numbers $\tilde{u}_0$ and $\tilde{u}_0'$ exist for the system

$$\frac{dx}{dt} = P - \mu Q, \quad \frac{dy}{dt} = Q + \mu P, \tag{D*}$$

where μ is sufficiently small in absolute magnitude, and

$$\tilde{u}_0 < \tilde{u}_0', \text{ if } \mu > 0,$$
$$\tilde{u}_0 > \tilde{u}_0', \text{ if } \mu < 0$$

(see (20) and (21)). This together with Theorem 44 lead to the following theorem on the creation of a closed path from a separatrix loop:

Theorem 46. If the separatrix L_0 of system (D) forms a stable or an unstable loop, then for any $\varepsilon > 0$ and $\delta > 0$ there exists a modified system $(\tilde{D})$ δ-close to (D) which has at least one closed path L_0^* in the ε-neighborhood of the loop L_0.

Remark. It is readily seen that if $\delta > 0$ is sufficiently small, the closed path L_0^*, whose existence is postulated by the theorem, crosses the arc l_0 at a point M^* which lies in $U_\varepsilon(M_0)$. The point M^* moreover lies on the segment $\tilde{M}_0B$ of the arc l_0 if $\tilde{u}_0' < \tilde{u}_0$, and on the segment $\tilde{M}_0'B$ if $\tilde{u}_0 < \tilde{u}_0'$ (the point M^* may not lie between the points $\tilde{M}_0$ and $\tilde{M}_0'$, if these are two different points, in virtue of Lemma 8).

We will say that the closed path L_0^* of system $(\tilde{D})$ introduced in Theorem 46 is created from the loop formed by the separatrix L_0 of system (D) (Figure 122).

Theorems 44 and 46 show that if

$$\sigma_0(x_0, y_0) = P_x'(x_0, y_0) + Q_y'(x_0, y_0) \neq 0 \tag{23}$$

for the saddle point $O(x_0, y_0)$ of system (D), there always exist modified systems, arbitrarily close to (D), in which the loop formed by the separatrix L_0 creates at least one closed path.

3. The uniqueness of the closed path created from a separatrix loop

We will now show that if the separatrix L_0 of the saddle point $O(x_0, y_0)$ forms a loop and condition (23) is satisfied, the separatrix loop may create at most one closed path.

Let us first prove one lemma.

Lemma 9. For any $C > 0$ *there exist* $\varepsilon > 0$ *and* $\delta > 0$ *such that if system* $(\tilde{D})$ *is* δ*-close to system* (D) *and has a closed path* L_0^* *contained in the* ε*-neighborhood of the loop* L_0*, the period* τ *of the path* L_0^* *is greater than* C.

Proof. Alongside with the arc without contact l_0, we consider, as before, an arc l_1 (Figure 139). The separatrix L_0 crosses the arcs l_0 and l_1 at points M_0 and M_1 corresponding to the time t_0 and $t_1 > t_0$.

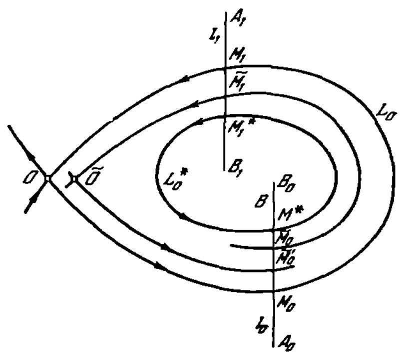

FIGURE 139

By the remark to Theorem 46, the closed path L_0^* of system $(\tilde{D})$ introduced in the lemma crosses the arc l_0 at a point M^*. Suppose that this intersection point corresponds to $t = t_0$. Clearly, if $\varepsilon > 0$ and $\delta > 0$ are sufficiently small, L_0^* also intersects the arc l_1 for some $t_1^* > t_0$ at a point M_1^* lying on the segment $\tilde{M}_1B_1$ of the arc l_1, and t_1^* is arbitrarily close to t_1 (Figure 139). Furthermore, for some $T^* > t_1^*$, the path L_0^* again crosses the arc l_0 at the point M^*.

Evidently,

$$T^* = t_0 + \tau,$$

where τ is the period of the closed path L_0^*.

If $\varepsilon > 0$ and $\delta > 0$ are sufficiently small, the point M^* is arbitrarily close to the point M_0 (and therefore to $\tilde{M}_0$ and $\tilde{M}_0'$), and the point M_1^* is arbitrarily close to M_1, i.e., by Lemma 13, §28.2, T^* is as large as desired. This means, in its turn, that for sufficiently small ε and δ, we have

$$\tau = T^* - t_0 > C.$$

Q.E.D.

Theorem 47. If $\sigma_0(x_0, y_0) \neq 0$, *there exist* $\varepsilon_0 > 0$ *and* $\delta_0 > 0$ *such that any system* $(\tilde{D})$ δ_0*-close to* (D) *may have at most one closed path in the* ε_0*-neighborhood of the loop* L_0*. If such a path exists, it is a limit cycle of the same stability as the loop of the original system, i.e., it is stable for* $\sigma_0(x_0, y_0) < 0$ *and unstable for* $\sigma_0(x_0, y_0) > 0$.

Proof. To fix ideas, let

$$\sigma_0 = \sigma_0(x_0, y_0) > 0.$$

By Theorem 44, the loop of system (D) is unstable in this case.

Let $\varepsilon_1 > 0$ be so small that at every point $M(x, y)$ of $U_{\varepsilon_1}(O)$ we have

$$\sigma(x, y) = P_x'(x, y) + Q_y'(x, y) > \frac{\sigma_0}{2}. \tag{24}$$

Let further $x = \varphi_0(t)$, $y = \psi_0(t)$ be the solution corresponding to the separatrix L_0. The points M_0 and M_1 corresponding to the time t_0 and t_1 and the arcs without contact l_0 and l_1 are chosen so that these arcs and the semipaths OM_0 and M_1O of the separatrix L_0 are entirely contained

in $U_{\varepsilon_1}(O)$. Let, as before, M_0B_0 and M_1B_1 be the segments of the arcs l_0 and l_1 which are entirely enclosed inside the loop L_0, except for their end points M_0 and M_1 (Figure 139).

Consider the integral

$$J = \int_{t_0}^{t_1} \sigma(\varphi_0(t), \psi_0(t))\, dt. \tag{25}$$

Let $\chi > 0$ be such that

$$|J| < \chi. \tag{26}$$

Together with the system (D), we will now consider modified systems ($\tilde{D}$).

Let $\delta_1 > 0$ be a number satisfying the following condition: if ($\tilde{D}$) is δ_1-close to (D), then at every point $M(x, y)$ of $U_{\varepsilon_1}(O)$ we have

$$\tilde{\sigma}(x, y) = \tilde{P}'_x(x, y) + \tilde{Q}'_y(x, y) > \frac{\sigma_0}{4}.$$

For any $C > 0$, there exist $\delta_2 > 0$ and $\varepsilon_2 > 0$ such that if ($\tilde{D}$) is δ_2-close to (D) and has a closed path L_0^* contained in $U_{\varepsilon_2}(L_0)$ which corresponds to a solution

$$x = \varphi^*(t), \quad y = \psi^*(t) \tag{L_0^*}$$

with period τ, and if this path crosses the arc l_0 at point M^* for $t = t_0$ and the arc l_1 at point M_1^* for $t = t_1^*$, $0 < t_1^* - t_0 < \tau$, then the following conditions are satisfied:

(a) $\tau > C$;

(b) $\left| \int_{t_0}^{t_1^*} \tilde{\sigma}(\varphi^*(t), \psi^*(t))\, dt \right| < 2\chi$;

(c) the points of the path L_0^* corresponding to $t_1^* \leqslant t \leqslant t_0 + \tau$ are entirely contained in $U_{\varepsilon_1}(O)$.

For sufficiently small δ_2 and ε_2, condition (a) is satisfied in virtue of Lemma 9, condition (b) in virtue of the theorem on the continuous dependence of solutions on the right-hand sides and in virtue of the continuity of $\sigma(x, y)$. Condition (c) is satisfied because the dynamic system (D) is structurally stable in a regular saddle-point region (this structural stability follows, e.g., from Lemma 4, §9.2).

Now let $\delta_0 = \min\{\delta_1, \delta_2\}$, $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. We will show that δ_0 and ε_0 defined in this way meet the conditions of the theorem. To this end, let L_0^* be a closed path of ($\tilde{D}$) which is δ_0-close to (D). We assume that this path lies in $U_{\varepsilon_0}(L_0)$ and proceed to evaluate the integral

$$J^* = \int_{t_0}^{t_0+\tau} \tilde{\sigma}(\varphi^*(t), \psi^*(t))\, dt. \tag{27}$$

We have

$$J^* = \int_{t_0}^{t_1^*} \tilde{\sigma}(\varphi^*(t), \psi^*(t))\, dt + \int_{t_1^*}^{t_0+\tau} \tilde{\sigma}(\varphi^*(t), \psi^*(t))\, dt.$$

By condition (b), the first integral on the right is greater than -2χ. The second integral is greater than $\frac{\sigma_0}{4}(t_0+\tau-t_1^*)$ by inequality (26). Therefore

$$J^* > \frac{\sigma_0}{4}(t_0+\tau-t_1^*)-2\chi$$

or, by condition (a)

$$J^* > \frac{\sigma_0}{4}(t_0+C-t_1^*)-2\chi.$$

Since for small δ_0, t_1^* is close to the constant number t_1, we conclude that for sufficiently large C (i.e., for sufficiently small δ_0 and ε_0), $J^*>0$. But then by Theorem 17 (§ 13.3), the closed path L_0^* of system $(\tilde{D})$ is an unstable limit cycle.

We have thus proved that if $\varepsilon_0>0$ and $\delta_0>0$ are sufficiently small, all the closed paths of a system $(\tilde{D})$ δ_0-close to (D) which lie in $U_{\varepsilon_0}(L_0)$ are unstable limit cycles. Suppose that more than one such path exists. We can then evidently find two closed paths L_1^* and L_2^* contained in $U_{\varepsilon_0}(L_0)$, such that one is enclosed by the other and the region delimited by the two paths contains no other closed paths. This is an obvious contradiction, however, since both paths L_1^* and L_2^* are unstable limit cycles (we naturally assume that $U_{\varepsilon_0}(L_0)$ contains no other equilibrium states of $(\tilde{D})$, except the saddle point $\tilde{O}$). This proves the theorem.

A similar proof can be given for $\sigma_0(x_0, y_0)<0$.

Remark. Theorem 47 can be generalized. Indeed, let system (D) have a closed contour γ consisting of the separatrices of the saddle points $O_i(x_i, y_i)$, $i=1, 2, \ldots, n$, $n\geqslant 2$, and the saddle points themselves. Reasoning along the same lines as before, we can show that if $\sigma(x_i, y_i)<0$ $(i = 1, 2, \ldots, n)$, the contour γ is stable, and if $\sigma(x_i, y_i)>0$, it is unstable, and that γ may create a single limit cycle, which is stable in the former case and unstable in the latter.

The following theorem follows almost immediately from Theorem 47.

Theorem 48. Let the separatrix L_0 of saddle point $O(x_0, y_0)$ of system (D) *form a loop and let condition* (23) *be satisfied, i.e.,* $\sigma_0(x_0, y_0)\neq 0$.

Then there exist $\varepsilon>0$ and $\delta>0$ such that if system $(\tilde{D})$ is δ-close to (D) and in the ε-neighborhood of the loop L_0 it has a separatrix $\tilde{L}_0$ of a saddle point $\tilde{O}$ which forms a loop, $(\tilde{D})$ can have no closed paths in the ε-neighborhood of the loop L_0.

Proof. For ε and δ we may take

$$\varepsilon=\frac{\varepsilon_0}{2} \text{ and } \delta=\frac{\delta_0}{2}, \tag{28}$$

where ε_0 and δ_0 are numbers defined by Theorem 46. Indeed, let $(\tilde{D})$ be δ-close to (D) and suppose that in the ε-neighborhood of the loop L_0 $(\tilde{D})$ has a separatrix $\tilde{L}_0$ which forms a loop and a closed path $\tilde{L}_1$. We assume that $\tilde{L}_1$ is not a simple limit cycle of $(\tilde{D})$. By Theorem 19 (§ 15.2), there exists a system $(\hat{D})$ δ-close to $(\tilde{D})$ which has at least two closed paths $\hat{L}_1$ and $\hat{L}_2$ in the ε-neighborhood of the path $\tilde{L}_1$. $(\hat{D})$ is evidently δ_0-close to (D), and the paths $\hat{L}_1$ and $\hat{L}_2$ lie in the ε_0-neighborhood of the loop L_0. Now, this contradicts Theorem 47. Thus, $\tilde{L}_1$ may not be a multiple limit cycle of $(\tilde{D})$. Now let $\tilde{L}_1$ be a simple, i.e., structurally stable, limit cycle. By

Theorem 46, there exists a system ($\dot{D}$) arbitrarily close to ($\tilde{D}$) which has a closed path $\hat{L}_0$ in any arbitrarily small neighborhood of the loop $\tilde{L}_0$. If ($\dot{D}$) is sufficiently close to ($\tilde{D}$), then in virtue of the structural stability of the cycle $\tilde{L}_1$, ($\dot{D}$) may have a limit cycle $\hat{L}_1$, which is arbitrarily close to the cycle $\tilde{L}_1$, while the two cycles $\hat{L}_0$ and $\hat{L}_1$ do not coincide and are both contained in the ε-neighborhood of the loop L_0. But this again contradicts Theorem 47.

We will prove another proposition which, in a sense, supplements Theorem 45.

Theorem 49. If the separatrix L_0 of the saddle point $O(x_0, y_0)$ of system (D) forms a loop and $\sigma_0(x_0, y_0) > 0$ (<0), there exist $\varepsilon > 0$ and $\delta > 0$ with the following property: any system ($\tilde{D}$) which is δ-close to (D) and for which $\tilde{u}_0 < \tilde{u}_0'$ (correspondingly, $\tilde{u}_0 > \tilde{u}_0'$) has no closed paths in $U_\varepsilon(L_0)$.

Proof. To fix ideas, let us consider the case $\sigma_0(x_0, y_0) < 0$. In this case, the loop formed by the separatrix L_0 is stable (§29.1, Theorem 44). Let $\varepsilon_0 > 0$ and $\delta_0 > 0$ be the numbers defined by Theorem 47.

I. Consider a point $M_1(u_1)$ of the arc l_0, where $u_1 > u_0$. If the point M_1 is sufficiently close to M_0, the path L_1 passing through this point for $t = t_0$ will cross the arc l_0 again at point N_1 as t increases. Let C_1 be a simple closed curve formed by the segment M_1N_1 of the path L_1 and the segment N_1M_1 of the arc l_0. We choose u_1 in such a way that the following conditions are satisfied:

(a) the curve C_1, and the ring region enclosed between the loop of the separatrix L_0 and the curve C_1 are contained in $U_{\varepsilon_0}(L_0)$;

(b) $d(u_1) < 0$.

Both conditions are clearly satisfied if u_1 is sufficiently close to u_0 (the second condition holds true in virtue of our assumption that the loop is stable).

II. We choose $\delta_1 > 0$ so small that if system ($\tilde{D}$) is δ_1-close to (D), the following conditions are satisfied:

(c) $\tilde{d}(u_1) < 0$;

(d) the curve $\tilde{C}_1$ and the region between this curve and the loop of separatrix L_0 are contained in $U_{\varepsilon_0}(L_0)$ (the function $\tilde{d}$ is analogous to d, and $\tilde{C}_1$ is the curve passing through M_1 which is analogous to C_1; Figure 140).

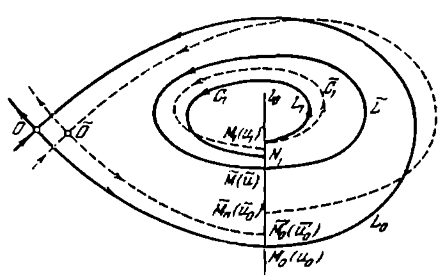

FIGURE 140

III. We choose $\varepsilon_2 > 0$ and $\delta_2 > 0$ so that they satisfy the following conditions:

(e) $\varepsilon_2 < \frac{\varepsilon_0}{2}$;

(f) if $(\tilde{D})$ is δ_2-close to (D), the curve $\tilde{C}_1$ and the region enclosed by this curve do not intersect with $U_{\varepsilon_2}(L_0)$.

IV. We choose $\varepsilon_3 > 0$ and $\delta_3 > 0$ so that the following conditions are satisfied:

(g) $\varepsilon_3 < \frac{\varepsilon_0}{2}$;

(h) if $(\tilde{D})$ is δ_3-close to (D), and $\tilde{L}$ is a closed path of $(\tilde{D})$ contained in $U_{\varepsilon_3}(L_0)$, the path $\tilde{L}$ crosses the arc without contact l_0 at point $\tilde{M}(\tilde{u})$, where $\tilde{u} > \tilde{u}_0$ ($\tilde{u}_0$ is the value of the parameter $\tilde{u}$ corresponding to the point $\tilde{M}_0$).

The existence of the numbers ε_3 and δ_3 follows from the fact that if $(\tilde{D})$ is sufficiently close to (D) and moreover $\tilde{u} < \tilde{u}_0$, the closed path $\tilde{L}$ should enclose the saddle point $\tilde{O}$ and, with it, all of its separatrices. But then it cannot be contained in a sufficiently small neighborhood of the loop.

We will prove that the numbers

$$\varepsilon = \min\{\varepsilon_2, \varepsilon_3\}, \quad \delta = \min\left\{\frac{\delta_0}{2}, \delta_1, \delta_2, \delta_3\right\} \tag{29}$$

satisfy the proposition of the theorem.

Suppose that this is not so. Then there exists a system $(\tilde{D})$ δ-close to (D) for which

$$\tilde{u}_0 > \tilde{u}_0' \tag{30}$$

and which has a closed path $\tilde{L}$ in $U_\varepsilon(L_0)$. Let this path cross the arc without contact l_0 at point $\tilde{M}(\tilde{u})$.

From (29) and conditions (f) and (h) it follows, as is readily seen, that

$$\tilde{u}_0 < \tilde{u} < u_1. \tag{31}$$

Consider the function $\tilde{d}(u)$ of system $(\tilde{D})$. Since $\tilde{L}$ is a closed path, we have

$$d(\tilde{u}) = 0. \tag{32}$$

Moreover, by condition (c)

$$\tilde{d}(u_1) < 0. \tag{33}$$

Finally, if u_2 is sufficiently close to $\tilde{u}_0$ and $\tilde{u}_0 < u_2 < \tilde{u}$, we have

$$\tilde{d}(u_2) < 0 \tag{34}$$

in virtue of inequality (30).

It follows from (32), (33), and (34) that either the function $\tilde{d}(u)$ has at least one more root $\tilde{u}^*$, $u_2 < \tilde{u}^* < u_1$, besides the root $\tilde{u}$, or $\tilde{d}'(\tilde{u}) = 0$. If the first alternative is true, a closed path $\tilde{L}^*$ of $(\tilde{D})$ will pass through the point $\tilde{M}^*(\tilde{u}^*)$; in virtue of condition (d), this path is contained in $U_{\varepsilon_0}(L_0)$, i.e., this neighborhood contains at least two closed paths. This clearly contradicts the choice of $\varepsilon_0 > 0$ and $\delta_0 > 0$.

If the second alternative is true, the closed path $\tilde{L}$ is not a simple limit cycle. But then, by Theorem 19, § 15.2, there exists a system $(\hat{D})$ arbitrarily close to $(\tilde{D})$, and in particular δ_0-close to (D), which has at least two closed paths in $U_{\varepsilon_0}(L_0)$, and this again contradicts the choice of δ_0 and ε_0. Q.E.D.

Remark. Let $O(x_0, y_0)$ be a saddle point of the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{D}$$

whose separatrix L_0 forms a loop, and let

$$\sigma_0(x_0, y_0) = P'_x(x_0, y_0) + Q'_y(x_0, y_0) \neq 0.$$

A rotation of the vector field of (D) through the angle $\tan^{-1}\mu$ produces a system of the form

$$\frac{dx}{dt} = P - \mu Q, \quad \frac{dy}{dt} = Q + \mu P. \tag{D_μ}$$

Depending on the sign of μ, we obtain either inequality (20) or (21), i.e.,

$$\tilde{u}_0 < \tilde{u}'_0 \text{ or } \tilde{u}_0 > \tilde{u}'_0.$$

It follows from these inequalities that when the vector field of (D) is rotated through a sufficiently small angle, the separatrix loop is broken. Theorems 45, 47, and 49 show that when the field is rotated in one of the two possible directions, the broken separatrix loop is replaced by a limit cycle of the same stability, which appears in the neighborhood of the loop; when the field is rotated in the opposite direction, no closed paths are observed in a sufficiently small neighborhood of the broken loop.

4. The case $P'_x(x_0, y_0) + Q'_y(x_0, y_0) = 0$

Let us now consider the case when the saddle point $O(x_0, y_0)$ of system (D) has a separatrix L_0 forming a loop, but $\sigma_0(x_0, y_0) = 0$.

We will show that in this case there exist dynamic systems arbitrarily close to (D) which have at least two closed paths in any arbitrarily small neighborhood of the loop L_0.

Lemma 10. Let (D) *be a dynamic system,* $O(x_0, y_0)$ *a saddle point of the system,* L_0 *the separatrix of this saddle point forming a loop, and let*

$$\sigma_0(x_0, y_0) = P'_x(x_0, y_0) + Q'_y(x_0, y_0) = 0. \tag{35}$$

Then for any $\varepsilon > 0$ *and* $\delta > 0$, *there exists a system* $(\tilde{D})$ *which satisfies the following conditions:*

(a) $(\tilde{D})$ *is* δ*-close to* (D).

(b) $O(x_0, y_0)$ *is a saddle point of* $(\tilde{D})$, *and*

$$\tilde{\sigma}_0(x_0, y_0) = \tilde{P}'_x(x_0, y_0) + \tilde{Q}'_y(x_0, y_0) > 0 \qquad (or < 0).$$

(c) The saddle point O *of* $(\tilde{D})$ *has a separatrix* $\tilde{L}_0$ *which forms a loop and is entirely contained in the* ε*-neighborhood of the loop* L_0 *of system* (D).

Proof. Without loss of generality, we take the saddle point O at the origin and assume that the system is given in canonical form. Then $x_0 = y_0 = 0$ and, by (35), the system has the form $u > u_0$.

$$\frac{dx}{dt} = \nu x + P_2(x, y), \quad \frac{dy}{dt} = -\nu y + Q_2(x, y), \tag{D_1}$$

where $\nu \neq 0$, and the functions P_2 and Q_2 together with their first derivatives vanish at the origin.

Let, as before, l_0 be an arc without contact through some point M_0 of the separatrix L_0, and u a parameter on this arc. The point M_0 corresponds to $u = u_0$, and the points of the arc l_0 lying inside the loop correspond to $u > u_0$.

Consider a modified system of the form

$$\frac{dx}{dt} = \nu x + P_2(x, y) = P_\alpha(x, y), \quad \frac{dy}{dt} = -(\nu - \alpha) y + Q_2(x, y) = Q_\alpha(x, y). \qquad (D_\alpha)$$

If α is sufficiently small in absolute magnitude, $O(0, 0)$ is a saddle point of this system, and

$$P'_{\alpha x}(0, 0) + Q'_{\alpha y}(0, 0) = \alpha. \qquad (36)$$

Together with (D_α), we consider another system

$$\begin{aligned} \frac{dx}{dt} &= P_\alpha - \mu Q_\alpha = \nu x + P_2(x, y) - \mu[-(\nu - \alpha) y + Q_2(x, y)] = P_{\alpha\mu}(x, y), \\ \frac{dy}{dt} &= Q_\alpha + \mu P_\alpha = -(\nu - \alpha) y + Q_2(x, y) + \mu[\nu x + P_2(x, y)] = Q_{\alpha\mu}(x, y), \end{aligned} \qquad (D_{\alpha\mu})$$

whose vector field is obtained by rotating the vector field of (D_α) through the angle $\tan^{-1}\mu$. For any sufficiently small (in absolute magnitude) μ, O is a saddle point of $(D_{\alpha\mu})$ and

$$P'_{\alpha\mu x}(0, 0) + Q'_{\alpha\mu y}(0, 0) = \alpha.$$

Let $\alpha > 0$.

Let, as before, L_0^+ and L_0^- be the semipaths comprising the separatrix L_0 which contain the point M_0. For any $\varepsilon > 0$ and $\delta > 0$, there exist $\alpha_0 > 0$ and $\mu_0 > 0$ such that if $|\alpha| < \alpha_0$ and $|\mu| < \mu_0$, $(D_{\alpha\mu})$ is δ-close to (D) and has two separatrices $L^+_{0\alpha\mu}$ and $L'^-_{0\alpha\mu}$ of the saddle point O in the ε-neighborhoods of the separatrices L_0^+ and L_0^-, respectively. Let these separatrices cross the arc l_0 at points $M_0(\alpha, \mu)$ and $M'_0(\alpha, \mu)$, which correspond to the values $u_0(\alpha, \mu)$ and $u'_0(\alpha, \mu)$ of the parameter u.

First let $\mu = 0$, i.e., consider the system (D_α). Two cases are possible a priori:

1) There exist arbitrarily small positive numbers α^* such that

$$u_0(\alpha^*, 0) = u'_0(\alpha^*, 0), \qquad (37)$$

i.e., the separatrices $L^+_{0\alpha^*0}$ and $L'^-_{0\alpha^*0}$ merge into a single separatrix of (D_{α^*}) which forms a loop. Clearly, for any $\varepsilon > 0$ and $\delta > 0$ and for a sufficiently small α^*, (D_{α^*}) is δ-close to (D), and this loop is contained in $U_\varepsilon(L_0)$, which proves the lemma.

2) There exists $\beta > 0$, such that for all α, $0 < \alpha < \beta$, either

$$u_0(\alpha, 0) > u'_0(\alpha, 0), \qquad (38)$$

or

$$u_0(\alpha, 0) < u'_0(\alpha, 0) \qquad (39)$$

(if there exist arbitrarily small α for which inequality (38) is satisfied and also arbitrarily small α for which (39) is satisfied, there also exist

arbitrarily small α^* for which (37) is satisfied, and we return to case 1).

To fix ideas, suppose that inequality (38) is satisfied for all α, $0<\alpha<\beta$.

Now consider the system $(D_{\alpha\mu})$ with $\alpha = 0$ (it is obtained from (D) by rotating its vector field through the angle $\tan^{-1}\mu$). By the lemma of §11.1, if $\mu>0$ and is sufficiently small, we have

$$u_0(0, \mu) < u_0'(0, \mu) \tag{40}$$

(see (20)).

Let $\delta>0$ and $\varepsilon>0$ be fixed and let α_0 and μ_0 be the positive numbers introduced above, corresponding to these δ and ε.

Let α_1 be a fixed positive number, $\alpha_1<\alpha_0$, $\alpha_1<\beta$. Then

$$u_0(\alpha_1, 0) > u_0'(\alpha_1, 0). \tag{41}$$

From the last inequality and remark to Lemma 10, §28.2, it follows that if μ^* is sufficiently small in absolute magnitude, we have

$$u_0(\alpha_1, \mu^*) > u_0'(\alpha_1, \mu^*). \tag{42}$$

Let $0<\mu^*<\mu_0$. By inequality (40), for a small positive μ^*,

$$u_0(0, \mu^*) < u_0'(0, \mu^*). \tag{43}$$

From inequalities (42) and (43) and continuity of the functions u_0 and u_0' it follows that for some α^*, $0<\alpha^*<\alpha_1$,

$$u_0(\alpha^*, \mu^*) = u_0'(\alpha^*, \mu^*).$$

This means that the system

$$\begin{aligned}\frac{dx}{dt} &= \nu x + P_2(x, y) - \mu^* [-(\nu-\alpha^*) y + Q_2(x, y)],\\ \frac{dy}{dt} &= -(\nu-\alpha^*) y + Q_2(x, y) + \mu^* [\nu x + P_2(x, y)]\end{aligned} \tag{$D_{\alpha^*\mu^*}$}$$

has a separatrix $\tilde{L}_0$ which forms a loop. Since then $0<\alpha^*<\alpha_0$, $0<\mu^*<\mu_0$, system $(D_{\alpha^*\mu^*})$ is δ-close to (D), and the loop $\tilde{L}_0$ lies in $U_{\varepsilon_0}(L_0)$. Moreover,

$$P'_{\alpha^*\mu^*x}(0, 0) + Q'_{\alpha^*\mu^*y}(0, 0) = \alpha^*.$$

This completes the proof of the lemma.

Theorem 50. *Let $O(x_0, y_0)$ be a saddle point of dynamic system* (D), *and L_0 its separatrix forming a loop. If $\sigma_0(x_0, y_0) = 0$, then for any $\varepsilon>0$ and $\delta>0$ there exists a modified system $(\tilde{D})$ which is δ-close to (D) and which has at least two closed paths in the ε-neighborhood of the loop L_0.*

Proof. For simplicity, let $x_0 = y_0 = 0$. If any neighborhood of the loop L_0 contains closed paths, (D) itself may be chosen as $(\tilde{D})$. It thus suffices to consider the case when some neighborhood of the loop L_0 contains no closed paths, i.e., the loop L_0 is either stable or unstable. Let l_0 be an arc without contact passing through the point M_0 of the loop L_0, u a parameter defined on l_0, u_0 the value of this parameter corresponding to M_0. We assume, as before, that the points of the arc l_0 which lie inside the loop correspond to u, $u_0 < u \leqslant b_0$, and the succession function

$\tilde{u} = f(u)$ of (D) on the arc l_0 is defined for all $u, u_0 < u \leqslant b$, where $b < b_0$. Since by assumption the loop L_0 of (D) is unstable, we see that for u greater than u_0 and sufficiently close to u_0 the following inequality is satisfied:

$$d(u) = f(u) - u > 0.$$

Let $\varepsilon > 0$, $\delta > 0$ be given.

I. Choose $u_1 > u_0$ sufficiently close to u_0 so that the following conditions are satisfied:

(a) $d(u_1) > 0$.

(b) The path L_1 which for $t = t_0$ passes through the point $M_1(u_1)$ of the arc without contact l_0 will again cross the arc l_0 at point N_1 as t increases, so that the curve C_1 consisting of the segment M_1N_1 of the path L_1 and the segment N_1M_1 of the arc l_0, together with the region enclosed between the curve C_1 and the loop of the separatrix L_0, are contained in the $\varepsilon/4$-neighborhood of the loop L_0 (Figure 141).

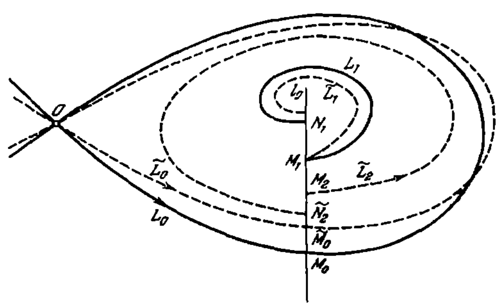

FIGURE 141.

II. Let η be the distance between the curve C_1 and the separatrix loop L_0. We choose $\delta_1 > 0$ so small that the following conditions are satisfied:

(a) $\delta_1 < \frac{\delta}{2}$.

(b) If $(\tilde{D})$ is δ_1-close to (D), then

$$\tilde{d}(u_1) > 0 \tag{44}$$

and curve $\tilde{C}_1$ — the analog of C_1 — is contained in $U_{\eta/4}(C)$ and also in $U_{\varepsilon/4}(C)$.

(c) If $(\tilde{D})$ is δ_1-close to (D) and $\tilde{M}_0(\tilde{u}_0)$ is the intersection point of the corresponding separatrix $\tilde{L}_0$ with the arc without contact l_0, we have $\tilde{u}_0 < u_1$ and the succession function $\tilde{f}(u)$ of $(\tilde{D})$ on the arc l_0 is defined for all u, $\tilde{u}_0 < u < u_1$.

III. We choose μ^* and α^* so that the following conditions are satisfied:

(a) The system $(D_{\alpha^*\mu^*})$ considered in Lemma 10 is δ-close to (D).

(b) The separatrix $\tilde{L}_0$ of the saddle point O of $(D_{\alpha^*\mu^*})$ forms a *stable* loop, which is entirely contained both in $U_{\varepsilon/4}(L_0)$ and in $U_{\eta/4}(L_0)$.

The existence of the numbers μ^* and α^* satisfying conditions (a) and (b) has been established in the proof to Lemma 10. In particular, for the loop $\tilde{L}_0$ to be stable, we should have $\alpha^* < 0$.

On the arc l_0, choose a point $M_2(u_2)$ sufficiently close to $\widetilde{M}_0(u_2>\widetilde{u}_0)$.

Let $\widetilde{d}(u)=\widetilde{f}(u)-u$, where $\widetilde{f}(u)$ is the succession function of $(D_{\alpha^*\mu^*})$. Since the loop $\widetilde{L}_0$ of $(D_{\alpha^*\mu^*})$ is stable by assumption, we have

$$\widetilde{d}(u_2)<0. \tag{45}$$

On the other hand, inequality (44) is satisfied. From these relations it follows that, for some u^*, $u_2<u^*<u_1$,

$$d(u^*)=0,$$

i.e., a closed path $\widetilde{L}^*$ of $(D_{\alpha^*\mu^*})$ passes through the point $M^*(u^*)$.

Let $(D_{\alpha^*\mu^*})$ be designated $(\widetilde{D}^*)$. From II(a) and III(a) it follows that $(\widetilde{D}^*)$ is $\delta/2$-close to (D). Furthermore, as we have seen, (D) has a separatrix $\widetilde{L}_0$ which forms a stable loop and a closed path $\widetilde{L}^*$. Both the loop $\widetilde{L}_0$ and the closed path $\widetilde{L}^*$ are contained in $U_{\varepsilon/2}(L_0)$ — the loop in virtue of III(b) and the path $\widetilde{L}^*$ in virtue of I(b), II(b), and III(b).

If the closed path $\widetilde{L}^*$ of $(\widetilde{D}^*)$ is not a simple limit cycle of this system, Theorem 19, §15.2, indicates that there exists a system $(\widetilde{D})$ $\delta/2$-close to $(\widetilde{D}^*)$ which has in $U_{\varepsilon/2}(\widetilde{L}^*)$ at least two closed paths $\widetilde{L}_1$ and $\widetilde{L}_2$. But then $(\widetilde{D})$ satisfies the proposition of the theorem. If $\widetilde{L}^*$ is a simple, i.e., structurally stable, limit cycle of $(\widetilde{D}^*)$, then Theorem 46 shows that there exists system $(\widetilde{D})$ arbitrarily close to $(\widetilde{D}^*)$ which has a closed path $\widetilde{L}_1$ in any arbitrarily small neighborhood of the loop $\widetilde{L}_0$. When $(\widetilde{D})$ is sufficiently close to $(\widetilde{D}^*)$, system $(\widetilde{D})$, in virtue of the structural stability of the cycle $\widetilde{L}^*$, has a limit cycle $\widetilde{L}_2$ which lies arbitrarily close to $\widetilde{L}^*$, and the paths $\widetilde{L}_1$ and $\widetilde{L}_2$ are different and are both contained in $U_\varepsilon(L_0)$. Thus, system (D) satisfies the proposition of the theorem. This completes the proof.

Remark. Theorem 50 proves that if $\sigma_0(x_0, y_0)=0$, the separatrix loop of the saddle point $O(x_0, y_0)$ may create at least two closed paths. The question of the largest number of closed paths which may be created from a separatrix loop in the case $\sigma_0(x_0, y_0)=0$ and the conditions determining this number requires a much more complex analysis. The problem was considered by E.A. Leontovich in his thesis and the results are presented in /21/.

Chapter XII

CREATION OF A LIMIT CYCLE FROM THE LOOP OF A SADDLE-NODE SEPARATRIX. SYSTEMS OF FIRST DEGREE OF STRUCTURAL INSTABILITY AND THEIR BIFURCATIONS

INTRODUCTION

The first of the two sections in this chapter, §30, deals with the creation of a limit cycle from the loop of a saddle-node separatrix. Let (A) be a dynamic system, M_0 (x_0, y_0) an equilibrium state of this system, which is a saddle-node of multiplicity 2. A canonical neighborhood of a saddle-node comprises a parabolic sector and two hyperbolic sectors, separated from one another by three separatrices. To fix ideas, suppose that the saddle-node M_0 has one α-separatrix L_0^- and two ω-separatrices L_1^+ and L_2^+.

Suppose that the α-separatrix L_0^- goes to M_0 for $t \to +\infty$, i.e., it forms a loop, whereas none of the separatrices L_1^+ and L_2^+ is a continuation of L_0^- (Figure 142).

Since M_0 is a double equilibrium state of (A), there exist arbitrarily close systems which have no equilibrium states in the neighborhood of M_0. The main result of § 30 states that if the equilibrium state M_0, and consequently the separatrix loop, disappear following a sufficiently small change in system (A), one and only one limit cycle is created in the neighborhood of the loop (Theorems 51 and 52, see Figure 143).

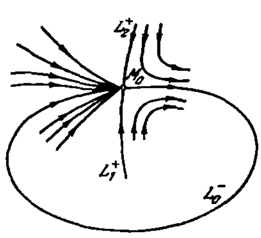

FIGURE 142

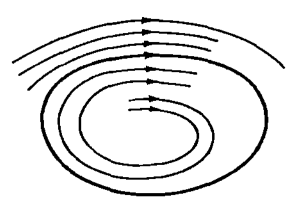

FIGURE 143

The second section, § 31, deals with the simplest structurally unstable systems, namely systems of the first degree of structural instability. A system of the first degree of structural instability inside a cycle without

contact was defined in Chapter VIII (§22). In §31, a definition of these systems is given for any bounded region, and the necessary and sufficient conditions are established for the system to be of the first degree of structural instability in that region (Theorem 67). These conditions are then applied to investigate the bifurcations of systems of the first degree of structural instability. All these bifurcations turn out to be particular cases of the bifurcations considered in the previous chapters and in §30.

§30. CREATION OF A LIMIT CYCLE FROM THE LOOP OF A SADDLE-NODE SEPARATRIX

1. The existence theorem

We consider an analytical dynamic system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{D}$$

which has an equilibrium state (x_0, y_0) of multiplicity 2 of the saddle-node type, with one of the characteristic numbers different from zero. Without loss of generality, we may assume that this equilibrium state is at the origin, i.e., $x_0 = y_0 = 0$. Thus,

$$\Delta(0, 0) = \begin{vmatrix} P'_x(0, 0) & P'_y(0, 0) \\ Q'_x(0, 0) & Q'_y(0, 0) \end{vmatrix} = 0, \tag{1}$$

and

$$\sigma(0, 0) = P'_x(0, 0) + Q'_y(0, 0) \neq 0. \tag{2}$$

Equilibrium states of this type were considered in Chapter VIII (§23, 1 and 2). A canonical neighborhood of such a state consists of one parabolic sector and two hyperbolic sectors. Suppose that the paths of the parabolic sector go to O for $t \to +\infty$. Then the equilibrium state O has one α-separatrix L_0^- and two ω-separatrices L_1^+ and L_2^+.

Suppose that the path L_0, from which the separatrix L_0^- is cut out, goes to the equilibrium state O for $t \to -\infty$, as well as for $t \to -\infty$, i.e., it forms a loop. We moreover assume that none of the separatrices L_1^+ and L_2^+ forms part of this loop (i.e., the separatrix L_0^- does not merge into a single path either with L_1^+ or with L_2^+, see Figure 142).

Since O is an equilibrium state of multiplicity 2, there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that if system $(\tilde{D})$ is δ_0-close to (D) to rank 2, it has at most two equilibrium states in $U_{\varepsilon_0}(O)$ (see Definition 15, §7.3 and Definition 5, §2.1). Thus, three cases are possible a priori:

1) System $(\tilde{D})$ has one equilibrium state $\tilde{O}$ in $U_{\varepsilon_0}(O)$.
2) System $(\tilde{D})$ has two equilibrium states $\tilde{O}_1$ and $\tilde{O}_2$ in $U_{\varepsilon_0}(O)$.
3) System $(\tilde{D})$ has no equilibrium states in $U_{\varepsilon_0}(O)$.

All the three cases are actually observed in practice: cases 2 and 3 in virtue of Theorem 34, §23.1; case 1 obtains if $(\tilde{D})$ is identified with (D) itself, say.

It is readily seen that if ($\tilde{\mathrm{D}}$) is sufficiently close to (D), the equilibrium state $\tilde{O}$ in case 1 is also of multiplicity 2 and is a saddle-node (this follows from the remark to Theorem 35, §23.2).

In case 2, one of the equilibrium states $\tilde{O}_1$ and $\tilde{O}_2$ into which the saddle-node O decomposes is a structurally stable node, and the other is a structurally stable saddle-point (§23.1, Lemma 1 and §23.2, Theorem 35).

We will now consider in more detail case 3, i.e., the case when the equilibrium state disappears on passing to a close system. We will show that a closed path necessarily forms in the neighborhood of the loop L_0 in this case. In fact, the following theorem can be stated:

Theorem 51. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if ($\tilde{\mathrm{D}}$) *is δ-close to* (D) *and has no equilibrium states in $U_\varepsilon(O)$,* ($\tilde{\mathrm{D}}$) *has at least one closed path contained in the ε-neighborhood of the loop L_0.*

Proof. Let $\varepsilon > 0$ be given. Consider a canonical neighborhood V of the equilibrium state O of system (D) delimited by the following path segments:

1) the segment K_1K_2 of the arc without contact l with end points R_1 and R_2 which meets the separatrices L_0^-, L_1^+, and L_2^+ at the points N_0, N_1, and N_2, respectively (Figure 144);

2) the arc without contact l_0 with end points M_1 and M_2 which meets the separatrix L_0^- at the point M_0;

3) the arcs of paths K_1M_1 and K_2M_2.

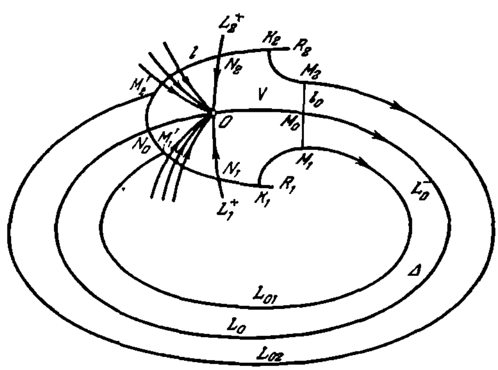

FIGURE 144

The concept of a canonical neighborhood has been introduced in QT (Chapter VIII, §19.2), where it is also shown that a canonical neighborhood can always be constructed for any arbitrarily small neighborhood of an equilibrium state. We may thus assume that the canonical neighborhood V and the arc without contact l lie in $U_{\varepsilon/2}(O)$ and that the following condition is satisfied: the paths L_{01} and L_{02} which for $t = t_0$ pass through the respective end points M_1 and M_2 of the arc l_0 cross, with increasing t, the arc without contact l at points M_1' and M_2', so that the quadrangle Δ limited by

the arc without contact l_0, the segment M_1M_1' of the arc without contact l, and the arcs M_1M_1' and M_2M_2' of the paths L_{01} and L_{02} is an elementary quadrangle contained in the $\varepsilon/2$-neighborhood of the loop L_0. Since the path L_0 crosses the arcs without contact l_0 and l at their interior points M_0 and N_0, this condition is satisfied whenever the arc l_0 is sufficiently small.

Let $\delta > 0$ be so small that if $(\tilde{D})$ is δ-close to (D), then

(a) the arcs l_0 and l are arcs without contact for $(\tilde{D})$;

(b) the paths $\tilde{L}_{01}$ and $\tilde{L}_{02}$ of $(\tilde{D})$ which for $t = t_0$ pass through the points M_1 and M_2, respectively, cross, with increasing t, the arc l at the points $\tilde{M}_1'$ and $\tilde{M}_2'$, and the resulting quadrangle $\tilde{\Delta}$ (the analog of Δ) is an elementary quadrangle of $(\tilde{D})$ contained in $U_\varepsilon(L_0)$ (Figure 145);

(c) as t decreases, the paths $\tilde{L}_{01}$ and $\tilde{L}_{02}$ cross the arc l at points $\tilde{K}_1$ and $\tilde{K}_2$, and the neighborhood $\tilde{V}$ of the point O, delimited by the arc without contact l_0, the segment $\tilde{K}_1\tilde{K}_2$ of the arc l, and the arcs of paths $\tilde{K}_1M_1$ and $\tilde{K}_2M_2$, is contained in $U_\varepsilon(L_0)$.

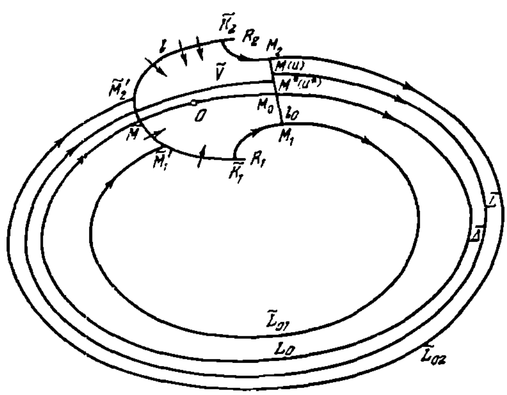

FIGURE 145

Conditions (a), (b), and (c) are satisfied for a sufficiently small δ in virtue of Lemmas 1 and 5 of § 4.1 and Lemma 7 of § 4.2.

We will show that the number δ selected in this way fulfills the proposition of the theorem.

Let $(\tilde{D})$ be a system δ-close to (D) which has no equilibrium states in $U_\varepsilon(O)$. Let $\tilde{W}$ be the union of the sets $\tilde{\Delta}$ and $\tilde{V}$ (these sets are assumed to be closed). Evidently, $\tilde{W}$ is a closed neighborhood of the loop L_0, and in virtue of (b) and (c) $\tilde{W} \subset U_\varepsilon(L_0)$. We define a parameter u on the arc l_0, so that the end points M_1 and M_2 of the arc correspond to the values u_1 and u_2 of the parameter, $u_1 < u_2$. Consider any point $M(u)$ of the arc l_0 $(u_1 \leqslant u \leqslant u_2)$ and the path $\tilde{L}$ of system $(\tilde{D})$ through that point. As t increases, this path crosses the arc without contact l at some point $\tilde{M}$ of the arc l and enters into the neighborhood $\tilde{V}$. Since there are no equilibrium states, and thus no limit continua of $(\tilde{D})$, in $\tilde{V}$, the path $\tilde{L}$ should leave $\tilde{V}$ as t further increases.

It may evidently leave $\tilde{V}$ only through the arc without contact l_0, i.e., it again crosses the arc l_0 at some point M^* (u^*). This means that for all u, $u_1 \leqslant u \leqslant u_2$, a succession function

$$u^* = \tilde{f}(u)$$

is defined on the arc without contact l_0, such that $f(u_1) > u_1$, $f(u_2) < u_2$.

It follows from the last inequalities that there exists a number $\hat{u}$, $u_1 < \hat{u} < u_2$, such that $f(\hat{u}) = \hat{u}$.

The path $\hat{L}$ through the point $\hat{M}(\hat{u})$ of the arc l_0 is a closed path. Clearly,

$$\hat{L} \subset \tilde{W} \subset U_\varepsilon(L_0).$$

This proves the theorem.

We shall say that the closed path $\hat{L}$ is created from the loop of the separatrix L_0 of the saddle-node O.

Remark. Theorem 51 and the above proof remain valid if the point O is any equilibrium state of multiplicity 4 (and not only 2) for which conditions 1 and 2 are satisfied (this equilibrium state is also a saddle-node, see §23.2, a).

2. The uniqueness theorem

We will prove the uniqueness of the closed path created from the loop of a saddle-node separatrix. As in the previous subsection, we consider an equilibrium state $O(0, 0)$ of multiplicity 4 which satisfies conditions 1 and 2, i.e., a saddle-node, and the path L_0 from which the α-separatrix L_0^- is cut out forms a loop, without merging with either of the ω-separatrices L_1^+ and L_2^+.

Lemma. Let M_0 be a point of the path L_0, and l_0 an arc without contact through M_0 for which M_0 is not an end point. There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if $(\tilde{D})$ is δ_0-close to (D), and $\tilde{L}$ is a closed path of $(\tilde{D})$ contained in $U_{\varepsilon_0}(L_0)$, $\tilde{L}$ crosses the arc l_0 at one and only one point.

Proof. It is readily seen that if the proposition of the lemma is true for any sufficiently small arc l_0 cut out from some fixed arc without contact which crosses the path L_0, it is also true for any fixed arc without contact l_0 (see Lemma 7, §4.2). By assumption, $\sigma(0, 0) = P'_x(0, 0) + Q'_y(0, 0) \neq 0$.

To fix ideas, let

$$\sigma(0, 0) = \sigma_0 > 0.$$

Let $\varepsilon > 0$ be so small that for any point (x, y), $(x, y) \subset U_\varepsilon(0)$, $\sigma(x, y) > 0$. Consider the canonical neighborhood V and the elementary quadrangle Δ of (D) described in the previous subsection (in our proof of Theorem 51), and also the canonical neighborhood $\tilde{V}$ and the quadrangle $\tilde{\Delta}$ of the close system $(\tilde{D})$, alongside with their union $\tilde{W}$. Let $V \subset U_\varepsilon(O)$. As l_0 we choose the arc without contact M_1M_2 entering the boundary of V (Figures 144 and 145).

Choose $\delta_0 > 0$ and $\varepsilon_0 > 0$ so that if $(\tilde{D})$ is δ_0-close to (D), the following conditions are satisfied:

(a) l_0 is an arc without contact for $(\tilde{D})$;

(b) for any point $(x, y) \in \overline{U_\varepsilon(O)}$,

$$\tilde{\sigma}(x, y) = \tilde{P}'_x(x, y) + \tilde{Q}'_y(x, y) > 0;$$

(c) $\tilde{V} \subset U_{\varepsilon}(O)$;

(d) $U_{\varepsilon_0}(L_0) \subset \tilde{W}$.

These conditions are evidently satisfied for sufficiently small ε_0 and δ_0.

Let $\tilde{L}$ be a closed path of system ($\tilde{D}$) which is δ_0-close to (D), and let $\tilde{L} \subset U_{\varepsilon_0}(L_0)$. Then, by (c), $\tilde{L} \subset \tilde{W}$.

If the path $\tilde{L}$ is entirely contained in $\tilde{V}$, it is contained in $U_{\varepsilon}(O)$. By condition (b) and Bendixson's criterion (QT, §12.3, Theorem 31, corollary), system ($\tilde{D}$) has no closed paths which are entirely contained in $U_{\varepsilon}(O)$. Therefore, the closed path $\tilde{L}$ has points which lie in the quadrangle $\tilde{\Delta}$. Then, by the properties of elementary quadrangles, the path $\tilde{L}$ crosses the arc l_0 (at one and only one point, by condition (a)). The arc without contact l_0 clearly can be chosen as small as desired in this case.

The proof of the lemma is complete.

Theorem 52. *Let O (0, 0) be a saddle-node of a dynamic system* (D) *for which $\sigma_0 = \sigma(0, 0) \neq 0$, and L_0 a separatrix of this saddle-node forming a loop.*

There exist two numbers $\varepsilon > 0$ and $\delta > 0$ such that if ($\tilde{D}$) *is δ-close to* (D), *it may have at most one closed path in $U_{\varepsilon}(L_0)$. If this closed path exists, it is a stable structurally stable limit cycle for $\sigma_0 < 0$ and an unstable structurally stable limit cycle for $\sigma_0 > 0$.*

Proof. Consider the case $\sigma_0 > 0$.

Let $\varepsilon_0 > 0$ be so small that for every point $(x, y) \in U_{\varepsilon_0}(O)$ we have

$$\sigma(x, y) = P'_x(x, y) + Q'_y(x, y) > \frac{\sigma_0}{2}. \tag{3}$$

Let l, l_0, V, Δ, W, $\tilde{V}$, etc., be the arcs without contact, the canonical neighborhoods, the elementary quadrangles, etc., considered in the previous subsection (Figures 144 and 145). Let the canonical neighborhood V be so small that

$$\bar{V} \subset U_{\varepsilon_0}(O), \tag{4}$$

and let Δ be the corresponding elementary quadrangle of (D) (Figure 146).

Choose $\delta_1 > 0$ so small that if ($\tilde{D}$) is δ_1-close to (D), the following conditions are satisfied:

(a) The arcs l and l_0 are arcs without contact for the paths of system ($\tilde{D}$).

(b) The canonical neighborhood $\tilde{V}$ corresponding to ($\tilde{D}$) lies in $U_{\varepsilon_0}(L_0)$ together with its closure.

(c) The paths of ($\tilde{D}$) passing through the points of the arc l_0 cross the arc l with the increase in t, and the segments of these paths confined between the arcs l_0 and l form an elementary quadrangle $\tilde{\Delta}$

(d) For all $(x, y) \in U_{\varepsilon_0}(O)$, $\tilde{\sigma}(x, y) > \frac{\sigma_0}{4}$.

Let $\tilde{W}$ be the union of the sets $\tilde{V}$ and $\tilde{\Delta}$. We choose ε_1, $0 < \varepsilon_1 < \varepsilon_0$, so small that for any set $\tilde{W}$ corresponding to a system ($\tilde{D}$) δ_1-close to (D), we have

$$U_{\varepsilon_1}(L_0) \subset \tilde{W}. \tag{5}$$

The existence of the appropriate δ_1 and ε_1 is self-evident.

Let $\tilde{L}$ be a closed path of system ($\tilde{D}$) which is δ_1-close to (D), and let $\tilde{L} \subset U_{\varepsilon_1}(L_0)$. Then, by (5), $\tilde{L} \subset \tilde{W} = \tilde{V} \cup \tilde{\Delta}$. We will show that the path $\tilde{L}$

cannot be entirely contained in $\tilde{V}$. Indeed, if $\tilde{L} \subset \tilde{V}$, then by (4), $\tilde{L} \subset U_{\varepsilon_0}(O)$. This is impossible, since relation (5) and Bendixson's criterion (QT, §12.3, Theorem 31, corollary) show that system ($\tilde{D}$) cannot have closed paths which are entirely contained in $U_\varepsilon(O)$. Thus, any closed path $\tilde{L}$ of ($\tilde{D}$) contained in $U_{\varepsilon_1}(L_0)$ has points which lie in the quadrangle $\tilde{\Delta}$. Then, from the properties of elementary quadrangles, the path $\tilde{L}$ crosses the arcs l and l_0, and all the points of the path which lie outside $\tilde{\Delta}$ are contained in $\tilde{V}$ and hence in $U_{\varepsilon_0}(O)$.

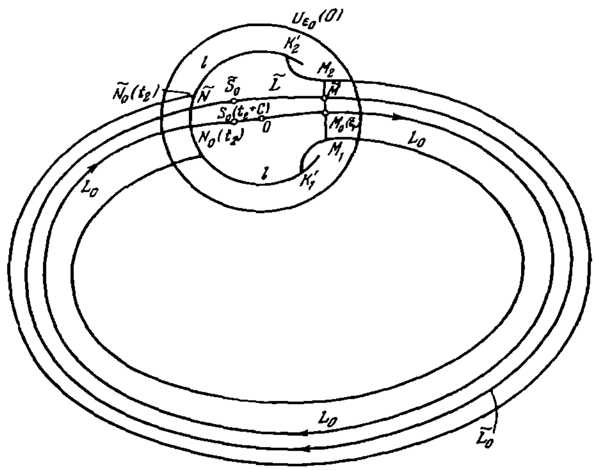

FIGURE 146

Let

$$x = \varphi_0(t), \quad y = \psi_0(t) \tag{6}$$

be a solution corresponding to the path L_0 of (D), M_0 and N_0 the intersection points of L_0 with the arcs without contact l_0 and l, respectively, t_1 and t_2, $t_1 < t_2$, the values of t corresponding to points M_0 and N_0 for motion (6) (Figure 146). Let, further,

$$J = \int_{t_1}^{t_2} [P'_x(\varphi_0(t), \psi_0(t)) + Q'_y(\varphi_0(t), \psi_0(t))]\, dt. \tag{7}$$

Let χ be any number such that

$$|I| < \chi, \tag{8}$$

and let C be any number such that

$$C > \frac{8\chi}{\sigma_0}. \tag{9}$$

Let S_0 be a point of the path L_0 corresponding to $t = t_2 + C$ for motion (6).

Let δ and ε be positive numbers such that $\delta<\delta_1$, $\varepsilon<\varepsilon_1$; let $(\tilde{D})$ be a dynamic system δ-close to (D), $\tilde{L}$ a closed path of $(\tilde{D})$ entirely contained in $U_\varepsilon(L_0)$, $\tilde{M}$ and $\tilde{N}$ the intersection points of $\tilde{L}$ with the arcs l_0 and l (Figure 146). Let

$$x=\tilde{\varphi}(t), \quad y=\tilde{\psi}(t) \tag{10}$$

be the motion along the path $\tilde{L}$ for which the point $\tilde{M}$ corresponds to $t=t_1$. Let $\tilde{N}_0$ and $\tilde{S}_0$ be the points of the closed path $\tilde{L}$ which correspond for motion (10) to $t=t_2$ and $t=t_2+C$, and let $\tilde{\tau}$ be the period of the solution of (10). If ε is sufficiently small, the point $\tilde{M}$ will be as close as desired to M_0. Hence and from the theorem of the continuous dependence of the solution on the initial values and the right-hand sides it follows that if δ and ε are sufficiently small, the following conditions are satisfied:

1) The point $\tilde{N}_0$, and all the points of the path $\tilde{L}$ between $\tilde{N}$ and $\tilde{N}_0$, are contained in $U_{\varepsilon_0}(O)$.

2) The arc $\tilde{M}\tilde{S}_0$ of the path $\tilde{L}$ corresponding to the values of t, $t_1\leqslant t\leqslant t_2+C$, is so close to the arc M_0S_0 of the path L_0 corresponding to the same values of t that

$$t_2+C<t_1+\tilde{\tau} \tag{11}$$

(we recall that $\tilde{\tau}$ is the period of the path $\tilde{L}$).

3)

$$|\tilde{I}|=\left|\int_{t_1}^{t_2}\tilde{\sigma}(\tilde{\varphi}(t),\ \tilde{\psi}(t))\,dt\right|=\int_{t_1}^{t_2}[\tilde{P}'_x(\tilde{\varphi}(t),\ \tilde{\psi}(t))+\tilde{Q}'_y(\tilde{\varphi}(t),\ \tilde{\psi}(t))]\,dt<2\chi. \tag{12}$$

The last equality is satisfied for sufficiently small δ and ε in virtue of (8).

Let δ and ε be chosen so that conditions 1 through 3 are satisfied. Let us estimate the number

$$h=\int_{t_1}^{t_1+\tilde{\tau}}\tilde{\sigma}(\tilde{\varphi}(t),\tilde{\psi}(t))\,dt=\int_{t_1}^{t_2}\tilde{\sigma}\,dt+\int_{t_2}^{t_1+\tilde{\tau}}\tilde{\sigma}\,dt.$$

By (12), $\int_{t_1}^{t_2}\tilde{\sigma}\,dt>-2\chi$. It follows from condition 1 that all the points of the path $\tilde{L}$ corresponding to $t_2\leqslant t\leqslant t_1+\tilde{\tau}$ are contained in $U_{\varepsilon_0}(O)$. Therefore, by (5) and (11),

$$\int_{t_2}^{t_1+\tilde{\tau}}\tilde{\sigma}\,dt>C\frac{\sigma_0}{4} \text{ and } h>-2\chi+C\frac{\sigma_0}{4}.$$

Hence and from inequality (9) it follows that $h>0$, i.e., the closed path $\tilde{L}$ of $(\tilde{D})$ is an unstable structurally stable limit cycle (§13.3, Theorem 17, §14, Theorem 18).

We have thus established that any closed path of system $(\tilde{D})$ δ-close to (D) which lies in $U_\varepsilon(L_0)$ is an unstable limit cycle. But then, $(\tilde{D})$ can have at most one closed path in $U_\varepsilon(L_0)$ (see the end of the proof to Theorem 47, §29.3).

The case $\sigma_0<0$ is considered along the same lines. The theorem is proved.

§31. DYNAMIC SYSTEMS OF THE FIRST DEGREE OF STRUCTURAL INSTABILITY AND THEIR BIFURCATIONS

1. The definition of a system of the first degree of structural instability

In the previous chapters and in §30 we considered bifurcations of the following types:

1) The decomposition of multiple equilibrium states into structurally stable equilibrium states.

2) The creation of limit cycles from a multiple focus.

3) The creation of limit cycles from a multiple limit cycle.

4) The creation of a limit cycle from the loop of a saddle-point separatrix.

5) The creation of a limit cycle from the loop of a saddle-node separatrix.

In this section we will consider dynamic systems of the first degree of structural instability and elucidate the conditions satisfied by these systems and the bifurcations that they may undergo. Since systems of the first degree of structural instability are, in a sense, the simplest structurally unstable systems, their bifurcations may naturally be considered as the simplest bifurcations. Any one of the simplest bifurcations creating a limit cycle proves to be a bifurcation of one of the types 2 through 4.

The definition of the degrees of structural instability of dynamic systems, and in particular of the first degree of structural instability, will be found in Chapter VIII (§22, Definition 23). It is assumed in this definition, however, that the system is considered in a region bounded by a cycle without contact. We therefore have to define a system of the first degree of structural instability in such a way that the definition will apply to any bounded closed region. The requirement of closure is not essential: a similar definition can be stated for any bounded region.

Like the concept of a structurally stable system (see §6.3), the concept of a *system of the first degree of structural instability* is associated with a certain space R^* of dynamic systems. For structurally stable systems, R^* can be identified with any of the spaces $R_a^{(r)}$ or $R_k^{(r)}$, $k \gg r \geqslant 1$. We have noted in §22 that the concept of a *dynamic system of the first degree of structural instability* is meaningful only in relation to the spaces $R_a^{(r)}$ or $R_k^{(r)}$, where $r \geqslant 3$. Therefore, in what follows it is invariably assumed, without any explicit indication, that we are dealing with structural instability in relation to one of the spaces $R_a^{(r)}$ or $R_k^{(r)}$, where $r \geqslant 3$. Let this space be R^*.

As in the case of a structurally stable system (§6.1, Definition 10 or §6.3, Definition 13), in defining a system of the first degree of structural instability in some W, we have to consider, alongside with W, some wider region. Let $\bar{G}$ be the region used to define the metric in the space R^*. Dynamic systems are henceforth understood as systems which belong to the space R^*, and closeness is interpreted as closeness in R^*. When considering some subregion of $\bar{G}$, we will always assume that its closure is contained entirely in G, i.e., its distance from the boundary of G is finite.

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system, and W a closed subregion of $\bar{G}$.

Definition 30. A dynamic system (A) *is called a system of the first degree of structural instability in* W *if it is not structurally stable in* W *and if there exists an open region* H,

$$W \subset H \subset \overline{H} \subset G,$$

satisfying the following condition: for any $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any system* $(\tilde{A})$ δ*-close to* (A), *which is structurally unstable in* W, *there exists a subregion* $\tilde{H}$ *for which*

$$(\tilde{H}, \tilde{A}) \overset{\varepsilon}{\equiv} (H, A).$$

The meaning of this somewhat clumsy definition can be elucidated as follows: (A) is a system of the first degree of structural instability in W if it is structurally unstable in W, whereas any sufficiently close system $(\tilde{A})$ is either structurally stable in W, or $(\tilde{A})$ and (A) have path partitions of the same topological structure in certain neighborhoods of W, and the transformation from one partition to another can be implemented by an arbitrarily small translation.

We will now derive the necessary conditions satisfied by any system of the first degree of structural instability. Their derivation is basically analogous to the derivation of the necessary conditions of structural stability of a system: this is a natural and relatively simple derivation, but it requires an examination of numerous alternatives.

Throughout the remaining part of this section, (A) is a system of the first degree of structural instability in W in the sense of Definition 30.

2. Equilibrium states of systems of the first degree of structural instability

We will first prove a number of theorems which establish the kind of equilibrium states that systems of the first degree of structural instability may have.

Lemma 1. Let $P(x, y)$ *and* $Q(x, y)$ *be functions of class* N, *defined in a closed bounded region* $\overline{G}$, *and* $M_0(x_0, y_0)$ *any point in that region. For any* $\delta > 0$ *and* $n \leqslant N$ (*n are natural numbers*), *there exist polynomials* $\tilde{P}(x, y)$ *and* $\tilde{Q}(x, y)$ *with the following properties:*

a) $\tilde{P}$ *and* $\tilde{Q}$ *are* δ*-close to rank* N *in* $\overline{G}$ *to the functions* P *and* Q, *respectively.*

b) $\tilde{P}$ *and* $\tilde{Q}$ *are irreducible, i.e.,* $(\tilde{P}, \tilde{Q}) = 1$.

c) The values of the polynomial $\tilde{P}(x, y)$ $(\tilde{Q}(x, y))$ *and all its derivatives to order* n *inclusive at the point* $M_0(x_0, y_0)$ *coincide with the corresponding values of the function* $P(x, y)$ $(Q(x, y))$ *and its derivatives at the same point.*

Proof. Let $x_0 = y_0 = 0$; this may be assumed without loss of generality. Let $\delta > 0$ be given. By the Weierstrass theorem (Theorem 2, § 1.1), there exist polynomials $P^*(x, y)$ and $Q^*(x, y)$ which are $\delta/2$-close to rank N to the functions P and Q, respectively, and which satisfy condition (c). If the polynomials P^* and Q^* are irreducible, they can be adopted as $\tilde{P}$ and $\tilde{Q}$, and the lemma is proved. Suppose that P^* and Q^* are reducible polynomials.

Then they can be written in the form

$$P^* = P_1(x, y) \cdot R(x, y), \qquad Q^* = Q_1(x, y) \cdot R(x, y),$$

where $(P_1, Q_1) = 1$, and R is a polynomial of higher than zero degree.

If $(R, Q_1) = 1$, we may take $\tilde{P}(x, y) = P^*(x, y)$. Now suppose that R and Q_1 are reducible. Let

$$Q_1(x, y) = \varphi_1(x, y) \cdot \varphi_2(x, y) \ldots \varphi_k(x, y) \cdot \psi_1(x, y) \cdot \psi_2(x, y) \ldots \psi_l(x, y) \tag{1}$$

be the factorization of Q_1 into irreducible factors. It is assumed that the following conditions are satisfied:

$$R(x, y) \text{ is divisible by } \psi_j,\ j = 1, 2, \ldots, l; \tag{2}$$

$$(R, \varphi_i) = 1, \quad i = 1, 2, \ldots, k \tag{3}$$

(the number k may be zero, i.e., the polynomials φ_i need not occur in the factorization (1)). Let further $\alpha x + \beta y$, $\alpha \neq 0$, $\beta \neq 0$ be a polynomial of the first degree which is irreducible with any of the polynomials ψ_j ($j = 1, 2, \ldots, l$), and r is a natural number.

Suppose that

$$R_1(x, y) = R(x, y) + (\alpha x + \beta y)^r \varphi_1 \varphi_2 \ldots \varphi_k,$$

and

$$\tilde{P}(x, y) = P_1 R_1 = P^*(x, y) + P_1(x, y)(\alpha x + \beta y)^r \varphi_1 \varphi_2 \ldots \varphi_k.$$

From (1), (2), (3) it follows that $(R_1, Q_1) = 1$. Since $(P_1, Q_1) = 1$, we have $(\tilde{P}, Q_1) = 1$. It is further evident that if the numbers α and β are sufficiently small, and r is sufficiently large, the polynomials $\tilde{P}$ and P^* are $\delta/2$-close to rank N and coincide at the point $(0, 0)$, together with their derivatives to order n, inclusive. We have thus constructed a polynomial $\tilde{P}$ which is irreducible with Q_1.

Now consider the polynomials $\tilde{P}$ and $Q^* = Q_1 R$. If $(\tilde{P}, R) = 1$, then $(\tilde{P}, Q^*) = 1$ and we may take $\tilde{Q}(x, y) = Q^*(x, y)$. If, however, $\tilde{P}$ and R are reducible, using the same construction as for $R_1(x, y)$, we can construct a polynomial $R_2(x, y)$ which will be irreducible with $\tilde{P}(x, y)$ and such that the polynomial

$$\tilde{Q}(x, y) = Q_1(x, y) R_2(x, y)$$

is $\delta/2$-close to $Q^*(x, y)$ to rank N, and $\tilde{Q}$ and Q^* coincide at the point $(0, 0)$, together with their derivatives to order n, inclusive. The polynomials $\tilde{P}$ and $\tilde{Q}$ constructed in this way fulfill conditions (a), (b), and (c) of the lemma. This completes the proof of the lemma.

Theorem 53. A system of the first degree of structural instability in W may have only a finite number of equilibrium states in this region, each of which is furthermore isolated.

Proof. Since W is a closed region, it suffices to prove that every equilibrium state of system (A) in W is isolated. Suppose this is not so, and that $M_0(x_0, y_0)$ is an unisolated equilibrium state in W. Then, by

Remark 3 to Theorem 5 (§ 2.2),

$$\Delta(x_0, y_0)=\begin{vmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0)\end{vmatrix}=0. \tag{4}$$

Since (A) is a system of the first degree of structural instability in W, there exists a region H, $W\subset H$, described in Definition 30. Let $\varepsilon>0$ be an arbitrary number. By Definition 30, there exists $\delta>0$ such that if system $(\tilde{\mathrm{A}})$ is δ-close to (A), then

either (a) $(\tilde{\mathrm{A}})$ is structurally stable in W,

or (b) $(\tilde{\mathrm{A}})$ is structurally unstable, and

$$(\tilde{H}, \tilde{A}) \overset{\varepsilon}{\equiv} (H, A), \tag{5}$$

where $\tilde{H}$ is some region.

Consider a dynamic system

$$\frac{dx}{dt}=\tilde{P}(x, y), \quad \frac{dy}{dt}=\tilde{Q}(x, y) \tag{$\tilde{\mathrm{A}}$}$$

δ-close to (A), where the right-hand sides are irreducible polynomials which coincide, together with their first derivatives, with the functions P and Q and their first derivatives at the point $M_0(x_0, y_0)$ (such a system exists by Lemma 1). $M_0(x_0, y_0)$ is then an equilibrium state of $(\tilde{\mathrm{A}})$, and

$$\tilde{\Delta}(x_0, y_0)=\Delta(x, y)=0. \tag{6}$$

Since $(\tilde{\mathrm{A}})$ is δ-close to (A), one of the conditions (a) or (b) should apply to this system. However, condition (a) is ruled out, since by (6), $(\tilde{\mathrm{A}})$ has a structurally unstable equilibrium state M_0 in W. Condition (b) is also unacceptable, since (A) has in W an infinite number of equilibrium states, whereas $(\tilde{\mathrm{A}})$ has a finite number of equilibrium states in the plane (since $(\tilde{P}, \tilde{Q})=1$) and relation (5) is therefore inapplicable. This contradiction proves the theorem.

Lemma 2. Let $O_1, O_2, \ldots, O_s$ be all the equilibrium states of (A) *in* W. *There exist $\varepsilon_0>0$ and $\delta_0>0$ with the following property: if $(\tilde{\mathrm{A}})$ is δ_0-close to* (A) *and is structurally unstable in* W, *the ε_0-neighborhood of each point O_i, $i=1, 2, \ldots, s$, contains precisely one equilibrium state $\tilde{O}_i$ of $(\tilde{\mathrm{A}})$ and the equilibrium states O_i and $\tilde{O}_i$ have the same topological structure. The number $\varepsilon_0>0$ may be taken as small as desired.*

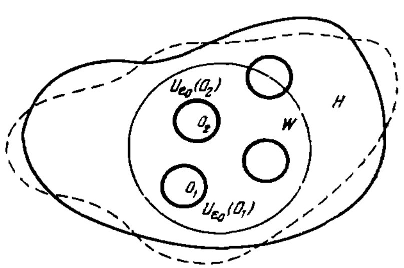

FIGURE 147

Proof. Let H be a region satisfying the condition of Definition 30, $H\supset W$ (Figure 147). Without loss of generality, we may assume that H contains no other equilibrium states of (A), except $O_1, O_2, \ldots, O_s$ (this is so if H is a sufficiently small neighborhood of the closed region W). $\varepsilon_0>0$ is chosen as a number satisfying the following conditions:

(a) The neighborhoods $U_{\varepsilon_0}(O_i)$, $i=1, 2, \ldots, s$, are contained in H and no two of these neighborhoods intersect.

(b) The distance of each neighborhood $U_{\varepsilon_0}(O_i)$ from the boundary of H is greater than $2\varepsilon_0$ (Figure 147).

By Definition 30, to this ε_0 corresponds $\delta_0 > 0$ such that if $(\tilde{A})$ is δ_0-close to (A) and is structurally unstable, we have

$$(\tilde{H},\ \tilde{A}) \overset{\varepsilon_0}{\equiv} (H,\ A), \tag{7}$$

where $\tilde{H}$ is some region.

It follows from (7) that $\tilde{H}$ contains precisely s equilibrium states of $(\tilde{A})$, which we designate $\tilde{O}_1, \tilde{O}_2, \ldots, \tilde{O}_s$.

Let f be the mapping of $\tilde{H}$ onto H implementing the relation (7), (i.e., f is an ε_0-translation moving $\tilde{H}$ into H and conserving the paths, see §4.1, Definitions 8 and 9). Let, further, $f(\tilde{O}_i) = O_i$, $i = 1, 2, \ldots, s$. Then

$$\tilde{O}_i \subset U_{\varepsilon_0}(O_i). \tag{8}$$

By condition (b), each of the neighborhoods $U_{\varepsilon_0}(O_i)$ is contained in $\tilde{H}$, i.e.,

$$U_{\varepsilon_0}(O_i) \subset \tilde{H} \tag{9}$$

(see footnote to p. 67). It follows from (8) and (9) that $\tilde{O}_i$ is the only equilibrium state of $(\tilde{A})$ contained in $U_{\varepsilon_0}(O_i)$. Since $f(\tilde{O}_i) = O_i$ and f is a path-conserving topological mapping, the equilibrium states O_i and $\tilde{O}_i$ have identical topological structures. The proof of the lemma is complete.

Lemma 3. Let O be an equilibrium state of (A), $O \subset W$. *There exists $\delta_0 > 0$ such that if $(\tilde{A})$ is structurally unstable, δ_0-close to* (A), *and has an equilibrium state at the point O, the topological structure of the equilibrium state O of* (A) *coincides with the topological structure of the equilibrium state O of* $(\tilde{A})$.

Lemma 3 follows directly from Lemma 2. δ_0 can be chosen as the δ_0 of Lemma 2.

Theorem 54. A system of the first degree of structural instability in W has no equilibrium states with $\Delta = 0$, $\sigma = 0$ in this region.

Proof. Let (A) be a system of the first degree of structural instability. Suppose that it has an equilibrium state $O(0, 0)$ in W for which $\Delta = 0$, $\sigma = 0$.

Let $\delta_0 > 0$ be the number defined by Lemma 3. By the Weierstrass theorem, there exists a system

$$\frac{dx}{dt} = \tilde{P}(x,\ y), \quad \frac{dy}{dt} = \tilde{Q}(x,\ y) \tag{$\tilde{A}$}$$

$\delta_0/2$-close to (A), where $\tilde{P}$ and $\tilde{Q}$ are polynomials, and the values of $\tilde{P}$ and $\tilde{Q}$ and their first derivatives at the point $O(0, 0)$ coincide with the values of the functions P and Q and their first derivatives at the same point. $O(0, 0)$ is an equilibrium state of $(\tilde{A})$ for which

$$\tilde{\Delta}(0,\ 0) = \Delta(0,\ 0) = 0, \quad \tilde{\sigma}(0,\ 0) = \sigma(0,\ 0) = 0. \tag{10}$$

At least one of the polynomials $\tilde{P}$, $\tilde{Q}$ has linear terms (otherwise, we replace $\tilde{P}$ with $\lambda y + \tilde{P}$, where λ is a sufficiently small number; the point O remains an equilibrium state with $\Delta = \sigma = 0$). Since (A) is a system of the first degree of structural instability, and $(\tilde{A})$ is $\delta_0/2$-close to (A) and

has an equilibrium state at point O, $(\tilde{A})$ is a structurally unstable system with an isolated equilibrium state at O.

Let us now apply the results of §23. In virtue of these results, $(\tilde{A})$ may be written in the form

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=ax^r[1+h(x)]+bx^n y[1+g(x)]+y^2 f(x,y), \qquad (\tilde{A})$$

where h, g, f are analytical functions, $h(0)=g(0)=0$, $r\geqslant 2$, $a\neq 0$, and $n\geqslant 1$ if $b\neq 0$ (see §23.2, (25)).

Consider the system

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=\lambda x^{r-1}+ax^r[1+h(x)]+bx^n y[1+g(x)]+y^2 f(x,y), \qquad (\tilde{A}_1)$$

where $\lambda<0$ and is so small in absolute magnitude that $(\tilde{A}_1)$ and $(\tilde{A})$ are $\delta_0/2$-close.

First consider the case $r>2$. The point O is then an equilibrium state of $(\tilde{A}_1)$ for which $\Delta=0$. By §23.2, II, p. 226, if r is odd (even), the point O for system $(\tilde{A})$ (for system $(\tilde{A}_1)$) is either a saddle-point, or a node, or a focus, or a center, or an equilibrium state with an elliptical region, and for system $(\tilde{A}_1)$ (for system $(\tilde{A})$) it is either a degenerate equilibrium state or a saddle-node. Thus, $(\tilde{A})$ and $(\tilde{A}_1)$ are structurally unstable systems δ_0-close to (A), and the topological structure of the equilibrium state O of $(\tilde{A})$ is different from the topological structure of the equilibrium state O of $(\tilde{A}_1)$. This contradicts Lemma 3.

Let now $r=2$. In this case $(\tilde{A}_1)$ has the form

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=\lambda x+ax^2[1+h(x)]+\ldots$$

The point $O(0,0)$ is an equilibrium state of this system with $\Delta=-\lambda>0$ and $\sigma=0$, i.e., a multiple focus or a center. This equilibrium state is structurally unstable, and $(\tilde{A}_1)$ is therefore a structurally unstable system. With regard to $(\tilde{A})$, the point O is a degenerate equilibrium state for $r=2$ (see QT, §22.2, Theorem 67). This leads to the same contradiction with Lemma 3 as for $r>2$. This completes the proof of the theorem.

It follows from Theorem 54 that every equilibrium state of system (A) in W is either a simple equilibrium state or an isolated equilibrium state with $\Delta=0$ and $\sigma\neq 0$.

As we know, a simple equilibrium state has a multiplicity of 1 (§7.3, Definition 15, §2.2, Theorem 6). Let us find the multiplicity of a multiple equilibrium state of system (A).

Theorem 55. If (A) *is a system of the first degree of instability in* W, *any multiple equilibrium state in* W *has a multiplicity of 2.*

Proof. Let $O(0,0)$ be a multiple equilibrium state of (A), $O\in W$. By the previous theorem $\Delta(0,0)=0$, $\sigma(0,0)\neq 0$. Without loss of generality we may take (A) in the form

$$\frac{dx}{dt}=p(x,y),\quad \frac{dy}{dt}=y+q(x,y), \qquad (11)$$

where

$$p(0,0)=p'_x(0,0)=p'_y(0,0)=q(0,0)=q'_x(0,0)=q'_y(0,0) \qquad (12)$$

(system (A) can be reduced to this form by a non-singular linear transformation, see QT, §21.2).

According to the theorem of implicit functions, the equation

$$y+q(x,\ y)=0 \tag{13}$$

has a unique solution for y in the neighborhood of $O(0,0)$. Let this solution be $\varphi(x)$. Then

$$\varphi(x)+q(x,\ \varphi(x))\equiv 0, \tag{14}$$

and

$$\varphi(0)=0. \tag{15}$$

It follows from (12), (14), and (15) that

$$\varphi'(0)=0. \tag{16}$$

Consider the function

$$\psi(x)=p(x,\ \varphi(x)). \tag{17}$$

Direct computations show that

$$\psi(0)=0, \quad \psi'(0)=0, \quad \psi''(0)=p''_{xx}(0,\ 0). \tag{18}$$

From Definition 15 (§7.3) and Theorem 7 (§2.3) it follows that the equilibrium state $O(0,0)$ of system (11) has a multiplicity of 2 if and only if $p''_{xx}(0,0)\neq 0$. We will now show that this condition is satisfied. Suppose that

$$p''_{xx}(0,\ 0)=0. \tag{19}$$

Let ε_0 and δ_0 be the positive numbers introduced in Lemma 2. As in our proof to Theorem 54, we use the Weierstrass theorem and construct a system

$$\frac{dx}{dt}=\widetilde{p}(x,\ y), \quad \frac{dy}{dt}=y+\widetilde{q}(x,\ y) \tag{$\tilde{A}$}$$

$\delta_0/2$-close to (11), where $\widetilde{p}$ and $\widetilde{q}$ are polynomials which together with their derivatives to second order inclusive coincide at the point $O(0,0)$ with the functions p and q and their derivatives. Then

$$\widetilde{p}(0,\ 0)=\widetilde{p}'_x(0,\ 0)=\widetilde{p}'_y(0,\ 0)=\widetilde{q}(0,\ 0)=\widetilde{q}'_x(0,\ 0)=\widetilde{q}'_y(0,\ 0)=0 \tag{20}$$

and

$$\widetilde{p}''_{xx}(0,\ 0)=0. \tag{21}$$

Since $\widetilde{\Delta}(0,0) = 0$, $(\tilde{A})$ is structurally unstable in W. Therefore, by Lemma 2, the equilibrium state $O(0,0)$ of $(\tilde{A})$ is isolated.

Let $\widetilde{\varphi}(x)$ and $\widetilde{\psi}(x)$ be the analogs of the functions φ and ψ, i.e., these are functions defined by the relations

$$\widetilde{\varphi}(x)+\widetilde{q}(x,\ \widetilde{\varphi}(x))=0 \tag{22}$$

and

$$\tilde{\psi}(x) = \tilde{p}(x, \tilde{\varphi}(x)). \tag{23}$$

Since $\tilde{p}$ and $\tilde{q}$ are analytical functions (polynomials), $\tilde{\varphi}$ and $\tilde{\psi}$ are also analytical functions. By (20) and (21),

$$\tilde{\varphi}(0) = \tilde{\varphi}'(0) = 0 \tag{24}$$

and

$$\tilde{\psi}(0) = \tilde{\psi}'(0) = \tilde{\psi}''(0) = 0. \tag{25}$$

Moreover, $\tilde{\psi}(x)$ cannot be identically zero. Indeed, if $\tilde{\psi}(x) \equiv 0$, then for all sufficiently small x, the point $(x, \tilde{\varphi}(x))$ is an equilibrium state of $(\tilde{A})$, so that O is no longer an isolated equilibrium state. Thus, near the point $x = 0$, the function $\tilde{\psi}(x)$ is expressible in the form

$$\tilde{\psi}(x) = ax^n + a_1 x^{n-1} + \ldots, \tag{26}$$

where $a \neq 0$ and $n > 3$.

Consider the dynamic system

$$\frac{dx}{dt} = \mu x^{n-1} + \tilde{p}(x, y) = \tilde{P}_\mu(x, y), \qquad \frac{dy}{dt} = y + \tilde{q}(x, y) = \tilde{Q}_\mu(x, y), \tag{$\tilde{A}_\mu$}$$

where $\mu \neq 0$ is so small that the systems $(\tilde{A}_\mu)$ and $(\tilde{A})$ are $\delta_0/2$-close. Since $O(0, 0)$ is a multiple equilibrium state of $(\tilde{A}_\mu)$, this system is structurally unstable in W.

Let us try to find some equilibrium states of $(\tilde{A}_\mu)$ near the point O. By (22), $y = \tilde{\varphi}(x)$ is the solution of the equation $\tilde{Q}_\mu(x, y) = 0$ for y. Inserting $\tilde{\varphi}(x)$ for y in the equation $\tilde{P}_\mu(x, y) = 0$, we obtain an equation for x, i.e., $\tilde{P}_\mu(x, \tilde{\varphi}(x)) = 0$, or explicitly

$$x^{n-1}[\mu + ax + a_1 x^2 + \ldots] = 0$$

or

$$x^{n-1} h(\mu, x) = 0, \tag{27}$$

where $h(\mu, x) = \mu + ax + a_1 x^2 + \ldots$.

Since $h(0, 0) = 0$, $h'_x(0, 0) = a \neq 0$, the theorem of implicit functions shows that the equation $h(\mu, x) = 0$ has a solution $x = x_\mu$ near the point $O(0, 0)$ which goes to zero for $\mu \to 0$. Clearly, if $\mu \neq 0$, then $x_\mu \neq 0$. The point $O_\mu(x_\mu, \tilde{\varphi}(x_\mu))$ is an equilibrium state of the system $(\tilde{A}_\mu)$ which is different from O and lies as close as desired to O, when μ is sufficiently small.

We have thus established that if $p''_{xx}(0, 0) = 0$, there exists a system $(\tilde{A}_\mu)$ δ_0-close to (A) which is structurally unstable in W and yet has two equilibrium states in any arbitrarily small neighborhood of O, namely O and O_μ. This contradicts Lemma 2. Thus, $p''_{xx}(0, 0) \neq 0$, i.e., the equilibrium state O has a multiplicity of 2. Q.E.D.

Theorem 56. If (A) *is a system of the first degree of structural instability in* W, *any multiple equilibrium state of this system in* W *is a saddle-node.*

Proof. Theorem 56 follows directly from Theorems 54 and 55 because of the following general proposition: a multiple equilibrium state of multiplicity 2 for which $\sigma \neq 0$ is a saddle-node. If (A) is an analytical system, this proposition is contained in I, §23.2 (or, equivalently, in QT, §21.2, Theorem 65). For a non-analytical system, the proof can be obtained by the same method as the proof of Theorem 65 in QT.

Theorems 54, 55, and 56 show that any multiple equilibrium state of a system of the first degree of structural instability is a saddle-node with $\sigma \neq 0$ and a multiplicity of 2.

Let us now consider simple equilibrium states, i.e., those with $\Delta \neq 0$. We have seen (Chapter IV) that a simple equilibrium state is stable, except if its characteristic numbers are pure imaginary. It is readily seen that a system of the first degree of structural instability may have a structurally stable equilibrium state of any type. We will therefore concentrate on equilibrium states with pure imaginary characteristic numbers. Each of these equilibrium states, as we know, is either a multiple focus, or a center, or a center-focus (a center-focus is possible only for a non-analytical system, remark at the end of §24.4).

Without loss of generality, we assume that the equilibrium state is the point $O(0,0)$ and that (A) is given in the canonical form

$$\frac{dx}{dt} = -\beta y + \varphi(x, y), \quad \frac{dy}{dt} = \beta x + \psi(x, y), \tag{28}$$

where $\beta > 0$, and φ and ψ are functions of class 3, which vanish together with their first order derivatives at the point O.

In Chapter IX we examined the succession function $f(\rho)$ defined on a ray extending from the point O, and the function $d(\rho) = f(\rho) - \rho$. Both these functions are of the same class as φ and ψ. The values of the derivatives of $d(\rho)$ at the point O are known as the focal values. We have seen (§24.2, Lemma 5 and (25)) that

$$d(0) = d'(0) = d''(0) = 0. \tag{29}$$

If $d'''(0) \neq 0$, $O(0,0)$ is a multiple focus of multiplicity 1 (see §24.2, Definition 26). If $d'''(0) = 0$, O is either a multiple focus of multiplicity $m > 1$, or a multiple focus of indefinite multiplicity, or a center, or a center-focus.

Theorem 57. If a system of the first degree of structural instability in W has equilibrium states with pure imaginary characteristic numbers in this region, it is a multiple focus of multiplicity 1, i.e., $d'''(0) \neq 0$ for this point.

Proof. Suppose that the theorem is not true, i.e., for the point $O(0,0)$ of system (28) we have

$$d'''(0) = 0. \tag{30}$$

As we know (§24.3, (35), (36)), system (28) may be written in the form

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) = -\beta y + P_2(x, y) + P_3(x, y) + \sum_{\alpha=0}^{3} x^{3-\alpha} y^{\alpha} P_{\alpha}^{*}(x, y), \\ \frac{dy}{dt} &= Q(x, y) = \beta x + Q_2(x, y) + Q_3(x, y) + \sum_{\alpha=0}^{3} x^{3-\alpha} y^{\alpha} Q_{\alpha}^{*}(x, y), \end{aligned} \tag{31}$$

where P_2 and Q_2 (P_3 and Q_3) are homogeneous polynomials of degree 2 (of degree 3), and $P_\alpha^*(x, y)$ and $Q_\alpha^*(x, y)$ are continuous functions which vanish for $x=y=0$.

Let $\delta_0>0$ be the number defined by Lemma 3. Let further $\widetilde{P}_\alpha^*$ and $\widetilde{Q}_\alpha^*$ be polynomials which provide an adequate fit of the functions P_α^* and Q_α^* and vanish at the point $O(0, 0)$. Then the system

$$\begin{aligned}\frac{dx}{dt}&=-\beta y+P_2(x, y)+P_3(x, y)+\sum_{\alpha=0}^{3} x^{3-\alpha}y^\alpha\widetilde{P}_\alpha^*(x, y)=\widetilde{P}(x, y),\\ \frac{dy}{dt}&=\beta x+Q_2(x, y)+Q_3(x, y)+\sum_{\alpha=0}^{3} x^{3-\alpha}y^\alpha\widetilde{Q}_\alpha^*(x, y)=\widetilde{Q}(x, y)\end{aligned}\tag{$\tilde{\mathrm{A}}$}$$

is $\delta_0/2$-close to (A) and the point O is its equilibrium state with pure imaginary characteristic numbers. Let $\tilde{d}(\rho)$ be the analog of $d(\rho)$ for the system ($\tilde{\mathrm{A}}$). It follows from equality (30) and from Lemma 6, §24.3 that

$$\tilde{d}'''(0)=0. \tag{32}$$

Since ($\tilde{\mathrm{A}}$) is an analytical system, the last equality shows that $O(0,0)$ is either a center or a multiple focus of multiplicity $\mu>1$ for this system. We will now show that it cannot be a center.

Indeed, suppose that O is a center of ($\tilde{\mathrm{A}}$). Consider the system

$$\frac{dx}{dt}=\widetilde{P}(x, y)+\mu(x^2+y^2)y,\quad \frac{dy}{dt}=\widetilde{Q}(x, y)+\mu(x^2+y^2)x, \tag{$\tilde{\mathrm{A}}_1$}$$

where $\mu\neq 0$, with the corresponding function $\tilde{d}_1(\rho)$. For a sufficiently small μ, system ($\tilde{\mathrm{A}}_1$) is δ_0-close to system (A) (i.e., to system (28)). By Lemma 7, §24.3, we see that

$$\tilde{d}_1'(0)=\tilde{d}_1''(0)=0,\quad \tilde{d}_1'''(0)=12\pi\frac{\mu}{\beta}\neq 0,$$

i.e., O is a multiple focus of system ($\tilde{\mathrm{A}}_1$). Thus, ($\tilde{\mathrm{A}}$) and ($\tilde{\mathrm{A}}_1$) are structurally unstable systems δ_0-close to (A), and the equilibrium state O is a focus for ($\tilde{\mathrm{A}}_1$) and a center for ($\tilde{\mathrm{A}}$). This clearly contradicts Lemma 3.

We have thus established that O is a multiple focus of ($\tilde{\mathrm{A}}$).

Then, by Lemma 3, O is also a multiple focus of the original system (A). We will show that this conclusion involves a contradiction.

Indeed, let O be a multiple focus of (A). Then, if $\eta>0$ is sufficiently small, all the paths of system (A) contained in $U_\eta(O)$ (with the exception of the point O) are spirals which wind onto O for $t\to-\infty$ or for $t\to+\infty$. Choose $\eta>0$ which satisfies this condition.

Let ε be a positive number, $\varepsilon<\frac{\eta}{4}$, δ, $0<\delta<\delta_0$, a number corresponding to ε in virtue of Definition 30. Then, if some system ($\tilde{\mathrm{B}}$) is structurally unstable in W and is δ-close to (A), we have

$$(H, A)\overset{\varepsilon}{\equiv}(\widetilde{H}_1, \widetilde{B}),$$

where $\widetilde{H}_1$ is some region, and H is the region introduced in Definition 30.

We may take $\eta>0$ so small that $U_\eta(O)\subset H$. Then, from the above relation

$$(U_\eta(O), A)\overset{\varepsilon}{\equiv}(\widetilde{V}, \widetilde{B}), \tag{33}$$

where $\widetilde{V}$ is some region.

Further suppose that O is an equilibrium state of $(\tilde{B})$. We will now try to extract some information about the neighborhood $\tilde{V}$.

By (33), all the paths of $(\tilde{B})$ contained in $\tilde{V}$ are spirals winding onto the point O (the point O itself excepted). It follows from the inequality $\varepsilon < \frac{\eta}{4}$ that $U_{\eta/2}(O) \subset \tilde{V}$. Thus, if $(\tilde{B})$ is a structurally unstable system δ-close to (A) with an equilibrium state at the point O, all its paths contained in $U_{\eta/2}(O)$ are spirals winding onto O.

Let the system $(\tilde{A})$ constructed above be $\delta/2$-close to (A). Since $\delta < \delta_0$, O is a multiple focus of multiplicity $m > 1$ of $(\tilde{A})$, as we have seen.

Let $(\tilde{B})$ be the system

$$\frac{dx}{dt} = \tilde{P}(x, y) + \mu(x^2 + y^2)y, \quad \frac{dy}{dt} = \tilde{Q}(x, y) + \mu(x^2 + y^2)x, \qquad (\tilde{A}_1)$$

where $\mu \neq 0$ is so small that $(\tilde{A}_1)$ is $\delta/2$-close to $(\tilde{A})$. In our proof of Theorem 40 we have seen that if η is taken sufficiently small and of an appropriate sign, $(\tilde{A}_1)$ will have at least one closed path in $U_{\eta/2}(O)$ (see §25.1, proof of Theorem 40, in particular equations (3)–(9); for our purposes $k = 2$). This contradicts the previous proposition that all the paths of $(\tilde{A}_1)$ in $U_{\eta/2}(O)$ are spirals. Q.E.D.

Theorems 53–57 provide a complete solution to the problem of equilibrium states of a system of the first degree of structural instability. They show, in particular, that if (A) is a dynamic system of the first degree of structural instability in W, it may have only a finite number of equilibrium states in this region, and each of these equilibrium states is either a structurally stable equilibrium state, or a multiple focus of multiplicity 1, or, finally, a saddle-node with $\sigma \neq 0$ and a multiplicity of 2.

3. Closed paths of systems of the first degree of structural instability

Closed paths of dynamic systems were considered in Chapters V and X. In the present subsection, we will use the earlier notation and some of the previous results.

First, we will prove a number of lemmas relating to the succession function on a normal to a closed path.

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \qquad (A)$$

be a dynamic system of class N or an analytical system, L_0 a closed path of this system,

$$x = \varphi(t), \quad y = \psi(t) \qquad (L_0)$$

the motion corresponding to this path, $\tau > 0$ the period of the functions φ and ψ.

In §13.1, we introduced curvilinear coordinates s, n in the neighborhood of the path L_0, defined by the relations

$$x = \bar{\varphi}(s, n), \quad y = \bar{\psi}(s, n),$$

where

$$\bar{\varphi}(s, n) = \varphi(s) + n \cdot \psi'(s), \qquad \bar{\psi}(s, n) = \psi(s) - n \cdot \varphi'(s). \tag{34}$$

System (A) is described in these coordinates by the differential equation

$$\frac{dn}{ds} = R(s, n), \tag{35}$$

where $R(s, n)$ is a function of class N (or an analytical function) which is periodic in s with the period τ and satisfies the initial condition

$$R(s, 0) \equiv 0.$$

Let $n = f(s, n_0)$ be the solution of equation (35) satisfying the initial condition $f(0, n_0) \equiv n_0$.

Then

$$f(n_0) = f(\tau, n_0)$$

is a succession function on the arc without contact l defined by the equations

$$x = \varphi(0) + n\psi'(0), \qquad y = \psi(0) - n\varphi'(0),$$

which is normal to the path L_0 at the point $s = 0$. Alongside with the succession function, we considered the function

$$d(n_0) = f(n_0) - n_0.$$

$f(n_0)$ and $d(n_0)$ are defined for all sufficiently small (in absolute magnitude) values of n_0, say for $|n_0| < n_0^*$, and they are functions of the same class as the original system (A) (see § 12.2, equation (5) and the text that follows).

Let us state three lemmas (Lemmas 4, 5, 6) regarding the succession functions of close systems. These lemmas follow directly from general considerations (the continuous dependence of the solutions of differential equations on the right-hand sides and the differentiability of solutions with respect to parameters and initial values), and we will omit the respective proofs. It is assumed that the modified systems

$$\frac{dx}{dt} = \tilde{P}(x, y), \qquad \frac{dy}{dt} = \tilde{Q}(x, y) \tag{$\tilde{A}$}$$

belong to the same class as system (A), and we consider closeness to rank k, where $1 \le k \le N$ (if (A) is an analytical system, k may be any natural number).

Lemma 4. For any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: if $(\tilde{A})$ is δ-close to (A) to rank k, then

(a) the normal l is an arc without contact of $(\tilde{A})$ also, and a succession function $\tilde{f}(n_0)$ of system $(\tilde{A})$ and thus also the function $\tilde{d}(n_0) = \tilde{f}(n_0) - n_0$ are defined on this arc for all n_0, $|n_0| < n_0^*$;

(b) the functions $f(n_0)$ and $\tilde{f}(n_0)$ and thus also the functions $\tilde{d}(n_0)$ and $d(n_0)$ are δ-close to rank k.

In what follows, we will use modified dynamic systems of a special form, namely

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) + \mu_1 p_1(x, y) + \ldots + \mu_n p_n(x, y), \\ \frac{dy}{dt} &= Q(x, y) + \mu_1 q_1(x, y) + \ldots + \mu_n q_n(x, y), \end{aligned} \tag{$\tilde{A}_{\mu, n}$}$$

where μ_i are parameters, and $p_i(x, y)$ and $q_i(x, y)$ are functions of one class with P and Q.

Lemma 5. There exists $\mu^*>0$ such that if $|\mu_i|<\mu^*$, the normal l is an arc without contact for $(\tilde{A}_{\mu,n})$ and a succession function $\tilde{f}(n_0, \mu_1, \mu_2, \ldots, \mu_n)$ of the same class as P and Q is defined on this arc for all n_0, $|n_0|<n_0^*$. Moreover,

$$\tilde{f}(n_0, 0, 0, \ldots, 0) \equiv f(n_0).$$

Together with $(\tilde{A}_{\mu,n})$, we consider the system

$$\begin{aligned} \frac{dx}{dt} &= P^*(x, y)+\mu_1 p_1^*(x, y)+\ldots+\mu_n p_n^*(x, y), \\ \frac{dy}{dt} &= Q^*(x, y)+\mu_1 q_1^*(x, y)+\ldots+\mu_n q_n^*(x, y), \end{aligned} \qquad (A^*_{\mu,n})$$

where P^*, Q^*, p_i^*, q_i^* are functions of the same class as P and Q. Let μ^* be a number satisfying the conditions of Lemma 5.

Lemma 6. For any $\varepsilon>0$, there exists $\delta>0$ with the following property: if $|\mu_i|<\mu^*$, and the functions P^*, Q^*, p_i^*, q_i^* are δ-close to rank k to the functions P, Q, p_i, q_i, respectively, the normal l is an arc without contact for the system $(A^*_{\mu,n})$ and a succession function $f^*(n_0, \mu_1, \mu_2, \ldots, \mu_n)$ is defined on this arc for all n_0, $|n_0|<n_0^*$. The function f^* is of the same class as P and Q in all its variables and it is ε-close to rank k to $\tilde{f}$.

The next lemma is more restricted. It deals with the system

$$\frac{dx}{dt} = P(x, y)+\mu p(x, y) = \tilde{P}(x, y, \mu), \quad \frac{dy}{dt} = Q(x, y)+\mu q(x, y) = \tilde{Q}(x, y). \qquad (\tilde{A})$$

Let $\tilde{f}(n_0, \mu)$ be the succession function for this system on the normal l.

Lemma 7. For the succession function $\tilde{f}(n_0, \mu)$ of system $(\tilde{A})$ on the normal l, we have the equality

$$\tilde{f}'_\mu(0,0) = \frac{1}{(\varphi'(0))^2+(\psi'(0))^2} e^{\int_0^\tau (P'_x+Q'_x)\,ds} \int_0^\tau [p\psi'(s)-q\varphi'(s)]\, e^{-\int_0^s (P'_x+Q'_y)\,ds} ds, \qquad (36)$$

where the values of the functions P'_x, Q'_y, p, and q are evaluated at the point $(\varphi(s), \psi(s))$.

Proof. In Chapter V we rewrote the differential equation (35) corresponding to system (A) in the form

$$\frac{dn}{ds} = R(s, n) = \frac{Q(\bar{\varphi}, \bar{\psi})\bar{\varphi}'_s - P(\bar{\varphi}, \bar{\psi})\bar{\psi}'_s}{P(\bar{\varphi}, \bar{\psi})\bar{\psi}'_n - Q(\bar{\varphi}, \bar{\psi})\bar{\varphi}'_n},$$

where $\bar{\varphi}$ and $\bar{\psi}$ are defined by (34) (see § 13.2, (12)). The corresponding equation for $(\tilde{A})$ is

$$\frac{dn}{ds} = \tilde{R}(s, n, \mu) = \frac{(Q+\mu q)\bar{\varphi}'_s - (P+\mu p)\bar{\psi}'_s}{(P+\mu p)\bar{\psi}'_n - (Q+\mu q)\bar{\varphi}'_n}, \qquad (37)$$

where the functions P, Q, p, q are evaluated at the point $(\bar{\varphi}(s, n), \bar{\psi}(s, n))$.

Let $n=\tilde{f}(s; n_0, \mu)$ be the solution of equation (37) satisfying the initial condition $\tilde{f}(0, n_0, \mu) \equiv n_0$. Then identically

$$\frac{\partial \tilde{f}(s, n_0, \mu)}{\partial s} \equiv \tilde{R}(s, \tilde{f}(s; n_0, \mu), \mu), \qquad (38)$$

and the succession function of $(\tilde{A})$ on the normal l is defined by the relation

$$\tilde{f}(n_0, \mu) \equiv \tilde{f}(\tau, n_0, \mu) \tag{39}$$

(the assignment of the same symbol $\tilde{f}$ to two different functions should not lead to confusion, since one of them is a function of two arguments, and the other is a function of three arguments).

Let

$$\theta(s) = \tilde{f}'_\mu(s, 0, 0). \tag{40}$$

Differentiating (38) with respect to μ, changing the order of differentiation in the left-hand side, setting $n_0 = \mu = 0$ and using (40), we obtain

$$\frac{d\theta(s)}{ds} = \left[\frac{\partial \tilde{R}(s; \tilde{f}(s; n_0, \mu), \mu)}{\partial n}\right]_{\substack{n_0=0\\ \mu=0}} \theta(s) + \left[\frac{\partial \tilde{R}(s; \tilde{f}(s, n_0, \mu), \mu)}{\partial \mu}\right]_{\substack{n_0=0\\ \mu=0}}. \tag{41}$$

Since for $\mu = 0$ system $(\tilde{A})$ reduces to (A), we have $\tilde{f}(s, 0, 0) \equiv 0$, and the coefficient before $\theta(s)$ in the right-hand side of (41) is equal to $\frac{\partial R(s, n)}{\partial n}$. The last expression was computed in Chapter V (§13.3, (28)). Thus,

$$\frac{\partial R(s, 0)}{\partial n} = P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s)) - \frac{d}{ds}[\ln(\varphi'(s))^2 + (\psi'(s))^2]. \tag{42}$$

The expression for $\left[\frac{\partial \tilde{R}(s; \tilde{f}(s; n_0, \mu), \mu)}{\partial \mu}\right]_{\substack{\mu=0\\ n_0=0}}$ can be obtained by direct differentiation, using (37) and (34). Using the relations $\varphi'(s) \equiv P(\varphi(s), \psi(s))$, $\psi'(s) \equiv Q(\varphi(s), \psi(s))$, we obtain

$$\left[\frac{\partial \tilde{R}(s; \tilde{f}(s; n_0, \mu), \mu)}{\partial \mu}\right]_{\substack{n_0=0\\ \mu=0}} = \frac{p(\varphi(s), \psi(s))\psi'(s) - q(\varphi(s), \psi(s))\varphi'(s)}{(\varphi'(s))^2 + (\psi'(s))^2}. \tag{43}$$

It follows from (41), (42), and (43) that

$$\frac{d\theta(s)}{ds} = \left\{P'_x + Q'_y - \frac{d}{ds}[\ln((\varphi'(s))^2 + (\psi'(s))^2]\right\}\theta(s) + \frac{p\psi'(s) - q\varphi'(s)}{(\varphi'(s))^2 + (\psi'(s))^2}, \tag{44}$$

where P'_x, Q'_y, p, q are the values of these functions at the point $(\varphi(s), \psi(s))$.

From the relation $\tilde{f}(0; n_0, \mu) \equiv n_0$ and (40), we find that

$$\theta(0) = \tilde{f}'_\mu(0, 0, 0) = 0.$$

Integrating the linear differential equation (44) with the initial condition $\theta(0) = 0$, setting $s = \tau$ and using (39), we find

$$\theta(\tau) = \tilde{f}'_\mu(\tau; 0, 0) = \tilde{f}'_\mu(0, 0) = \frac{e^{\int_0^\tau (P'_x + Q'_y)\,ds}}{(\varphi'(0))^2 + (\psi'(0))^2} \int_0^\tau [p\psi'(s) - q\varphi'(s)]\, e^{-\int_0^s (P'_x + Q'_y)\,ds}\, ds,$$

where the functions P'_x, Q'_y, p, and q are evaluated at the point $(\varphi(s), \psi(s))$. This completes the proof of the lemma.

Corollary. If $(\tilde{A})$ has the form

$$\frac{dx}{dt} = P(x, y) - \mu Q(x, y), \quad \frac{dy}{dt} = Q(x, y) + \mu P(x, y),$$

i.e., if the vector field of the system is a rotated field of (A), then $\tilde{f}'_{\mu}(0,\ 0)\neq 0$.

Proof. The validity of the last inequality follows directly from (36) since in the present case

$$p\psi'(s)-q\varphi'(s)=-Q(\varphi(s),\ \psi(s))\psi'(s)-P(\varphi(s),\ \psi(s))\varphi'(s)=-(\varphi'(s))^2-(\psi'(s))^2\neq 0.$$

In what follows we will also consider modified systems of the form

$$\frac{dx}{dt}=P+\mu F^m F'_x,\qquad \frac{dy}{dt}=Q+\mu F^m F'_y, \tag{A^m_μ}$$

where $F(x,\ y)$ is a function of class $N+1$ satisfying the following conditions:

(a) $F(\varphi(s),\ \psi(s))\equiv 0$;

(b) $[F'_x(\varphi(s),\ \psi(s))]^2+[F'_y(\varphi(s),\ \psi(s))]^2\neq 0$.

These systems were considered in Chapters V and X.

Lemma 8. If the function $d(n_0)=f(n_0)-n_0$ of system (A) satisfies the equalities

$$d'(0)=d''(0)=\ldots=d^{(m-1)}(0)=d^{(m)}(0)=0,$$

where $1<m\leqslant N$, the corresponding function of system (A^m_μ) satisfies the equalities

$$\tilde{d}(0)=\tilde{d}'(0)=\ldots=\tilde{d}^{(m-1)}(0)=0,\qquad \tilde{d}^{(m)}(0)=\mu\beta, \tag{45}$$

where $\beta\neq 0$, i.e., for $\mu\neq 0$, L_0 is a multiple limit cycle of multiplicity m of system (A^m_μ).

If $d'(0)=0$, the function $\tilde{d}(n_0)$ of the system

$$\frac{dx}{dt}=P+\mu FF'_x,\qquad \frac{dy}{dt}=Q+\mu FF'_y \tag{A^1_μ}$$

satisfies the equalities

$$\tilde{d}(0)=0,\qquad \tilde{d}'(0)=e^{\mu I}-1, \tag{46}$$

where

$$I=\int_0^\tau [(F'_x(\varphi(s),\ \psi(s)))^2+(F'_y(\varphi(s),\ \psi(s)))^2]\,ds,$$

i.e., for $\mu\neq 0$, L_0 is a simple limit cycle of the system (A^1_μ).

Proof. The first proposition of the lemma is contained in §26.2, Lemma 1, and the second proposition was established incidentally in our proof of Theorem 19 (§15.2 (18)). The number β in the last equality in (45) is a constant depending on the functions P, Q, F, φ, and ψ only.

Remark. If (A) is a system of class N and for its closed path L_0

$$d'(0)=d''(0)=\ldots=d^{(m-1)}(0)=0,$$

where $1\leqslant m\leqslant N$, then for all sufficiently small $\mu\neq 0$, the path L_0 is an m-tuple limit cycle of (A^m_μ). Indeed, if $d^{(m)}(0)=0$, our proposition follows from Lemma 8. If, however, $d^{(m)}(0)\neq 0$, then for sufficiently small μ, the derivative $\tilde{d}^{(m)}(0)$ does not vanish either, and the values of $\tilde{d}'(0)$, $\tilde{d}''(0),\ \ldots,\ \tilde{d}^{(m-1)}(0)$ are respectively equal to $d'(0)$, $d''(0),\ \ldots,\ d^{(m-1)}(0)$, i.e., they all vanish (see §26.2, proof of Lemma 1).

The next lemma is of fundamental importance for what follows.

Lemma 9. Let (A) *be a dynamic system of class* $N \geqslant 2$, *and* m *and* k *natural numbers,* $2 \leqslant m \leqslant k \leqslant N$. *If* (A) *has a limit cycle* L_0 *of multiplicity* m, *then for any* $\varepsilon > 0$ *and* $\delta > 0$, *there exists an analytical system* $(\hat{A})$ δ-*close to rank* k *to* (A) *which has a limit cycle of multiplicity* m *in* $U_\varepsilon(L_0)$.

Proof. The proof will be given for the case $m = 3$. The proof for $m = 2$ or $m > 3$ is completely analogous. Furthermore, in our treatment of systems of the first degree of structural instability, we will only require the case $m = 3$.

Thus, let L_0 be a limit cycle of multiplicity 3 of the system (A). Then

$$d(0) = 0, \quad d'(0) = 0, \quad d''(0) = 0, \quad d'''(0) \neq 0. \tag{47}$$

We fix the numbers $\varepsilon > 0$ and $\delta > 0$ and choose $F(x, y)$ to be the function of class $N+1$ defined above (satisfying conditions (a) and (b)). Consider the modified system

$$\begin{aligned}\frac{dx}{dt} &= P(x, y) - \mu_0 Q(x, y) + \mu_1 F(x, y) F'_x(x, y),\\ \frac{dy}{dt} &= Q(x, y) + \mu_0 P(x, y) + \mu_1 F(x, y) F'_y(x, y)\end{aligned} \tag{$\tilde{A}_\mu$}$$

and the corresponding function $\tilde{d} = \tilde{d}(n_0, \mu_0, \mu_1)$. Let μ_0 and μ_1 be so small that $(\tilde{A}_\mu)$ is $\delta/2$-close to rank k to (A), and the function $\tilde{d}$ is defined for all n_0, $|n_0| < n_0^*$, for which $d(n_0)$ is defined. Clearly,

$$\tilde{d}(n_0, 0, 0) \equiv d(n_0). \tag{48}$$

From (47) and (48) we see that the equations

$$\tilde{d}(n_0, \mu_0, \mu_1) = 0, \quad \tilde{d}'_{n_0}(n_0, \mu_0, \mu_1) = 0, \quad \tilde{d}''_{n_0^2}(n_0, \mu_0, \mu_1) = 0 \tag{49}$$

have a simultaneous solution $n_0 = 0$, $\mu_0 = 0$, $\mu_1 = 0$. Consider the Jacobian of this system of equations. We have

$$\Delta(n_0, \mu_0, \mu_1) = \frac{D(\tilde{d}, \tilde{d}'_{n_0}, \tilde{d}''_{n_0^2})}{D(n_0, \mu_1, \mu_0)} = \begin{vmatrix} \tilde{d}'_{n_0} & \tilde{d}'_{\mu_1} & \tilde{d}'_{\mu_0} \\ \tilde{d}''_{n_0^2} & \tilde{d}''_{n_0\mu_1} & \tilde{d}''_{n_0\mu_0} \\ \tilde{d}'''_{n_0^3} & \tilde{d}'''_{n_0^2\mu_1} & \tilde{d}'''_{n_0^2\mu_0} \end{vmatrix}.$$

From (48) and (47) it follows that $\tilde{d}'_{n_0}(0, 0, 0) = 0$, $\tilde{d}''_{n_0^2}(0, 0, 0) = 0$, $\tilde{d}'''_{n_0^3}(0, 0, 0) \neq 0$. Let us compute the elements $\tilde{d}'_{\mu_1}$, $\tilde{d}'_{\mu_0}$, and $\tilde{d}''_{n_0\mu_1}$ at the point $(0, 0, 0)$. In the computation of the numbers $\tilde{d}'_{\mu_1}(0, 0, 0)$ and $\tilde{d}''_{n_0\mu_1}(0, 0, 0)$ we may evidently take from the start $\mu_0 = 0$, i.e., we may consider the function d corresponding to the system $\frac{dx}{dt} = P + \mu_1 FF'_x$, $\frac{dy}{dt} = Q + \mu_1 FF'_y$. Lemma 8 is thus applicable. By the first equation in (46), $\tilde{d}(0, 0, \mu_1) = 0$. Therefore, $\tilde{d}'_{\mu_1}(0, 0, 0) = 0$. Now, by the second equation in (46), $\tilde{d}'_{n_0}(0, 0, \mu_1) = e^{\mu_1 I} - 1$, where $I \neq 0$. Therefore, $\tilde{d}''_{n_0\mu_1}(0, 0, 0) = I \neq 0$.

For the computation of $\tilde{d}'_{\mu_0}(0, 0, 0)$ we may assume from the start $\mu_1 = 0$, i.e., we may consider the function d corresponding to the system

$$\frac{dx}{dt} = P - \mu_0 Q, \quad \frac{dy}{dt} = Q + \mu_0 P.$$

Then, by the corollary from Lemma 7, $d'_\mu(0, 0, 0) \neq 0$.

Thus, for $n_0 = \mu_0 = \mu_1 = 0$, none of the elements of the Jacobian $\Delta(n_0, \mu_0, \mu_1)$ along the second diagonal vanish, whereas all the elements left of this diagonal are zero, i.e., $\Delta(0, 0, 0) \neq 0$. Therefore, by the theorem of implicit functions, $n_0 = \mu_0 = \mu_1 = 0$ is the only solution of system (49) in a sufficiently small neighborhood of the point $(0, 0, 0)$.

Consider the analytical system

$$\frac{dx}{dt} = \hat{P} - \mu_0 \hat{Q} + \mu_1 \hat{F}\hat{F}'_x, \quad \frac{dy}{dt} = \hat{Q} + \mu_0 \hat{P} + \mu_1 \hat{F}\hat{F}'_y, \qquad (\hat{A}_\mu)$$

where $\hat{P}, \hat{Q}$ are polynomials which are δ_1-close to rank k to the functions P and Q, respectively, and $\hat{F}$ is a polynomial δ_1-close to rank $k+1$ to the function F. Let $\hat{d}(n_0, \mu_0, \mu_1)$ be the function corresponding to this system, and

$$\hat{d}(n_0, \mu_0, \mu_1) = 0, \quad \hat{d}'_{n_0}(n_0, \mu_0, \mu_1) = 0, \quad \hat{d}''_{n_0^2}(n_0, \mu_0, \mu_1) = 0 \qquad (50)$$

the equations corresponding to system (49). By Theorem 4 (the theorem of a small increment of implicit functions, § 1.2), system (50) has a unique solution in a sufficiently small neighborhood of the point $(0, 0, 0)$ if $\hat{d}$ is sufficiently close to rank 2 to $\tilde{d}$. On the other hand, by Lemma 6, if δ_1 is sufficiently small, $\hat{d}$ is as close as desired to rank k to $\tilde{d}$, where $k > m = 3$. Therefore, we can choose $\delta_1 > 0$ so small that the following conditions are satisfied:

1) $\delta_1 < \frac{\delta}{2}$;

2) system (50) has a unique solution $(\hat{n}_0, \hat{\mu}_0, \hat{\mu}_1)$ in a certain neighborhood of the point $(0, 0, 0)$ which is as close as desired to zero.

Furthermore, if $\delta_1, \hat{\mu}_0, \hat{\mu}_1$ are sufficiently small, an additional condition is satisfied:

3) $$\hat{d}'''_{n_0^3}(\hat{n}_0, \hat{\mu}_0, \hat{\mu}_1) \neq 0. \qquad (51)$$

This follows from (47).

Suppose that all these conditions are satisfied. Then

$$\frac{dx}{dt} = \hat{P} - \hat{\mu}_0 \hat{Q} + \hat{\mu}_1 \hat{F}\hat{F}'_x, \quad \frac{dy}{dt} = \hat{Q} + \hat{\mu}_0 \hat{P} + \hat{\mu}_1 \hat{F}\hat{F}'_y \qquad (\hat{A})$$

is an analytical system δ-close to (A) which by (50) and (51) has a triple limit cycle $\hat{L}_0$ corresponding to the value $\hat{n}_0$ of the parameter n_0 and contained in $U_\varepsilon(L_0)$. This completes the proof of the lemma for the case $m = 3$.

For $m = 2$, the system $(\tilde{A}_\mu)$ should be taken in the form

$$\frac{dx}{dt} = P - \mu_0 Q, \quad \frac{dy}{dt} = Q + \mu_0 P,$$

and in the general case, we should take the system

$$\frac{dx}{dt} = P - \mu_0 Q + \mu_1 FF'_x + \mu_2 F^2 F'_x + \ldots + \mu_{m-2} F^{m-2} F'_x,$$
$$\frac{dy}{dt} = Q + \mu_0 P + \mu_1 FF'_y + \mu_2 F^2 F'_y + \ldots + \mu_{m-2} F^{m-2} F'_y.$$

In all other respects, the proof for $m \neq 3$ is the same as for $m = 3$.

Theorem 58. If the dynamic system (A) *is a system of the first degree of structural instability in* W, *it may only have isolated closed paths in this region.*

Proof. Let L_0 be a closed path of (A) in W. We will consider, as before, the normal l to the path L_0 at one of its points and the function $d(n_0)$. Let this function be defined for all n_0, $|n_0| < n_0^*$, and let the path L_0 correspond to $n_0 = 0$. H is the region introduced in the definition of a system of the first degree of structural instability, $H \supset W$.

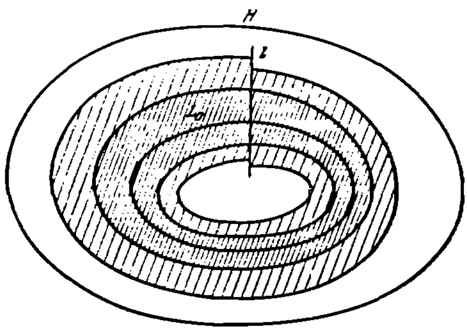

FIGURE 148

Let Ω, $\Omega \subset H$, be a neighborhood of the path L_0 with the following property: any path of (A) passing through a point of Ω crosses the normal l both for increasing and decreasing t in the segment $|n_0| < n_0^*$. A sufficiently small canonical neighborhood of L_0 may be chosen as Ω (Figure 148).

Consider a neighborhood U of the path L contained inside Ω at a positive distance from its boundary (in Figure 148, Ω is the entire diagonally hatched region and U is the densely hatched region). Let $\varepsilon > 0$ be so small than an ε-translation leaves the neighborhood U inside Ω. Let further $\delta > 0$ be the number corresponding to ε according to the terms of Definition 30. Let δ_1, $0 < \delta_1 < \delta$, be so small that if $(\hat{A})$ is δ_1-close to (A), each path of $(\hat{A})$ passing through points of the region Ω crosses the arc without contact l in the segment $|n_0| < n_0^*$ both for increasing and decreasing t, and the function $\hat{d}(n_0)$ of $(\hat{A})$ is defined for the corresponding values of the parameter n_0. By assumption $d(0) = 0$. Since (A) is a system of class 3, the numbers $d'(0), d''(0), d'''(0)$ exist. If at least one of these numbers does not vanish, L_0 is a limit cycle (a simple, a double, or a triple one), i.e., it is an isolated closed path, and the theorem is proved.

Now suppose that

$$d(0) = d'(0) = d''(0) = d'''(0) = 0.$$

Consider the system

$$\frac{dx}{dt} = P + \mu F^3 F'_x, \quad \frac{dy}{dt} = Q + \mu F^3 F'_y. \tag{$\tilde{A}$}$$

For $\mu \neq 0$ and a sufficiently small δ_1, $(\tilde{A})$ is $\delta_1/2$-close to (A), and by Lemma 8, L_0 is a limit cycle of multiplicity 3 of this system.

By Lemma 9, there exists an analytical system $(\hat{A})$ $\delta_1/2$-close to $(\tilde{A})$ which has a limit cycle $\hat{L}_0$ of multiplicity 3 in any arbitrarily small neighborhood of the path L_0. System $(\hat{A})$ is δ_1-close to (A) and is structurally unstable. Therefore, $(H, A) \overset{\varepsilon}{\equiv} (\hat{H}, \hat{A})$, and consequently

$$(U, A) \overset{\varepsilon}{\equiv} (\hat{U}, \hat{A}), \tag{52}$$

where $\hat{U}$ is some region. Since $\hat{U}$ is obtained by ε-translation from U, we have $\hat{U} \subset \Omega$.

Consider the function $\hat{d}(n_0)$ corresponding to the system ($\hat{\text{A}}$). It is defined for all n_0, $|n_0| < n_0^*$, and is analytical. Let $\hat{n}_0$ be the value of the parameter n_0 corresponding to the cycle $\hat{L}_0$. We may take $|\hat{n}_0| < n_0^*$. Since $\hat{L}_0$ is a limit cycle of multiplicity 3, $\hat{d}'''(\hat{n}_0) \neq 0$ and therefore $\hat{d}(n_0) \not\equiv 0$. But then, because of analyticity, the function $\hat{d}(n_0)$ may have only a finite number of roots for $|n_0| < n_0^*$, i.e., ($\hat{\text{A}}$) may only have a finite number of closed paths in Ω, and therefore in U. These paths are isolated, i.e., they are limit cycles. The mapping of $\hat{U}$ onto U which implements the relation (52) moves one of these cycles into the path L_0. Therefore, L_0 is also a limit cycle, i.e., an isolated closed path. This completes the proof of the theorem.

Theorem 59. If (A) *is a system of the first degree of structural instability in* W, *every closed path* L_0 *of the system contained in* W *is either a simple (i.e., structurally stable) or a double (of multiplicity 2) limit cycle.*

Proof. Suppose that the theorem is not true, i.e., (A) has a closed path L_0 in W which is neither a simple nor a double limit cycle of the system. Let L_0 correspond to the value $n_0 = 0$ of the parameter on the normal l to the path. Let $f(n_0)$ be the succession function on l, which is a priori defined for all n_0, $|n_0| < n_0^*$, and $d(n_0) = f(n_0) - n_0$.

Then

$$d(0) = d'(0) = d''(0) = 0. \tag{53}$$

According to the previous theorem, L_0 is a limit cycle of (A). Therefore, if $\eta > 0$ is sufficiently small, L_0 is the only closed path in $U_\eta(L_0)$, and all the other paths passing through this neighborhood wind onto L_0. Let this condition be satisfied. Moreover, let $U_\eta(L_0) \subset H$, where H is the region introduced in Definition 30.

Let V be a neighborhood of the path L_0 such that $\bar{V} \subset U_\eta(L_0)$; let $\varepsilon > 0$ be so small that if $\tilde{U}$ is generated from U_η by an ε-translation, then $\tilde{U} \supset V$. Let further $\delta > 0$ be the number corresponding to this ε in virtue of Definition 30. Then for any structurally unstable system ($\tilde{\text{A}}$) δ-close to (A) we have $(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A})$ and therefore

$$(U_\eta(L_0), A) \overset{\varepsilon}{\equiv} (\tilde{U}, \tilde{A}), \tag{54}$$

where $\tilde{U}$ is some region containing V.

Since L_0 is a limit cycle of (A), $d(n_0)$ retains a constant sign for all sufficiently small $n_0 > 0$. Suppose that $d(n_0) > 0$ and $n_0^{(1)}$ is a sufficiently small number. Then $d(n_0^{(1)}) > 0$.

Consider the system

$$\frac{dx}{dt} = P + \mu F^2 F'_x, \quad \frac{dy}{dt} = Q + \mu F^2 F'_y, \tag{$\tilde{\text{A}}$}$$

where $F(x, y)$ is the function of class 4 repeatedly encountered in the preceding. $\tilde{d}(n_0)$ is the function corresponding to ($\tilde{\text{A}}$).

Let $\mu \neq 0$ be so small that the following conditions are satisfied:

1) ($\tilde{\text{A}}$) is δ-close to (A).
2) $\tilde{d}(n_0^{(1)}) > 0$.

By (53) and Lemma 8 we see that the path L_0 of (A) is a double limit cycle of ($\tilde{A}$) and that $\tilde{d}(0) = \tilde{d}'(0) = 0$, $\tilde{d}''(0) = \mu\beta$, where $\beta \neq 0$ is a constant independent of μ. If the sign of μ is chosen as minus the sign of β, an additional condition is satisfied:

3) $\tilde{d}''(0) = \mu\beta < 0$.

By Maclaurin's formula $d(n_0) = \frac{d''(0)}{2!} n^2 + o(n_0^2)$, and for sufficiently small n_0, $d(n_0) < 0$.

Let $n_0^{(2)} > 0$, $n_0^{(2)} < n_0^{(1)}$ be sufficiently small. Then

4) $\tilde{d}(n_0^{(2)}) < 0$.

From conditions 2 and 4 it follows that there exists $n_0^{(3)}$, $n_0^{(2)} < n_0^{(3)} < n_0^{(1)}$, such that $\tilde{d}(n_0^{(3)}) = 0$. The number $n_0^{(3)}$ corresponds to a closed path $\tilde{L}$ of system ($\tilde{A}$) which does not coincide with L_0. If $n_0^{(1)}$ and μ are sufficiently small, we have $\tilde{L} \subset V$. Thus, for a sufficiently small μ of an appropriate sign, ($\tilde{A}$) has at least two closed paths L_0 and $\tilde{L}$ in V and hence in $\tilde{U}$. Since L_0 is a limit cycle of multiplicity 2, ($\tilde{A}$) is structurally unstable. This contradicts relation (54), which indicates that $\tilde{U}$ contains only one closed path of ($\tilde{A}$). The theorem is proved.

R e m a r k . We considered systems of class 3, i.e., structural instability of the first degree was treated in relation to the space $R_3^{(1)}$ (see § 5.1). This proof is inapplicable to the class of analytical functions, since the analytical function $F(x, y)$ with the desired properties in general can be constructed only in a neighborhood of the path L_0, and not in the entire region $\bar{G}$. To prove the theorem in the analytical case, we may proceed as follows: construct, as before, a system ($\tilde{A}$) of class 3 which has a double cycle L_0 and a closed path $\tilde{L}$ in V. By Lemma 2, § 15.2, there exists an arbitrarily close system of class 3, ($\tilde{A}_1$), which coincides with ($\tilde{A}$) everywhere except a small neighborhood of the path $\tilde{L}$ and which has in that neighborhood, and therefore in V also, a structurally stable limit cycle $\tilde{L}_1$. By Lemma 9, on the other hand, there exists an analytical system ($\tilde{A}_2$) arbitrarily close to ($\tilde{A}_1$) which has a double limit cycle in a neighborhood of the path L_0. If ($\tilde{A}_2$) is sufficiently close to ($\tilde{A}_1$), it has a structurally stable limit cycle $\tilde{L}_2$ in the neighborhood of the cycle $\tilde{L}_1$. Thus, the analytical system ($\tilde{A}_2$) has two closed paths in V, and we again end up with a contradiction to identity (54).

4. A saddle-point separatrix forming a loop

A saddle-to-saddle separatrix was considered in detail in Chapters IV and XI. In Chapter IV (§ 11) we established that a structurally stable system can have no such separatrices. In Chapter XI we considered a separatrix forming a loop and derived some of its properties.

In this subsection we will deal with a saddle-to-saddle separatrix of a system of the first degree of structural instability. The case of a separatrix between two different saddle points is of no interest for our purposes, since these separatrices are not characterized by any additional new properties in systems of the first degree of structural instability. We will therefore consider the case when a saddle-point separatrix goes to the same saddle point for both $t \to -\infty$ and $t \to +\infty$, i.e., it forms a loop. The results follow almost directly from the findings of Chapter XI.

Let (A) be a system of the first degree of structural instability in W, $O(x_0, y_0)$ a saddle point of (A), L_0 a separatrix of the saddle point O which is contained in W and forms a loop there. As in Chapter XI, we assume that the other two separatrices of the saddle point O lie outside the loop formed by the separatrix L_0.

Lemma 10. There exists $\varepsilon_0 > 0$ such that the ε_0-neighborhood of the loop L_0 does not contain any closed paths of (A).

Proof. Suppose that the lemma is not true, i.e., any neighborhood of the loop L_0 contains closed paths of (A). Then by Theorem 44, § 29

$$\sigma_0 = P'_x(x_0, y_0) + Q'_y(x_0, y_0) = 0. \tag{55}$$

Closed paths passing sufficiently close to L_0 clearly may only lie inside the loop.

Let H be the region introduced in Definition 30, $H \supset W$. Let further $\eta > 0$ be so small that the neighborhood $U_\eta(L_0) \subset H$ does not contain any equilibrium states of (A), except the point O, nor any closed paths which lie inside the loop L_0 (Figure 149). Take some ε, $0 < \varepsilon < \frac{\eta}{4}$. Definition 30 assigns a certain number $\delta > 0$ to this ε such that if $(\tilde{A})$ is δ-close to (A) and is structurally unstable, we have $(H, A) \stackrel{\varepsilon}{\equiv} (\tilde{H}, \tilde{A})$. Then

$$(U_\eta(L_0), A) \stackrel{\varepsilon}{\equiv} (\tilde{V}, \tilde{A}), \tag{56}$$

where $\tilde{V}$ is some region.

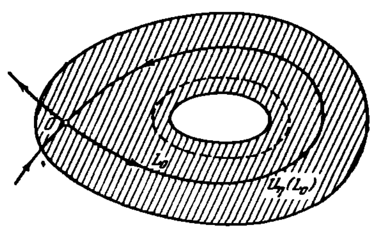

FIGURE 149

From identity (56) and the inequality $\varepsilon < \frac{\eta}{4}$ we have

$$U_{\eta/2}(L_0) \subset \tilde{V}.$$

We moreover see from (56) that $\tilde{V}$ contains precisely one saddle point $\tilde{O}$ of system $(\tilde{A})$ and precisely one loop $\tilde{L}_0$ formed by the path of $(\tilde{A})$ corresponding to the path L_0. Finally, by the same relation (56), any neighborhood of the loop $\tilde{L}_0$ should contain closed paths of $(\tilde{A})$. This, however, contradicts the results of Chapter XI. Indeed, by Lemma 10, § 29.4, there exists a system $(\tilde{A})$ δ-close to (A) for which O is a saddle point; this saddle point has a separatrix $\tilde{L}_0$ entirely contained in $U_{\frac{\eta}{2}}(L_0)$ and moreover

$$\tilde{\sigma}_0(x_0, y_0) = \tilde{P}'_x(x_0, y_0) + \tilde{Q}'_y(x_0, y_0) > 0.$$

By Theorem 44, §29.1, on the other hand, the loop L_0 is unstable, i.e., a sufficiently small neighborhood of the loop can contain no closed paths. This proves the lemma.

Lemma 10 evidently implies that if (A) has in W a separatrix which forms a loop, this loop is either stable or unstable.

Theorem 60. If a system of the first degree of structural instability in W has a saddle-point O (x_0, y_0) whose separatrix forms a loop contained in W, then $\sigma_0(x_0, y_0) \neq 0$.

Proof. Suppose that the theorem is not true, i.e., system (A) has a saddle point O (x_0, y_0) whose separatrix L_0 is entirely contained in W and forms a loop, and yet $\sigma_0(x_0, y_0) = 0$.

In our proof to Theorem 50 (§29.4) we showed that if the loop L_0 is stable (or unstable), and $\sigma_0(x_0, y_0) = 0$, then for any $\eta > 0$ and $\delta > 0$ there exists a system $(\tilde{A})$ δ-close to (A) which has in $U_{\frac{\eta}{2}}(L_0)$ a separatrix that forms a loop and at least one closed path.

$(\tilde{A})$ is a structurally unstable system, and therefore for appropriate ε and δ, relation (56) should be satisfied for this system. On the other hand, this relation cannot be true since for sufficiently small η and ε, $U_\eta(L_0)$ does not contain closed paths of (A) and $\tilde{V}$ contains at least one closed path of $(\tilde{A})$. The contradiction establishes the validity of the theorem.

Theorem 60 signifies that if a system of the first degree of structural instability in W has a saddle point O (x_0, y_0) for which $\sigma_0(x_0, y_0) = 0$, no separatrix exists for this saddle point which forms a loop and is contained entirely in W.

5. The simplest structurally unstable paths

As we know, singular paths of a dynamic system, i.e., equilibrium states, limit cycles, and separatrices, are the most important elements for the analysis of the topological structure of a dynamic system on a plane. By Theorem 23, § 18.2, structurally stable systems may only have singular paths of the following types:

(a) structurally stable equilibrium states — nodes, saddle points, and simple foci;

(b) structurally stable limit cycles;

(c) saddle-point separatrices going to a simple node, a simple focus, or a simple limit cycle or leaving the region of definition.

Any structurally unstable system, in particular a system of the first degree of structural instability, should have at least one singular path which does not fit the above classification. Consider the following additional types of paths:

1. A multiple focus of multiplicity 1.
2. A saddle-node of multiplicity 2 with $\sigma = P'_x + Q'_y \neq 0$.
3. A limit cycle of multiplicity 2.
4. A separatrix from one saddle point to another saddle point.
5. A separatrix of saddle point M (x_0, y_0) which forms a loop when $\sigma(x_0, y_0) \neq 0$.

The paths of these five types will be called the *simplest structurally unstable paths*. The existence of these paths in systems of the first degree of structural instability does not contradict Theorems 53 through 60. On the other hand, a dynamic system (A) of the first degree of structural instability in W should have in this region at least one simplest structurally unstable path (otherwise, it would be structurally stable in W).

We will show in this subsection that a dynamic system of the first degree of structural instability in W cannot have more than one simplest structurally unstable path in this region.

First we shall prove a number of lemmas.

Lemma 11. Let L_0 be a separatrix of the saddle point O of a dynamic system (A) *of class N which extends to another saddle point O_1. There exists a simple closed curve of class k (k being a given number $\leqslant N+1$) which passes through the points O and O_1, encloses L_0, and does not enclose any other separatrix or any equilibrium state of system* (A).

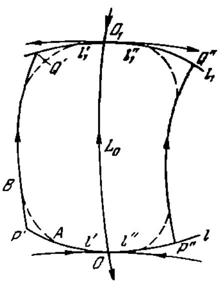

FIGURE 150

Proof. We shall first show that an arc l of a parabola can be passed through the point O, which has no contacts with the paths of system (A) except at the point O (Figure 150). Without loss of generality, we may place the point O at the origin and represent the system (A) in the canonical form

$$\frac{dx}{dt}=\lambda_1 x+\varphi(x,\ y),\qquad \frac{dy}{dt}=\lambda_2 y+\psi(x,\ y), \tag{57}$$

where φ and ψ are functions which vanish at the point $O(0,0)$ together with their first order derivatives (see §8.1, (1), (2)), and $\lambda_1\lambda_2<0$.

By assumption, all our systems, system (A) included, are systems of class 3. The functions φ and ψ, by Theorem 5 of the Appendix (see Appendix, subsection 2), may therefore be written in the form

$$\begin{aligned}\varphi(x,\ y)&=a_{11}x^2+a_{12}xy+a_{22}y^2+\sum_{\alpha=0}^{2}x^{2-\alpha}y^{\alpha}P_{\alpha}^{*}(x,\ y),\\ \psi(x,\ y)&=b_{11}x^2+b_{12}xy+b_{22}y^2+\sum_{\alpha=0}^{2}x^{2-\alpha}y^{\alpha}Q_{\alpha}^{*}(x,\ y),\end{aligned} \tag{58}$$

where P_{α}^{*} and Q_{α}^{*} are continuous functions, which vanish at the point $O(0,0)$.

Consider the parabola $y=Cx^2$, where the value of the coefficient C will be chosen at a later stage. The condition that the parabola $y=Cx^2$ is tangent to a path of system (57) can be written at the point of contact in the form

$$2Cx\,[\lambda_1 x+\varphi(x,\ y)]-[\lambda_2 y+\psi(x,\ y)]=0.$$

Inserting for φ and ψ their expressions from (58) and substituting Cx^2 for y, we obtain

$$x^2\,[(2\lambda_1-\lambda_2)\,C-b_{11}]+o\,(x^2)=0. \tag{59}$$

Since $\lambda_1\lambda_2<0$, we have $2\lambda_1-\lambda_2\neq 0$. Therefore, for any $C\neq\frac{b_{11}}{2\lambda_1-\lambda_2}$ and for all x, $|x|<x_0$, where x_0 is a sufficiently small positive number, the arc

of the parabola $y = Cx^2$ is without contact with the paths of system (57), except at the point O. We designate this arc l.

The arc l clearly can be drawn so that the separtrix L_0 of the saddle point O lies on one side of the arc l (on the concave side of the parabola), and the other three separatrices lie on the other side of the arc. An analogous arc l_1 of the parabola can be drawn through the saddle point O_1 (Figure 150).

Let O be an α-limit point of the path L_0, and O_1 its ω-limit point. The points O and O_1 respectively divide each of the arcs l and l_1 into two parts: l' and l'', l_1' and l_1''. It is readily seen that any path crossing the arc l' sufficiently close to the point O will also cross the arc l_1', and any path crossing the arc l'' will cross l_1''. Choose one arc of each of these pairs. We obtain a simple piecewise-smooth closed curve made up of the segments $P'P''$ and $Q'Q''$ of the arcs l and l_1 and the arcs $P'Q'$ and $P''Q''$ of paths. This closed curve evidently encloses the separatrix L_0.

On the arc OP' we choose a point A which does not coincide either with O or with P', and on the arc $P'Q'$ of a path we choose a point B. Since l is an arc without contact, we can always join the point A to the point B by an arc (without contact), which at the point A and at the point B has a point of contact of any desired order $k \leqslant N+1$ with the arc l and respectively with the arc of the path (this is readily proved using QT, § 3.5, Lemma 8). Drawing three other analogous arcs (Figure 150) we obtain a simple closed curve of class k which completely encloses the separatrix L_0 and does not enclose other separatrices or any equilibrium states of the system. Q.E.D.

Remark. The number k, in general, should be $\leqslant N+1$, since the paths of systems of class N are curves of class $N+1$, and according to the construction used in the proof of the lemma, segments of these curves are included in the closed curve.

Lemma 12. Let L_0 be a separatrix of system (A) *of class N which originates and ends in the saddle point O (forming a loop L), and let there be a neighborhood of L which does not contain any closed paths (so that the loop is stable or unstable). Then there exist two simple closed curves C' and C'' of class k, where k is any fixed integer, $k \leqslant N+1$, one of which encloses the loop L and the other is enclosed by the loop L, such that the annular region between C' and C'' contains no equilibrium states, no closed curves, and no separatrices (except L) of system* (A).

Proof. Suppose that the other two separatrices of the saddle-point O (different from L_0) lie outside the loop formed by the path L.

We can always pass through the saddle point O a segment of a straight line which will be without contact at all points sufficiently close to O (other than O itself) (see QT, § 7.3, p. 155). Let l be such a segment which contains O and is without contact with the paths of (A) at all points other than O (Figure 151).

Let l' and l'' be the two segments into which l is divided by the saddle point O. It is readily seen that any path crossing the segment l' at a point P' sufficiently close to O will cross l'' at some point P''.

We thus obtain a simple closed piecewise-smooth curve enclosing the loop L, such that the region between this curve and L contains no equilibrium states, no closed paths, and no separatrices (this is readily

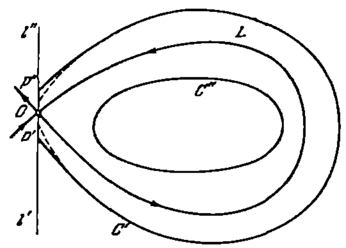

FIGURE 151

verified, since arcs of paths crossing the segments l' and l'' of the arc l pass through all the points of this annular region). "Smoothing" this piecewise-smooth curve by the same technique as in the previous lemma, we obtain one of the two simple closed curves, C' say, postulated by the lemma.

The second curve C'' can be chosen as a cycle without contact, sufficiently close to the "loop" (Figure 151). The existence of this cycle is established in QT, §24.3, Lemma 2.

The same proof can be used in the case when the two other separatrices of the saddle point O lie inside the loop formed by the separatrix L.

This completes the proof of the lemma.

L e m m a 13. *Let γ be a simple closed curve of class k in some region $\bar{G}$. There exists a function $z = \Phi(x, y)$ of class $k-1$, defined in $\bar{G}$, which vanishes at the points of the curve γ, takes on positive values inside γ and negative values outside γ, and at the points of the curve γ satisfies the relation*

$$\Phi_x'^2 + \Phi_y'^2 \neq 0.$$

P r o o f . Choose the arc length s reckoned from some fixed point of the curve as the parameter of γ. Then, as is readily seen, γ can be written in parametric form $x = \varphi(s)$, $y = \psi(s)$, where φ and ψ are functions of class $k+1$, s varies from 0 to some τ, and $\varphi(\tau) = \varphi(0)$, $\psi(\tau) = \psi(0)$. Since γ is smooth, $\varphi'(s)$ and $\psi'(s)$ do not vanish simultaneously for any s, $0 \leqslant s \leqslant \tau$.

In the neighborhood of γ we define a curvilinear system of coordinates (s, n) by the equalities

$$x = \varphi(s) + n\psi'(s), \qquad y = \psi(s) - n\varphi'(s), \tag{60}$$

where the right-hand sides are evidently functions of class k. According to Chapter V (see §15.1, Lemma 1 and Remark 1 to the lemma), equations (60) define n as a single-valued function of the coordinates x and y in some neighborhood of the curve γ:

$$n = F(x, y),$$

where F is a function of class $k-1$ which vanishes on γ, is positive on one side of γ and negative on the other side of γ, and on γ satisfies the relation $F_x'^2 + F_y'^2 \neq 0$. We can choose the direction of the normals so that the function $F(x, y)$ is positive inside the curve γ and negative outside this curve. Consider any two closed curves γ_1 and γ_2 defined by the equations $n = n_1$ and $n = n_2$, respectively, i.e., $F(x, y) = n_1$ and $F(x, y) = n_2$, where the numbers n_1 and n_2 are sufficiently small and $0 < n_1 < n_2$ (Figure 152).

Let $f = f(n)$ be a function of class $k-1$ defined for $0 \leqslant n < \infty$, which satisfies the following conditions:

1) $f(n) \equiv n$ for $0 \leqslant n \leqslant n_1$;

2) $f(n) \equiv n_2$ for $n \geqslant n_2$;

3) $n_1 \leqslant f(n) \leqslant n_2$ for $n_1 < n < n_2$ (a specimen graph of the function $f(n)$ is shown in Figure 153; a similar function was constructed in §15.2

in the proof of Lemma 2). The function $\Phi(x, y)$ will be defined inside the curve γ by the following conditions:

1) in the ring between γ and γ_2,

$$\Phi(x, y) = f(n) = f(F(x, y));$$

2) inside γ_2,

$$\Phi(x, y) \equiv n_2.$$

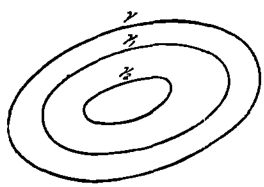

FIGURE 152

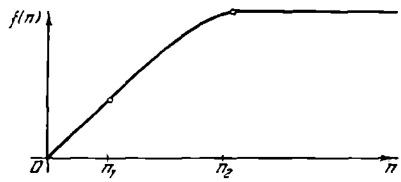

FIGURE 153

$\Phi(x, y)$ is similarly defined outside the curve γ. The function Φ constructed in this way evidently meets all the conditions of the lemma. Q.E.D.

Lemma 14. Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system of class N, L its separatrix extending from saddle point O' to saddle point O'' (O' and O'' may be the same point). For any $\delta > 0$ and $r \leqslant N$ (r is a natural number), there exists an analytical system $(\tilde{A})$ δ-close to rank r to (A) which has a saddle-to-saddle separatrix.

Proof. Through some point M of the separatrix L, we draw an arc without contact l and define a parameter u on this arc, so that the point M corresponds to $u = 0$. To fix ideas, let the separatrix L make a positive angle with the arc l (Figure 154).

By the fundamental theorem of the structural stability of dynamic systems (Theorem 23, § 18.2), the dynamic system (A) is structurally stable in sufficiently small neighborhoods of the segments $O'M$ and $O''M$ of the separatrix L. Hence it follows, as is readily seen, that a number $\eta > 0$ exists with the following property: if $(\tilde{A})$ is η-close to (A), then in a sufficiently small neighborhood of the saddle point O' (O'') there exists a single saddle point $\tilde{O}'$ ($\tilde{O}''$) of the system $(\tilde{A})$ and the separatrix $\tilde{L}'$ ($\tilde{L}''$) of the saddle point $\tilde{O}'$ ($\tilde{O}''$) crosses the arc without contact l at the point $\tilde{M}'$ ($\tilde{M}''$) corresponding to the value $\tilde{u}'$ ($\tilde{u}''$) of the parameter u, so that the segment $\tilde{O}'\tilde{M}'$ ($\tilde{O}''\tilde{M}''$) of the separatrix $\tilde{L}'$ ($\tilde{L}''$) is contained inside an arbitrarily small neighborhood of the segment $O'M$ ($O''M$) of the separatrix L (Figure 154).

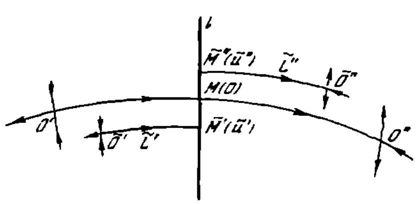

FIGURE 154

Let δ_1 be a positive number, $\delta_1 < \frac{\eta}{2}$, $\delta_1 < \frac{\delta}{2}$.

Consider the system

$$\frac{dx}{dt} = P - \lambda Q, \quad \frac{dy}{dt} = Q + \lambda P. \qquad (A_\lambda)$$

Let $\lambda_0 > 0$ be so small that for all λ, $|\lambda| < \lambda_0$, (A_λ) is δ_1-close to rank r to (A). Let $\lambda_1 > 0$, $\lambda_1 < \lambda_0$, $\lambda_2 < 0$, $|\lambda_2| < \lambda_0$. On passing from (A) to (A_{λ_i}) $(i = 1, 2)$, the separatrix L branches into two separatrices L_i' and L_i'' which cross the arc l at the points N_i' and N_i'', respectively. Let these points correspond to the values $u'(\lambda_i)$ and $u''(\lambda_i)$ of the parameter u. By the lemma of §11.1, we have

$$\begin{aligned} &u'(\lambda_1) > 0, \quad u''(\lambda_1) < 0,\\ &u'(\lambda_2) < 0, \quad u''(\lambda_2) > 0, \end{aligned} \qquad (61)$$

(Figure 155).

Let P^* and Q^* be polynomials which provide a sufficiently close fit of the functions P and Q so that the following conditions be satisfied:

1) For all λ, $|\lambda| < \lambda_0$, the system

$$\frac{dx}{dt} = P^* - \lambda Q^*, \quad \frac{dy}{dt} = Q^* + \lambda P^* \qquad (\tilde{A}_\lambda)$$

is δ_1-close to rank r to (A).

2) The separatrices $\tilde{L}'(\lambda_1)$ and $\tilde{L}''(\lambda_1)$ of $(\tilde{A}_{\lambda_1})$ cross the arc without contact l at the points $\tilde{N}'(\lambda_1)$ and $\tilde{N}''(\lambda_1)$ corresponding to the values $\tilde{u}'(\lambda_1)$ and $\tilde{u}''(\lambda_1)$ of the parameter which are so close to the points N_1' and N_1'' that

$$\tilde{u}'(\lambda_1) > 0, \quad \tilde{u}''(\lambda_1) < 0. \qquad (62)$$

3) The separatrices $\tilde{L}'(\lambda_2)$ and $\tilde{L}''(\lambda_2)$ of system $(\tilde{A}_{\lambda_2})$ cross the arc without contact l at the points $\tilde{N}'(\lambda_2)$ and $\tilde{N}''(\lambda_2)$ corresponding to the values $\tilde{u}'(\lambda_2)$ and $\tilde{u}''(\lambda_2)$ of the parameter which are so close to the points N_2' and N_2'' that

$$\tilde{u}'(\lambda_2) < 0, \quad \tilde{u}''(\lambda_2) > 0. \qquad (63)$$

By (62) and (63), we have

$$\begin{aligned} &\tilde{u}'(\lambda_1) - \tilde{u}''(\lambda_1) > 0,\\ &\tilde{u}'(\lambda_2) - \tilde{u}''(\lambda_2) < 0. \end{aligned}$$

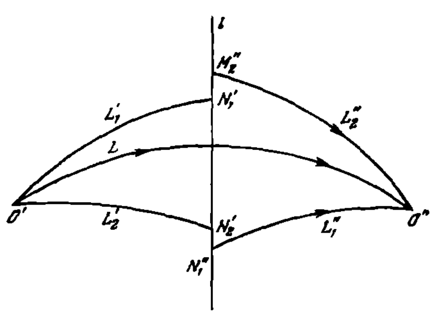

FIGURE 155

Since $\tilde{u}'$ and $\tilde{u}''$ are continuous functions of the parameter λ, there exists a number $\bar{\lambda}$, $\lambda_2 < \bar{\lambda} < \lambda_1$, such that

$$\tilde{u}'(\bar{\lambda}) = \tilde{u}''(\bar{\lambda}).$$

This implies that the separatrix of the system $(\tilde{A}_{\bar{\lambda}})$ extends from saddle point $\tilde{O}'$ to saddle point $\tilde{O}''$. Since $(\tilde{A}_{\bar{\lambda}})$ is δ-close to rank r to (A) and is an analytical system, it satisfies all the propositions of the lemma. Q.E.D.

Theorem 61. If (A) *is a dynamic system of the first degree of structural instability in* W, *it may have at most one simplest structurally unstable path in this region.*

Proof. Suppose that (A) has two simplest structurally unstable paths in W. In our proof, we do not distinguish between paths of type 4 and type 5 (see p. 351), i.e., a saddle-to-saddle separatrix may be a separatrix between two different saddle points or a separatrix forming a loop for a saddle point with $\sigma \neq 0$. We moreover assume that all the systems are analytical, i.e., we are dealing with structural instability of the first degree with respect to the space $R_a^{(r)}$, where $r > 3$. Our proof remains in force, as is readily seen, in the nonanalytical case as well (i.e., for the space $R_N^{(r)}$, $3 \leqslant r \leqslant N$), and some arguments can actually be simplified in the nonanalytical case.

Let us consider successively all the possible cases of two simplest structurally unstable paths in W.

1) System (A) has two saddle-nodes in W.

Let one of these saddle-nodes be the point $O(0, 0)$, and the other $O_1(a, b)$, and let the system (A) have the form

$$\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = y + q(x, y), \tag{A}$$

where $p(0, 0) = p'_x(0, 0) = p'_y(0, 0) = q(0, 0) = q'_x(0, 0) = q'_y(0, 0) = 0$ (see § 31.1, (11) and (12)).

Since $O_1(a, b)$ is a saddle-node, we have

$$\Delta(a, b) = \begin{vmatrix} p'_x(a, b) & p'_y(a, b) \\ q'_x(a, b) & 1 + q'_y(a, b) \end{vmatrix} = 0. \tag{64}$$

Consider the modified system

$$\frac{dx}{dt} = p(x, y) + \mu\,[(x^2 - a^2)^2 + (y^2 - b^2)^2] = \tilde{P}(x, y), \; \frac{dy}{dt} = y + q(x, y) = \tilde{Q}(x, y), \tag{$\tilde{A}$}$$

where $\mu \neq 0$. For any μ, the point $O_2(a, b)$ is a multiple equilibrium state of $(\tilde{A})$, since $\tilde{\Delta}(a, b) = \Delta(a, b) = 0$. $(\tilde{A})$ is therefore structurally unstable.

Let $y = \varphi(x)$ be a solution of the equation $y + q(x, y) = 0$ in the neighborhood of the point $O(0, 0)$, and let $\psi(x) = P(x, \varphi(x)) = p(x, \varphi(x))$. Since O is an equilibrium state of multiplicity 2 for system (A), the function $\psi(x)$ has the form

$$\psi(x) = \alpha_2 x^2 + \alpha_3 x^3 + \ldots, \tag{65}$$

where $\alpha_2 \neq 0$ (Theorem 33, § 23.1).

We will now search for the equilibrium states of $(\tilde{A})$ in the neighborhood of $O(0, 0)$. To this end, we have to solve the simultaneous equations

$$\tilde{P}(x, y) = 0, \quad \tilde{Q}(x, y) = 0$$

or, equivalently,

$$y = \varphi(x), \quad Q(x, \varphi(x)) = 0.$$

Since $\varphi(x) = \varphi'(x) = 0$ (see (15), (16)), the series expansion of the function $\varphi(x)$ does not contain any terms below second order. Therefore,

$$\tilde{Q}(x, \varphi(x)) = p(x, \varphi(x)) + \mu[(x^2 - a^2)^2 + (\varphi^2(x) - b^2)^2] =$$
$$= \alpha_2 x^2 + \alpha_3 x^3 + \ldots + \mu[a^4 + b^4] - 2\mu a^2 x^2 + \ldots = \mu(a^4 + b^4) + \alpha_2 x^2 + \ldots,$$

where $\alpha_2 \neq 0$, and the missing terms are of third or higher order in all the variables μ and x jointly.

We thus have to find the roots of the equation

$$\mu(a^4 + b^4) + \alpha_2 x^2 + \ldots = 0 \tag{66}$$

which are close to zero; in this equation $\mu \neq 0, \alpha_2 \neq 0$. A detailed investigation of the roots of this equation will be given in the next chapter, in §32. In particular, it follows from Lemma 2 of this section that if μ is sufficiently small and its sign is minus the sign of α_2, equation (66) has precisely two real roots, which both go to zero for $\mu \to 0$. This signifies that for a sufficiently small μ of an appropriate sign, system $(\tilde{A})$ has at least two equilibrium states in any neighborhood of the point $O(0, 0)$. Since $(\tilde{A})$ is structurally unstable, this contradicts Lemma 2 of the present section.

2) System (A) has a multiple focus (of multiplicity 1) and a saddle-node in W.

Without loss of generality, we may place the focus at the origin $O(0, 0)$ and write (A) in the form

$$\frac{dx}{dt} = -y + \varphi(x, y) = P(x, y), \quad \frac{dy}{dt} = x + \psi(x, y) = Q(x, y), \tag{A}$$

where φ and ψ are analytical functions whose series expansions start with quadratic terms. Let $O_1(a, b) \in W$ be the saddle-node of (A).

The modified system is taken in the form

$$\frac{dx}{dt} = P - \mu Q = \tilde{P}, \quad \frac{dy}{dt} = Q + \mu P = \tilde{Q}, \tag{$\tilde{A}$}$$

which is obtained from system (A) by rotating its vector field through the angle $\tan^{-1}\mu$.

Since $O_1(a, b)$ is a saddle-point of (A), we have $P(a, b) = Q(a, b) = \Delta(a, b) = 0$. From these relations and from the obvious equality $\tilde{\Delta}(x, y) = \Delta(x, y)(1 + \mu^2)$ we conclude that $\tilde{P}(a, b) = \tilde{Q}(a, b) = \tilde{\Delta}(a, b) = 0$, i.e., the point O_1 is a structurally unstable equilibrium state of $(\tilde{A})$.

$(\tilde{A})$ is thus a structurally unstable system. Reasoning as in the proof to Theorem 57 (§31.2, (33) and (34)), we can show that if μ is sufficiently small, $(\tilde{A})$ can have no closed paths in some neighborhood $\tilde{V}$ of O. By Remark 3 to Theorem 14, §10.3, for a sufficiently small μ of an appropriate

sign, $(\tilde{A})$ has at least one closed path in any arbitrarily small neighborhood of O. We have established a contradiction, which proves the theorem for this case.

3) System (A) has two multiple foci in W.

Let one of the foci be $O(0, 0)$ and the other $O_1(a, b)$. We write the system in the form

$$\frac{dx}{dt} = -y + \varphi(x, y) = P(x, y), \quad \frac{dy}{dt} = x + \psi(x, y) = Q(x, y). \tag{A}$$

The modified system is

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) - \mu[(x-a)^2 + (y-b)^2]Q = \tilde{P}(x, y), \\ \frac{dy}{dt} &= Q(x, y) + \mu[(x-a)^2 + (y-b)^2]P = \tilde{Q}(x, y). \end{aligned} \tag{$\tilde{A}$}$$

Since the point $O_1(a, b)$ is a multiple focus of (A), we have $\Delta(a, b) > 0$, $\sigma(a, b) = 0$. A direct check will show that $\tilde{\Delta}(a, b) = \Delta(a, b)$ and $\tilde{\sigma}(a, b) = \sigma(a, b)$. Therefore, $O_1(a, b)$ is an equilibrium state of $(\tilde{A})$ with pure imaginary characteristic numbers, i.e., $(\tilde{A})$ is structurally unstable.

$(\tilde{A})$ can be written in the form

$$\frac{dx}{dt} = -y - \mu(a^2 + b^2)x + \ldots, \quad \frac{dy}{dt} = x - \mu(a^2 + b^2)y + \ldots,$$

where the missing terms are of second or higher order. The point $O(0, 0)$ is therefore a structurally stable focus of $(\tilde{A})$, stable or unstable depending on the sign of μ. Then, as in Remark 3 to Theorem 14, we can show that for a sufficiently small μ of an appropriate sign, $(\tilde{A})$ has at least one closed path in any arbitrarily small neighborhood of O. We again reach a contradiction as in case 2.

4) System (A) has a saddle-node and a limit cycle of multiplicity 2 in W.

Let $O(0, 0)$ be the saddle-node and L the limit cycle. The modified system is again taken in the form

$$\frac{dx}{dt} = P - \mu Q = \tilde{P}, \quad \frac{dy}{dt} = Q + \mu P = \tilde{Q}, \tag{$\tilde{A}$}$$

which is obtained by an appropriate rotation of the vector field of (A). $(\tilde{A})$ is structurally unstable, since $O(0, 0)$ is its equilibrium state and $\Delta(0, 0) = 0$.

We shall again use one of the results of the next chapter, namely Theorem 71 (§ 32.4). By this theorem, for a sufficiently small μ of an appropriate sign, $(\tilde{A})$ has two closed paths in any neighborhood of the limit cycle L of multiplicity 2. On the other hand, L is a limit cycle of (A), and $(\tilde{A})$ is a structurally unstable system. Therefore, reasoning as we have done often before, we can show that for a sufficiently small μ, $(\tilde{A})$ may have only one closed path in a sufficiently small neighborhood of L. We have again established a contradiction.

5) System (A) has a saddle-node $O(0, 0)$ and a saddle-to-saddle separatrix L in W.

As in the previous case, $(\tilde{A})$ is chosen as the modified system.

The point $O(0, 0)$ is a multiple equilibrium state of $(\tilde{A})$, and therefore $(\tilde{A})$ is structurally unstable. Therefore, for every $\varepsilon > 0$, there exists μ_0, such that for $|\mu| < \mu_0$ we have the relation $(H, A) \overset{\varepsilon}{\equiv} (\tilde{H}, \tilde{A})$ and hence the

relation $(U(L), A) \overset{\varepsilon}{\equiv} (\widetilde{V}, \widetilde{A})$, where $U(L)$ is a neighborhood of the separatrix L. From the last relation it follows that $\widetilde{V}$ contains a saddle-to-saddle separatrix of (A). On the other hand, in our proof to Theorem 16, §11.2, we have seen that if the neighborhood $U(L)$ of the path L and the numbers ε and μ are sufficiently small, $\widetilde{V}$ may not contain any saddle-to-saddle separatrices of $(\tilde{A})$. We have established a contradiction.

6) System (A) has a multiple focus and a limit cycle of multiplicity 2 in W, and

7) System (A) has a multiple focus and a saddle-to-saddle separatrix in W.

The proof which rules out cases 6 and 7 is the same. Let $O(0,0)$ be the multiple focus. The modified system is taken in the form

$$\begin{aligned}\frac{dx}{dt} &= P(x, y) - \mu(x^2+y^2)Q(x, y) = \widetilde{P}(x, y),\\ \frac{dy}{dt} &= Q(x, y) + \mu(x^2+y^2)P(x, y) = \widetilde{Q}(x, y).\end{aligned} \qquad (\tilde{A})$$

The point $O(0,0)$ is a multiple focus of $(\tilde{A})$ also, and $(\tilde{A})$ is therefore structurally unstable. Furthermore, everywhere (with the exception of the point O), the vector field of $(\tilde{A})$ is obtained from the vector field of (A) by a rotation in the same sense (through an angle equal to $\tan^{-1}\mu(x^2+y^2)$). Therefore, case 6 leads to the same contradiction as case 4, using Theorem 71 and Remark 2 to Theorem 72. In case 7, the contradiction is established as in case 5, using Remark 2 to Theorem 16 (§11.2).

8) System (A) has two cycles of multiplicity 2 in W.

Let L_1 and L_2 be the two cycles, $U = U(L_2)$ an arbitrarily small neighborhood of L_2. Using the theorem of the creation of a closed path from a multiple limit cycle (§27.1, Theorem 42) and employing the same construction as in the proof of Lemma 2, §15.2, we obtain a system (A_1) of class $N \geqslant r$ as close as desired to (A) to rank $r \geqslant 3$, which coincides with (A) outside the neighborhood U and has two closed paths, which are structurally stable limit cycles, inside U. Furthermore, using Lemma 9 of this section, we conclude that there exists an analytical system $(\tilde{A})$ as close as desired to (A_1) to rank r, which has a limit cycle of multiplicity 2 in any arbitrarily small neighborhood of the cycle L_1 and is therefore structurally unstable. If (A_1) and $(\tilde{A})$ are sufficiently close, $(\tilde{A})$ will have at least two closed paths in $U(L_2)$. The contradiction is established as in case 4.

9) System (A) has a limit cycle L_1 of multiplicity 2 and a saddle-to-saddle separatrix L_2 in W (Figure 156a).

Let the separatrix L_2 extend from saddle point O_1 to saddle point O_2. We choose a sufficiently small neighborhood $U = U(L_2)$ of the separatrix L_2 which contains no equilibrium states except O_1 and O_2. Modifying the system (A) by a rotation of the vector field through a small constant angle and employing the same construction as for the proof of Lemma 2, §15.2, we obtain a system (A_1) of class $N \geqslant r$ arbitrarily close to rank $r \geqslant 3$ to (A) which

(a) coincides with (A) outside U;

(b) has no equilibrium states in U, except the saddle points O_1 and O_2;

(c) has no saddle-to-saddle separatrices in U (Figure 156b).

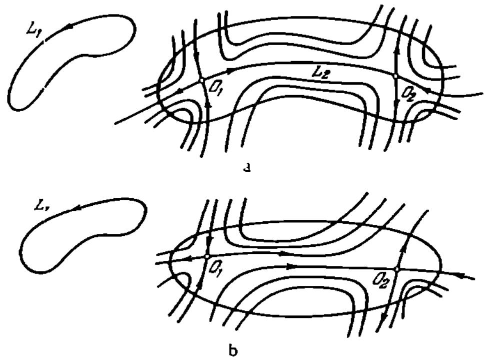

FIGURE 156

By Lemma 9 of the present section, there exists an analytical system $(\tilde{A})$ as close as desired to (A_1) to rank r which has a limit cycle of multiplicity 2 in any arbitrarily small neighborhood of the cycle L_1, and is therefore structurally unstable. But (A_1) is structurally stable in U. Therefore, if $(\tilde{A})$ is sufficiently close to (A_1), conditions (b) and (c) are satisfied for $(\tilde{A})$. We have thus established the existence of a structurally unstable system $(\tilde{A})$ arbitrarily close to (A) without any saddle-to-saddle separatrices in U. This leads to the same contradiction as in case 5.

10) System (A) has two saddle-to-saddle separatrices L_1 and L_2 in W.

Two different cases are possible:

(a) at least one of the separatrices L_1 and L_2 extends between two different saddle points;

(b) each separatrix forms a loop for its saddle point.

Let us first consider case (a). Let L_1 be a separatrix extending from saddle point O to another saddle point O_1, which does not coincide with O.

The separatrix L_2 either passes at a finite distance from L_1 or at least one of the two equilibrium states of this separatrix coincides with O or O_1. Since none of the points of the separatrix L_1 is a limit point for L_2, the simple closed curve of class k (where k is a priori known to be equal to 4) whose existence is established in Lemma 11 may be chosen so that it encloses L_1 without enclosing L_2.

In case (b), several subcases should be considered. Specifically, the distance between the loops formed by the separatrices L_1 and L_2 may be either positive or zero, one of the two loops may enclose the other loop or they may lie one outside the other (Figures 157 and 158).

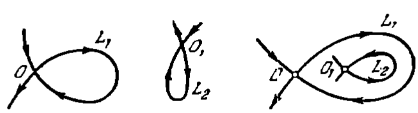

FIGURE 157

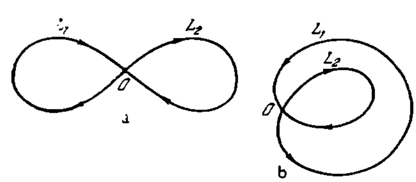

FIGURE 158

Applying Lemma 12, however, it is readily seen that for each of these subcases, there exists a simple closed curve of class k which encloses one of the separatrices L_1 and L_2, without enclosing the other.

Thus, a simple closed curve γ of class k can always be chosen so that it encloses the separatrix L_1, say, and leaves the separatrix L_2 outside.

Let $z = \Phi(x, y)$ be the function of class $k-1$ defined by Lemma 13, i.e., a function which is zero on the curve γ, positive inside γ, and negative outside γ.

Consider the modified system

$$\dot{x} = P - \lambda Q - \mu\Phi Q, \quad \dot{y} = Q + \lambda P + \mu\Phi P. \qquad (A_{\lambda\mu})$$

Through a point M_0 of the separatrix L_1 we draw an arc without contact l_1, and through a point N_0 of the separatrix L_2 we draw an arc without contact l_2 (Figure 159).

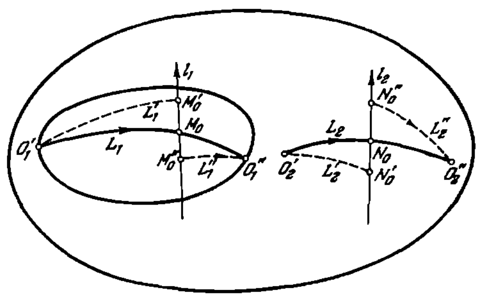

FIGURE 159

Let u and v be the parameters defined on the arcs l_1 and l_2, respectively, so that $u = u_0$ at the point M_0 and $v = v_0$ at the point N_0.

For all sufficiently small λ and μ, there exist separatrices L_1' and L_1'' of $(A_{\lambda\mu})$ which cross the arc l_1 at the points $M_0'(u_0')$ and $M_0''(u_0'')$, as close as desired to M_0, and separatrices L_2' and L_2'' which cross the arc l_2 at the points $N_0'(v_0')$ and $N_0''(v_0'')$, as close as desired to N_0. In particular, the points M_0' and M_0'' (or N_0' and N_0'') may coincide, so that the separatrices L_1' and L_1'' (L_2' and L_2'', respectively) coincide and form a loop.

By assumption, $\Phi(x, y) > 0$ inside γ. Hence it follows by the lemma of § 11.1 and by the remark to this lemma that if λ and μ are positive and sufficiently small, $(A_{\lambda\mu})$ has no saddle-to-saddle separatrices in a sufficiently small neighborhood of the separatrix L_1 (i.e., $u_0' \neq u_0''$; see also the proof of Theorem 16, § 11.2).

Outside the closed curve γ, $\Phi(x, y) < 0$. Therefore, outside γ, the vector field of $(A_{\lambda\mu})$ is rotated relative to the vector field of (A)

$$\begin{array}{l}\text{through a negative angle for } \mu > 0,\ \lambda = 0;\\ \text{through a positive angle for } \mu = 0,\ \lambda > 0.\end{array} \qquad (67)$$

To fix ideas, suppose that the separatrix L_2 forms a positive angle with the arc without contact l_2. Then, by (67) and the lemma of § 11.1,

$$v_0' < v_0'' \text{ if } \lambda = 0,\ \mu > 0; \qquad v_0' > v_0'' \text{ if } \lambda > 0,\ \mu = 0. \tag{68}$$

(The dashed curves in Figure 159 are the separatrices L_1', L_1'', L_2', L_2'' for $\lambda = 0, \mu > 0$.) v_0' and v_0'' are functions of the parameters λ and μ: $v_0' = v_0'(\lambda, \mu)$, $v_0'' = v_0''(\lambda, \mu)$, and by remark to Lemma 3, § 9.2, these functions are continuous. We choose a sufficiently small fixed $\lambda_1 > 0$. By (68), $v_0'(\lambda_1, 0) > v_0''(\lambda_1, 0)$, and by the continuity of v_0' and v_0'', we have for a sufficiently small $\bar{\mu} > 0$

$$v_0'(\lambda_1, \bar{\mu}) > v_0''(\lambda_1, \bar{\mu}). \tag{69}$$

By (68), $v_0'(0, \bar{\mu}) < v_0''(0, \mu)$ when $\bar{\mu} > 0$ is sufficiently small. But then, for a sufficiently small $\lambda_2 > 0$,

$$v_0'(\lambda_2, \bar{\mu}) < v_0''(\lambda_2, \mu). \tag{70}$$

By (69) and (70) there exists $\bar{\lambda} > 0$, $\lambda_2 < \bar{\lambda} < \lambda_1$, such that $v_0'(\bar{\lambda}, \bar{\mu}) = v_0''(\bar{\lambda}, \bar{\mu})$. The separatrices L_2' and L_2'' of $(A_{\bar{\lambda}\bar{\mu}})$ therefore coincide, forming a single saddle-to-saddle separatrix. The system $(A_{\bar{\lambda}\bar{\mu}})$ is thus structurally unstable. The numbers $\bar{\lambda}$ and $\bar{\mu}$ can be taken as small as we desire, in particular, they can be made sufficiently small for $(A_{\bar{\lambda}\bar{\mu}})$ to have no saddle-to-saddle separatrices in a sufficiently small neighborhood of the separatrix L_1.

Furthermore, if (A) is a system of class $N \geqslant 3$, Lemmas 11 and 12 indicate that γ can be chosen as a curve of class $N + 1$ and Lemma 13 shows that $\Phi(x, y)$ can be chosen as a function of class N. Then $(A_{\bar{\lambda}\bar{\mu}})$ is also a system of class N.

We have thus established that if (A) is a system of the first degree of structural instability of class $N \geqslant 3$ with two saddle-to-saddle separatrices, there exists a structurally unstable system $(A_{\bar{\lambda}\bar{\mu}})$ of the same class, as close as desired to (A), which has no saddle-to-saddle separatrices in a sufficiently small neighborhood of the separatrix L_1. This, however, contradicts the definition of a system of the first degree of structural instability. Therefore, case 10 is ruled out for a system of the first degree of structural instability in relation to the space $R_N^{(r)}$, $3 \leqslant r \leqslant N$.

For systems of the first degree of structural instability in relation to the space $R_a^{(r)}$ (the space of analytical system with closeness to rank r), Lemma 14 also must be applied to prove inapplicability of case 10. Let (A) be an analytical system of the first degree of structural instability which has two saddle-to-saddle separatrices L_1 and L_2. Let further $r \geqslant 3$ be a natural number, and δ and ε positive numbers, ε being sufficiently small. As before, we can construct a system $(A_{\bar{\lambda}\bar{\mu}})$ of class $N \geqslant r$ which is $\delta/2$-close to rank r to (A) and which has a saddle-to-saddle separatrix in $U_\varepsilon(L_2)$ and no such separatrix in $U_\varepsilon(L_1)$.

Let $\delta_1 > 0$ be an arbitrary number $\delta_1 < \delta/2$. By Lemma 14, there exists an analytical system $(\tilde{A})$ δ_1-close to rank r to $(A_{\bar{\lambda}\bar{\mu}})$ which has a saddle-to-saddle separatrix. If δ_1 is sufficiently small, $(\tilde{A})$ evidently has no saddle-to-saddle separatrices in $U_\varepsilon(L_1)$ and we arrive, as before,

at a contradiction with the definition of a system of the first degree of structural instability. Case 10 is thus ruled out for systems of the first degree of instability in relation to the space $R_a^{(r)}$. This completes the proof of Theorem 61. 3

6. The properties of the separatrices of a saddle-node in systems of the first degree of structural instability

Let (A) be a system of the first degree of structural instability in W which has a saddle-node $O(0, 0)$ in this region. Consider some separatrix L of this saddle-node. To fix ideas, let this be an α-separatrix, i.e., a separatrix which goes to O for $t \to -\infty$. By Theorem 61, the saddle-node O is the only structurally unstable path of (A) in W, and therefore as t increases, one of the following cases is a priori possible for the separatrix L:

1) L will leave W.

2) L will go to a structurally stable node or focus, or to a structurally stable limit cycle.

3) L will go to the equilibrium state O without being its ω-separatrix (i.e., the positive semipath making up L is one of the interior semipaths of the parabolic sector of the saddle-node O).

4) L will be the ω-separatrix of the equilibrium state O.

5) L will go to a structurally stable saddle point.

We will prove that cases 4 and 5 are unfeasible.

First we will formulate, without proof, two lemmas that will be useful later on.

Lemma 15. Let the separatrix L of the saddle-node O of system (A) *cross an arc without contact l at the interior point M_0 of the arc. For every $\varepsilon > 0$, there exists $\delta > 0$ such that if* $(\tilde{A})$ *is δ-close to* (A) *and O is a saddle-node of* $(\tilde{A})$, *then*

(a) there exists a single separatrix $\tilde{L}$ of the saddle-node O of $(\tilde{A})$ *which crosses the arc l at the point $\tilde{M}_0$ contained in $U_\varepsilon(M_0)$;*

(b) if the points M_0 and $\tilde{M}_0$ on the separatrices L and $\tilde{L}$ correspond to the same time $t = t_0$, the points of the segments M_0O and $\tilde{M}_0O$ of the separatrices L and $\tilde{L}$ corresponding to the same times t are distant less than ε from each other.

The proof of Lemma 15 is analogous to the proof of Lemma 3, §9.2 and the remark following the lemma.

Let again L be an α-separatrix of the saddle-node O of (A), M_0 any point of the separatrix, l an arc without contact through M_0 which has no common points, except M_0, with the separatrix L and with the other separatrices of O. The arc l is defined by the parametric equations

$$x = f(u), \quad y = g(u)$$

and the point M corresponds to the value u_0 of the parameter.

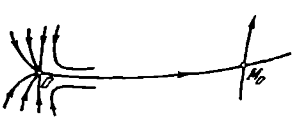

FIGURE 160

Moreover, let the positive direction on l correspond to the increasing parameter and let the paths of (A) make positive angles with the arc l (Figure 160).

We will consider modified systems ($\tilde{A}$) of the form

$$\frac{dx}{dt} = P - \mu Q = \tilde{P}, \quad \frac{dy}{dt} = Q + \mu P = \tilde{Q}, \qquad (\tilde{A}_\mu)$$

which are obtained from (A) by an appropriate rotation of the vector field.

Lemma 16. If $\mu \neq 0$ is sufficiently small, O is a saddle-point of $(\tilde{A}_\mu)$ and the arc without contact l has precisely one point $\tilde{M}_0$ in common with one of the separatrices $\tilde{L}$ of the saddle-node O of $(\tilde{A}_\mu)$ and has no common points with other separatrices of O. If the point $\tilde{M}_0$ on the arc l corresponds to the value $\tilde{u}_0 = u_0(\mu)$ of the parameter u, then $\tilde{u}_0(\mu) \to u_0$ for $\mu \to 0$. If $\mu > 0$ $(\mu < 0)$, then $\tilde{u}_0(\mu) > u_0$ $(\tilde{u}_0(\mu) < u_0)$. A similar proposition, with suitably modified wording, applies when L is an ω-separatrix of the saddle-node O.

The proof of Lemma 16 is analogous to the proof of the corresponding lemma for the separatrix of a saddle point (§ 11.1), and it is therefore omitted.

Theorem 62. If (A) is a system of the first degree of structural instability in W, it cannot have in this region a separatrix which goes from a saddle-node to a saddle point.

Proof. Suppose that the theorem is not true, i.e., system (A) has in W a separatrix L of the saddle-node O which at the same time is a separatrix of the saddle point O_1 (Figure 161). Let l be an arc without contact passing through the point M_0 of the separatrix L. Consider a system (A_μ). Since O is a saddle-node of (A_μ), (A_μ) is structurally unstable. Lemma 16 and the corresponding proposition for a saddle point (§ 11.1) then show that for a sufficiently small $\mu \neq 0$ (A_μ) does not have in a sufficiently small neighborhood of the path L a separatrix which goes from O to O_1. The contradiction is now established by the usual argument using the relation $(U(L), A) \overset{\varepsilon}{\equiv} (\tilde{V}, A_\mu)$. This proves the theorem.

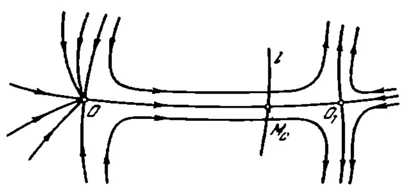

FIGURE 161

Theorem 63. If (A) is a dynamic system of the first degree of structural instability in W, it cannot have in this region a path L which is at the same time an α-separatrix and an ω-separatrix of a saddle point of (A).

Proof of Theorem 63 is analogous to the proof of the previous theorem. It is conducted by reductio ad absurdum, using a system (A_μ), Lemma 16, and the fact that a path forming a loop may cross a segment without contact at most in one point.

Theorems 62 and 63 show that cases 4 and 5 listed at the beginning of this subsection are unfeasible.

Let us now return to closed paths of a system of the first degree of structural instability and derive a further property of these paths on the basis of the above theorems.

Theorem 64. System (A) *of the first degree of structural instability in W may have only a finite number of closed paths in this region.**

Proof of Theorem 64 is analogous to the proof of the corresponding proposition for structurally stable systems (Theorem 21, §16.1). Suppose that (A) has an infinity of closed paths in W. Consider a sequence L_1, L_2, L_3, ... of these paths and choose a single point on each. Let M_1, M_2, M_3, ... be the sequence of these points, $M_i \in L_i$. Since W is a compact region, we may assume without loss of generality that the sequence M_i is convergent. Let this sequence converge to some point M^*. We will now show that no such point may exist.

Two cases are possible a priori:

1) M^* is an equilibrium state.

2) M^* is not an equilibrium state.

In case 1, M^* is neither a node nor a focus (whether simple or multiple), since a sufficiently small neighborhood of a node or a focus may not contain points of closed paths. Hence, M^* is either a saddle point or a saddle-node. But then an infinity of points M_i belong to one of the hyperbolic sectors of the equilibrium state O, and we can show that there exists a sequence of points which belong to the closed paths L_i and which have a condensation point which is not an equilibrium state (see §16.1, proof of Theorem 21, case 2). Thus, case 1 is reduced to case 2. Let us consider the second case.

Let L^* be the path of (A) through M^*. L^* clearly cannot leave the region W, nor can it be a path which goes to a node, a focus (whether simple or multiple), or a limit cycle (otherwise, closed paths L_i would pass arbitrarily close to the node, the focus, or the limit cycle, which is impossible). Similarly, L^* may not be an interior path of the parabolic sector of a saddle-node. Finally, by Theorems 62 and 63, L^* may not be a separatrix of a saddle-node, and by Theorem 58, L^* may not be a closed path. We are thus left with one last possibility, namely that L^* is a path which for $t \to -\infty$ goes to the saddle point O_1 and for $t \to +\infty$ goes to the saddle point O_2.

Let us first assume that the points O_1 and O_2 coincide, i.e., L^* is a saddle-point separatrix which forms a loop.

Let (x_0, y_0) be the coordinates of the saddle point O_1. By Theorem 60, in this case $\sigma(x_0, y_0) = P'_x(x_0, y_0) + Q'_y(x_0, y_0) \neq 0$. But then by Theorem 44, §29.1, the loop L^* is either stable or unstable, i.e., a sufficiently small neighborhood of the loop may not contain points of closed paths. We have thus established a contradiction.

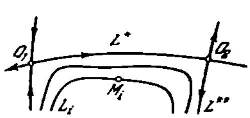

FIGURE 162

Let now O_1 and O_2 be two different saddle points, L^* going to O_1 for $t \to -\infty$ and to O_2 for $t \to +\infty$ (Figure 162). Consider that hyperbolic sector of the saddle point O_1 which contains an infinity of points M_i. Let L^{**} be the ω-continuation of the separatrix L^* on the side of this hyperbolic sector. Clearly, L^{**} is analogous to the path L^* in all respects, i.e., it is a saddle-to-saddle separatrix. But then (A) has two structurally unstable paths L^* and L^{**}, and this contradicts Theorem 61. This completes the proof of the theorem.

* Theorem 64 is not a direct consequence of Theorem 58 (which states that the closed paths of a system of the first degree of structural instability are isolated).

7. Properties of separatrices of saddle points of systems of the first degree of structural instability

We have already derived some properties of separatrices of saddle points of systems of the first degree of structural instability. Indeed, we have seen that, first, the separatrix of a saddle point $O(x_0, y_0)$ may not form a loop if $\sigma(x_0, y_0) = 0$ (Theorem 60) and, second, an ω-separatrix (α-separatrix) of a saddle point may not be at the same time an α-separatrix (ω-separatrix) of a saddle-node (Theorem 62). In this subsection, we will establish two further properties of saddle-point separatrices of systems of the first degree of structural instability.

Theorem 65. A system of the first degree of structural instability in W may not have in this region two saddle-point separatrices which go to a limit cycle of multiplicity 2, one for $t \to -\infty$ and the other for $t \to +\infty$.

Proof. Suppose that system (A) has a limit cycle L_0 of multiplicity 2 in W and two saddle-point separatrices L_1 and L_2, of which L_1 goes to L_0 for $t \to +\infty$ and L_2 goes to the same limit cycle for $t \to -\infty$. One of these separatrices evidently lies outside L_0 and the other lies inside L_0.

Let $\varepsilon_0 > 0$ be such that $U_{\varepsilon_0}(L_0)$ does not contain any equilibrium states of (A) and any closed paths, except L_0.

An arc without contact l is passed through some point M of the cycle L_0. Let s be the parameter on the arc l, chosen so that the point M corresponds to $s = 0$. Let further

$$\bar{s} = f(s)$$

be the succession function on the arc l defined for all s, $|s| < \eta$, where η is some positive number. Since the separatrices L_1 and L_2 by assumption go to L_0 (for $t \to +\infty$ and $t \to -\infty$, respectively), the arc l contains an infinite number of points which belong to L_2 and an infinite number of points which belong to L_1. Let $M'_0(s'_0)$ and $M'_1(s'_1)$ be two successive (in terms of t) points at which the path L_1 crosses the arc l, and $M''_0(s''_0)$ and $M''_1(s''_1)$ two successive (in terms of t) points at which L_2 crosses l. We assume that M'_0 and M''_0 are so close to M that the coils $M'_0M'_1$ and $M''_0M''_1$ of the paths L_1 and L_2, respectively, are contained entirely in $U_{\varepsilon_0}(L_0)$ and $|s'_i| < \eta$, $|s''_i| < \eta$, $i = 1, 2$.

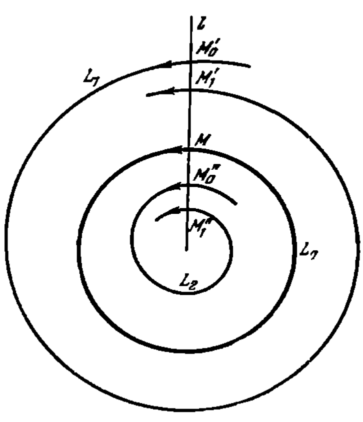

FIGURE 163

To fix ideas, let the parameter s on the arc l be chosen so that $s'_0 < 0$, $s''_0 > 0$ (Figure 163). Then

$$s'_0 < s'_1 < 0 < s''_0 < s''_1.$$

Also note that $s'_1 = f(s'_0)$, $s''_1 = f(s''_0)$.

Alongside with (A), consider the system

$$\frac{dx}{dt} = P - \mu Q = P_\mu, \quad \frac{dy}{dt} = Q + \mu P = Q_\mu, \qquad (A_\mu)$$

and let $\mu_0 > 0$ be so small that for all μ, $|\mu| < \mu_0$, the following conditions are satisfied:

1) l is an arc without contact of (A_μ), and for all $s, |s| < \eta$, the succession function $\bar{s} = f_\mu(s)$ of this system is defined on the arc.

2) $U_\varepsilon(L_0)$ contains no equilibrium states of (A_μ) and – either for all $\mu > 0$ or for all $\mu < 0$ – it contains no closed paths of (A_μ) (the latter proposition is true by Theorem 71, § 32.4).

3) There exist separatrices $L_{1\mu}$ and $L_{2\mu}$ of (A_μ) which cross the arc l at the points $M'_{0\mu}$, $M'_{1\mu}$, $M''_{0\mu}$, $M''_{1\mu}$ corresponding to the values $s'_{0\mu}$, $s'_{1\mu}$, $s''_{0\mu}$, $s''_{1\mu}$ of the parameter s, such that

$$s'_{0\mu} < s'_{1\mu} < s''_{0\mu} < s''_{1\mu}$$

and

$$s'_{1\mu} = f_\mu(s'_{0\mu}), \quad s''_{1\mu} = f_\mu(s''_{0\mu}).$$

4) The coils $M'_{0\mu}M'_{1\mu}$ and $M''_{0\mu}M''_{1\mu}$ of the paths $L_{1\mu}$ and $L_{2\mu}$ are contained in $U_{\varepsilon_0}(L_0)$ (Figure 164).

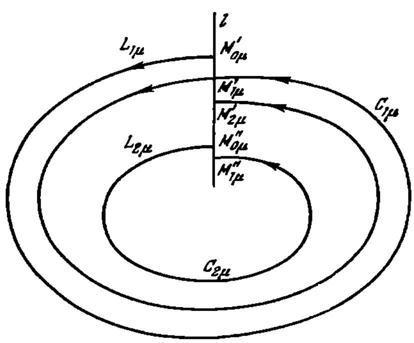

FIGURE 164

Consider successive iterations of the function f_μ, i.e., the functions

$$f_{2\mu} = f_\mu(f_\mu), \quad f_{3\mu} = f_\mu(f_{2\mu}),$$

and let

$$f_{k\mu}(s'_{0\mu}) = s'_{k\mu}$$

($k = 2, 3, \ldots$).

In virtue of our assumptions, if $\mu = 0$, then for any $k \geqslant 2$,

$$s''_{0\mu} > s'_{k\mu} = f_{k\mu}(s'_{0\mu}). \tag{71}$$

Let $C_{1\mu}$ and $C_{2\mu}$ be the simple closed curves formed respectively by the coils $M'_{0\mu}M'_{1\mu}$ and $M''_{0\mu}M''_{1\mu}$ of the paths $L_{1\mu}$ and $L_{2\mu}$ and the segments $M'_{1\mu}M'_{0\mu}$ and $M''_{1\mu}M''_{0\mu}$ of the arc l.

It is readily seen that in our case the curve $C_{2\mu}$ is enclosed by $C_{1\mu}$, and if μ_0 is sufficiently small, the region Γ_μ between these two curves is

contained in $U_{\varepsilon_0}(O)$. We assume that this condition is indeed satisfied. The limit cycle L_0 is clearly enclosed between the curves C_{10} and C_{20}.

By condition 2, for all μ of a certain sign (if $0<|\mu|<\mu_0$), $U_{\varepsilon_0}(L_0)$ contains no closed paths of (A_μ). Let this be so for positive μ. Consider a path $L_{1\mu}$, where $0<\mu<\mu_0$. As t increases, the path $L_{1\mu}$ enters into the region Γ_μ between the curves $C_{1\mu}$ and $C_{2\mu}$ $(\Gamma_\mu \subset U_{\varepsilon_0}(L_0))$, crossing through the point $M_{2\mu}'$. Since $U_{\varepsilon_0}(L_0)$ contains neither equilibrium states nor closed paths of (A_μ), the path $L_{1\mu}$ will leave $U_{\varepsilon_0}(L_0)$, and thus the region Γ_μ, as t increases further. This may occur only if $L_{1\mu}$ crosses the segment $M_{0\mu}''M_{1\mu}''$ of the arc without contact l. Consequently, there exists a natural number $N \geqslant 1$, such that

$$f_{N\mu}(s_{0\mu}') \geqslant s_{0\mu}''. \tag{72}$$

On the other hand, by (71), we have for $\mu = 0$

$$f_{N\mu}(s_{0\mu}') < s_{0\mu}''. \tag{73}$$

Since $s_{0\mu}'$, $s_{0\mu}''$, and $f_{N\mu}$ are continuous functions of μ (by remark to Lemma 3, § 9.2), inequalities (72) and (73) prove the existence of a number μ^*, $0<\mu^*<\mu$, such that

$$f_{N\mu^*}(s_{0\mu^*}') = s_{0\mu^*}'',$$

i.e., such that

$$s_{N\mu^*}' = s_{0\mu^*}''.$$

The last equality shows that the separatrix $L_{1\mu^*}$ of (A_{μ^*}) coincides with the separatrix $L_{2\mu^*}$, i.e., (A_{μ^*}) has a saddle-to-saddle separatrix. The number μ^* may be taken as small as desired. We have thus established that if (A) has a limit cycle of multiplicity 2 in W to which one separatrix goes for $t \to -\infty$ and the other separatrix for $t \to +\infty$, there exists a system (A_{μ^*}) as close as desired to (A) which has a saddle-to-saddle separatrix in W. (A_{μ^*}) is a structurally unstable system. Therefore, since (A) is a system of the first degree of structural instability, we should have

$$(H, A) \overset{\varepsilon}{\equiv} (\bar{H}, A_{\mu^*}).$$

This relation is clearly unfeasible, since (A_{μ^*}) has a saddle-to-saddle separatrix and (A) has no such separatrices in the neighborhood of W. We have thus established a contradiction, which proves the theorem.

Theorem 66. A system of the first degree of structural instability in W cannot have in this region a saddle-point separatrix which goes (for $t \to -\infty$ or for $t \to +\infty$) to a saddle-point separatrix forming a loop.

Proof. Again suppose that the theorem is not true, i.e., system (A) of the first degree of structural instability has in W a separatrix L_0 of the saddle-point $O(x_0, y_0)$ which forms a loop and a separatrix L_1 of the saddle point $O_1(x_1, y_1)$ which goes to the loop L_0 for $t \to +\infty$ (the saddle points O and O_1 are clearly different). By Theorem 60, $\sigma(x_0, y_0) \neq 0$ for the saddle point O, and by Theorem 44, § 29.1,

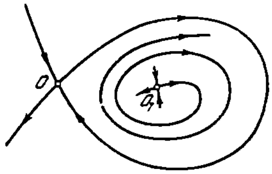

FIGURE 165

$$\sigma(x_0, y_0) < 0. \tag{74}$$

To fix ideas, suppose that two separatrices of the saddle point O which are different from L_0 lie

outside the loop formed by the separatrix L_0, so that the separatrix L_1 goes to the loop L_0 from the inside (Figure 165).

Let $\varepsilon_0 > 0$ be such that $U_{\varepsilon_0}(L_0)$ contains no equilibrium states, except O, and no closed paths of (A).

Through a point P of the path L_0 draw an arc without contact l, and let s be the parameter on the arc l chosen so that the point P corresponds to $s = 0$ and the points of the arc l lying outside the loop correspond to positive values of s (Figure 166). Since by assumption the loop is stable, we can always find a point A of the arc l which lies inside the loop such that all the paths crossing the segment AP of the arc l go to the loop L_0 and therefore cross the segment AP at infinitely many points. In particular, the separatrix L_1 of the saddle point O_1 crosses the segment AP of the arc l at infinitely many points.

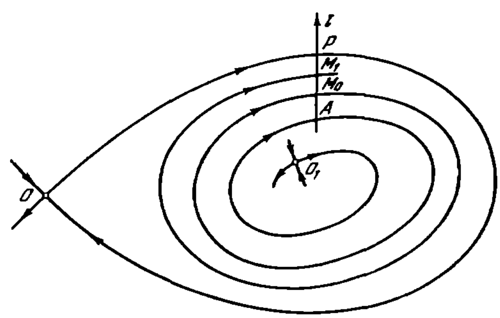

FIGURE 166

Let the value of s corresponding to the point A be a. The succession function

$$\bar{s} = f(s)$$

is thus defined on the segment AP of the arc l, i.e., for $a \leqslant s < 0$.

Moreover, for every integer N, there is an N-th successive point for every point of the arc AP,

$$\bar{s}_N = f_N(s).$$

Consider two successive points M_0 and M_1 among the intersection points of the separatrix L_1 with the arc AP. Let s_0 and s_1 be the values of the parameter s corresponding to these points, and t_0 and t_1 the corresponding times on the path L_1. Clearly $t_0 < t_1$, and $s_0 < s_1$. Moreover, let the point M_0 be so close to the loop L_0 that the segment M_0M_1 of the path L_0 is contained in $U_{\varepsilon_0}(L_0)$ (Figure 166).

Consider the modified system

$$\frac{dx}{dt} = P(x, y) - \mu Q(x, y), \quad \frac{dy}{dt} = Q(x, y) + \mu P(x, y), \qquad (\mathrm{A}_\mu)$$

which is obtained from (A) by a rotation of the vector field. For all sufficiently small $\mu \neq 0$, (A_μ) has no separatrices forming loops in $U_{\varepsilon_0}(L_0)$,

but it has two different separatrices $L_0'(\mu)$ and $L_0''(\mu)$ crossing the arc l in this neighborhood.

Let $L_0'(\mu)$ be an ω-separatrix and $L_0''(\mu)$ an α-separatrix of the saddle point O.

Let $P'(\mu)$ and $P''(\mu)$ be the "first" intersection points of the separatrices $L_0'(\mu)$ and $L_0''(\mu)$, respectively, with the arc l, so that the segment $OP'(\mu)$ of the path $L_0'(\mu)$ and the segment $OP''(\mu)$ of the path $L_0''(\mu)$ contains no points of the arc l, except $P'(\mu)$ and $P''(\mu)$. Let $s'(\mu)$ and $s''(\mu)$ be the values of the parameter s corresponding to the points $P'(\mu)$ and $P''(\mu)$. Clearly,

$$\lim_{\mu\to 0} s'(\mu) = \lim_{\mu\to 0} s''(\mu) = 0. \tag{75}$$

Let $\mu_0 > 0$ be sufficiently small. Then, if $|\mu| < \mu_0$, we have $s'(\mu) > 0$, $s''(\mu) > a$, and the segments $OP'(\mu)$ and $OP''(\mu)$ of the separatrices $L_0'(\mu)$ and $L_0''(\mu)$ lie in $U_{\varepsilon_0}(L_0)$. If, moreover, $\mu > 0$, and ε_0 is sufficiently small, we have by the lemma of § 11.1 $s''(\mu) > 0$ and $s'(\mu) < 0$. By Theorem 49 and the remark to the theorem (§ 29.3), (A_μ) has no closed paths in $U_{\varepsilon_0}(L_0)$. We will assume in what follows that these various conditions are satisfied.

By Lemma 7, § 29.2, a succession function $\bar{s} = f(s, \mu)$ is defined for all $|\mu| < \mu_0$, where $\mu_0 > 0$ is an appropriately chosen number, on the segment $AP'(\mu)$ of the arc without contact l, i.e., for all s, $a \leqslant s < s'(\mu)$.

Since $U_{\varepsilon_0}(L_0)$ contains no closed paths, we readily conclude that $f(s, \mu) > s$. It is moreover obvious that

$$\lim_{s\to s'(\mu)} f(s, \mu) = s''(\mu)$$

and that for $\mu = 0$ the function $\bar{s} = f(s, \mu)$ reduces to the succession function of the original system $(f(s, 0) \equiv f(s))$ defined on the segment AP of the arc l.

Since for the original system, every point of the segment AP of the arc l has an N-th successor for every integer N, then for *any given* N every *fixed point* of the segment AP of the arc l will have an N-th successor for the system A_μ, provided μ is sufficiently small. We will designate this successor by

$$\bar{s}_N = f_N(s, \mu).$$

Clearly, $f_N(s, \mu)$ is a continuous function of μ (for those values of μ for which it is defined).

By Lemma 3, § 9.2, and the remark to this lemma, we conclude that if $\mu \neq 0$ is sufficiently small, (A_μ) has a separatrix $L_1(\mu)$ of the saddle point O_1 which crosses the arc without contact l at the points $M_0(\mu)$ and $M_1(\mu)$, where $M_0(\mu) \to M_0$ and $M_1(\mu) \to M_1$ for $\mu \to 0$. Let $s_0(\mu)$ and $s_1(\mu)$ be the values of the parameter s corresponding to $M_0(\mu)$ and $M_1(\mu)$. By the lemma of § 11.1, if $\mu > 0$, we have $s_0(\mu) > s_0$ and $s_1(\mu) > s_1$.

We moreover assume that μ_0 is so small that if $|\mu| < \mu_0$, the coil of the path $L_1(\mu)$ between the points $M_0(\mu)$ and $M_1(\mu)$ is entirely contained in $U_{\varepsilon_0}(L_0)$.

By the remark to Lemma 3, § 9.2, $s_0(\mu)$, $s_1(\mu)$, $s'(\mu)$, and $s''(\mu)$ are continuous functions of μ. Also note that the equilibrium states of (A) and (A_μ) coincide, and therefore (A_μ) has no equilibrium states, except the saddle point O, in $U_\varepsilon(L_0)$.

It follows from the above that if $\varepsilon_0 > 0$ and $\mu_0 > 0$ are sufficiently small, and $0 < \mu \leqslant \mu_0$, the following inequalities are satisfied

$$s_0(\mu) < s_1(\mu) < s'(\mu) < 0, \quad s''(\mu) > 0$$

and $U_{\varepsilon_0}(L_0)$ contains no closed paths of (A_μ) and no equilibrium states, except the saddle point O.

Since $M_1(\mu)$ is the successor of $M_0(\mu)$, we have

$$s_1(\mu) = f(s_0(\mu), \mu).$$

Moreover, it is readily seen from the above that for a given integer N and for all sufficiently small μ, there exists an N-th successor of the point $M_0(\mu)$, i.e., the function

$$\bar{s}_N(\mu) = f_N(s_0(\mu), \mu)$$

exists, and it is a continuous function of μ (naturally for all μ for which it exists).

Let us now consider the behavior of the separatrix $L_1(\mu)$ prescribed by the above assumptions.

Two cases are possible:

1) there exist arbitrarily small μ, $|\mu| < \mu_0$, such that for some N

$$f_N(s_0(\mu), \mu) = s'(\mu),$$

i.e., there exist arbitrarily small μ such that $L_1(\mu)$ coincides with the separatrix $L_0'(\mu)$ (Figure 167);

2) there exists $\mu_1 > 0$, $\mu_1 \leqslant \mu_0$, such that for all $|\mu| \leqslant \mu_1$, the separatrix $L_1(\mu)$ does not coincide with $L_0'(\mu)$.

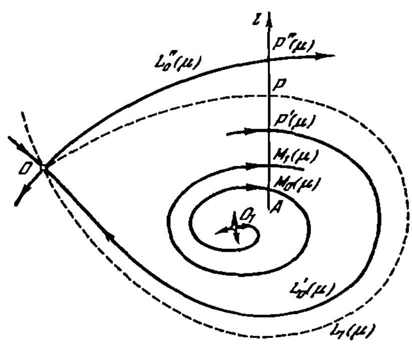

FIGURE 167

Let us first establish the behavior of the separatrix in case 2.

We will consider

(a) a simple closed curve C' consisting of the coil of the path $L_1(\mu)$ between the points $M_0(\mu)$ and $M_1(\mu)$ and the segment of the arc l between the points $M_0(\mu)$ and $M_1(\mu)$;

(b) a simple closed curve C'' consisting of the segment $P'(\mu)O$ of the separatrix $L_0'(\mu)$, the point O, the segment $OP''(\mu)$ of the separatrix $L_0''(\mu)$, and the segment of the arc l between the points $P'(\mu)$ and $P''(\mu)$.

Let Γ be the ring enclosed between the curves C' and C'' (Figure 168). It is readily seen that Γ is entirely contained in $U_{\varepsilon_0}(L_0)$ (since each point of this region belongs to a coil of a path of system (A_μ) entirely contained in $U_{\varepsilon_0}(L_0)$) and therefore contains no closed paths and no equilibrium states, except the saddle point O. The separatrix $L_1(\mu)$ evidently enters into Γ through the point $M_1(\mu)$ as t increases. Since by assumption it does not coincide with the separatrix $L_0'(\mu)$ of the saddle point O, further increase of t will inevitably cause it to leave Γ. It may leave Γ only by crossing the segment $P'(\mu)\,P''(\mu)$ of the arc l.

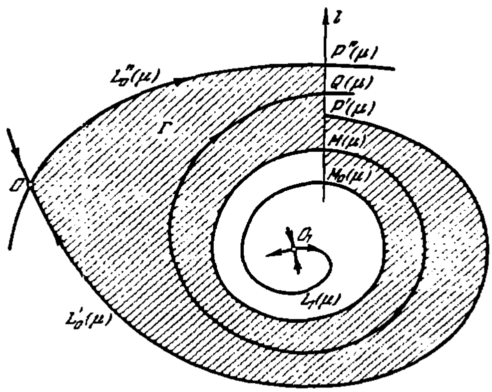

FIGURE 168

Let $Q(\mu)$ be the intersection point of the separatrix $L_1(\mu)$ with the segment $P'(\mu)\,P''(\mu)$ of the arc l. The point $Q(\mu)$ is clearly the N-th successor of $M_0(\mu)$, where N is some natural number dependent on μ.

Thus, if we write $s_Q(\mu)$ for the coordinate of the point $Q(\mu)$ on the arc l, then for any $\mu^* > 0$, $\mu^* < \mu_1$ we have

$$s_Q(\mu^*) = f_{N^*}(s_0(\mu^*),\ \mu^*),$$

where N^* depends on μ^*. The following inequality is also satisfied:

$$f_{N^*}(s_0(\mu^*),\ \mu^*) > s'(\mu^*). \tag{76}$$

Choose a fixed μ^* and the corresponding N^*.

For $\mu = 0$, for every N, and in particular for $N = N^*$, there exists a negative number $f_{N^*}(s_0(0),\ 0)$. But $s_0(\mu)$, $s'(\mu)$, $s''(\mu)$, $f_{N^*}(s_0(\mu),\ \mu)$ are continuous functions of μ. Therefore, if μ^{**}, $0 < \mu^{**} < \mu^*$, is sufficiently small, the number $f_{N^*}(s_0(\mu^{**}),\ \mu^*)$ is close to $f_{N^*}(s_0(\mu^*),\ \mu^*)$, and the number $s'(\mu^{**})$ is close to zero, so that

$$f_{N^*}(s_0(\mu^{**}),\ \mu^*) < s'(\mu^{**}). \tag{77}$$

From (76) and (77), using the continuity of all the relevant functions, we conclude that there exists a number $\bar{\mu}$, $\mu^{**}<\bar{\mu}<\mu^{*}$, such that

$$f_{N^*}(s_0(\bar{\mu}), \bar{\mu}) = s'(\bar{\mu}),$$

i.e., the separatrix $L_1(\bar{\mu})$ of the saddle point O_1 coincides with the separatrix $L_0'(\bar{\mu})$ of the saddle point O. This, however, contradicts assumption 2.

We have thus established that if a system of the first degree of structural instability (A) contains in W a separatrix L_0 of the saddle point O which forms a loop and a separatrix L_1 of the saddle point O_1 which goes to this loop, there exists a system (A_μ) as close as desired to (A) which

(a) has a separatrix extending from saddle point O to saddle point O_1;

(b) does not have separatrices forming a loop and contained in $U_{\varepsilon_0}(L_0)$, where ε_0 is a sufficiently small positive number.

By (a), (A_μ) is a structurally unstable system, and by (b), the relation $(\tilde{H}, A_\mu) \overset{\varepsilon_0}{\equiv} (H, A)$ is impossible for the systems (A) and (A_μ). But then, by Definition 30 (§ 31.1), (A) may not be a system of the first degree of structural instability. This contradiction proves the theorem.

8. The fundamental theorem (the necessary and sufficient conditions for systems of the first degree of structural instability)

Collecting all the previous results, we see that a dynamic system (A) of the first degree of structural instability in a closed region W satisfies the following conditions:

I. (A) has in W one and only one simplest structurally unstable path, i.e., a path of one of the following types:

1) a multiple focus of multiplicity 1;
2) a saddle-node of multiplicity 2 with $\sigma_0 = P'_x + Q'_y \neq 0$;
3) a limit cycle of multiplicity 2;
4) a separatrix from one saddle point to another;
5) a separatrix forming a loop for a saddle point with $\sigma_0 \neq 0$.

II. (A) does not have in W any structurally unstable limit cycles, saddle-point separatrices forming a loop, or equilibrium states other than those listed in I.

III. If (A) has a saddle-node in W, none of the separatrices of this saddle-node may go to a saddle point and no two separatrices of the saddle-node are a continuation of each other.

IV. The separatrix of a saddle point of (A) contained in W may not go for $t \to -\infty$ or for $t \to +\infty$ to a separatrix forming a loop. W may not contain two saddle-point separatrices going to the same limit cycle of multiplicity 2, one for $t \to -\infty$ and the other for $t \to +\infty$.

Conditions I through IV prove to be not only necessary but also sufficient for (A) to be a system of the first degree of structural instability in W, i.e., we have the following theorem:

Theorem 67. ***For a dynamic system* (A) *to be a system of the first degree of structural instability in a closed region W, it is necessary and sufficient that the above conditions I through IV be satisfied.***

Proof. The necessity of conditions I through IV follows directly from Theorems 54–57, 59–63, 65, and 66. The proof of sufficiency is omitted here. Note, however, that although the proof of sufficiency is fairly lengthy, the underlying idea is simple and it follows the proof of Theorem 23 (the necessary and sufficient conditions of structural stability of systems, §18.2).*

9. Bifurcations of systems of the first degree of structural instability

The above properties enable us to identify without any further difficulties all the possible bifurcations of a dynamic system (A) in a region W where (A) is of the first degree of structural instability. These bifurcations evidently depend on the particular simplest structurally unstable path that the system has.

Let us consider the different cases.

1) (A) has a multiple focus of multiplicity 1 in W. Only one bifurcation is possible in this case, namely the creation of a limit cycle from the multiple focus. This bifurcation transforms the multiple focus into a simple (structurally stable) focus and changes its stability, while the created cycle is structurally stable and its stability is identical to the stability of the original focus.

2) (A) has a limit cycle of multiplicity 2 in W. Bifurcations of two types are possible in this case: the disappearance of the limit cycle and decomposition of the limit cycle into two limit cycles. In the latter case, the two new cycles are structurally stable, one being stable and the other unstable.

3) (A) has a separatrix from one saddle point to another saddle point in W. In this case, the saddle-to-saddle separatrix may decompose into two separatrices which are not a continuation of one another. This is the only possible bifurcation of such systems.

4) (A) has a saddle-point separatrix forming a loop in W. Bifurcations occur only if the separatrix loop disappears on moving to close systems. Two different bifurcations are possible: either a structurally stable limit cycle is created in the neighborhood of the disappearing separatrix loop (of the same stability as the loop), or the loop disappears without creating a limit cycle.

5) (A) has a saddle-node M_0 with $\sigma_0 \neq 0$ in W. Two subcases should be considered here:

5a) The saddle-node M_0 has no separatrix forming a loop.

Two bifurcations are possible: disappearance of the equilibrium state M_0 or its decomposition into two structurally stable equilibrium states — a saddle point and a node. In either case, no limit cycles are created.

5b) The saddle-node M_0 has a separatrix forming a loop.

Again birfurcations of two types are possible:

1) Decomposition of the equilibrium state M_0 into a structurally stable saddle point and a structurally stable node. No limit cycle is created.

* Dynamic systems of the first degree of structural instability on a torus were considered by Aranson (see /38/).

2) Disappearance of the equilibrium state M_0. The separatrix loop naturally disappears, and a limit cycle is created in its neighborhood.

The above bifurcations cover all the possible simplest bifurcations, i.e., bifurcations which may occur in systems of the first degree of structural instability. This conclusion follows from the results of the previous chapters and from § 30.

The above list shows that all the simplest bifurcations involving creation or disappearance of a limit cycle are particular cases of bifurcations considered in Chapters IX, X, XI and in § 30 of the present chapter. Note, however, that these chapters, and Chapter VIII, are by no means restricted to the analysis of the simplest bifurcations.

Chapter XIII

LIMIT CYCLES OF SOME DYNAMIC SYSTEMS DEPENDING ON A PARAMETER

INTRODUCTION

The subject of this chapter are analytical dynamic systems depending on a parameter, i.e., systems of the form

$$\frac{dx}{dt}=\bar{P}(x, y, \mu), \quad \frac{dy}{dt}=\bar{Q}(x, y, \mu), \tag{A_μ}$$

where $\bar{P}$ and $\bar{Q}$ are analytical functions. We will investigate the topic of limit cycles created from a closed path L_0 of the "original" system (A_0) on passing from $\mu = 0$ to close values of the parameter μ.

The chapter is divided into two sections, § 32 and § 33. The first three subsections of § 32 are of auxiliary character. In § 32.1, we investigate the succession function on an arc without contact crossing a closed path L_0 and derive a number of formulas for the coefficients in the series expansion of this function. In § 32.2 a more detailed statement of the problem is given. § 32.3 is devoted to one classical problem of the theory of analytical functions, and it is thus of independent interest. It considers the equation $F(w, z) = 0$, where F is an analytical function satisfying the condition $F(0,0) = 0$, and the so-called Newton's polygon is applied to investigate the number and the behavior of the solutions of this equation in the neighborhood of the point $w = 0$, $z = 0$.

The main results of this section are presented in § 32.4, in the form of Theorems 71 and 72. In these theorems, system (A_μ) is taken in the form

$$\frac{dx}{dt}=P-\mu Q, \quad \frac{dy}{dt}=Q+\mu P,$$

whose vector field is obtained from the vector field of (A_0) by a rotation through a constant angle. It is proved that if L_0 is a limit cycle of even multiplicity of (A_0), rotation of the vector field in one direction decomposes this cycle into two structurally stable cycles, and rotation in the opposite direction makes the cycle disappear (Theorem 71). If, however, L_0 is a limit cycle of odd multiplicity of (A_0), any system obtained by a small rotation of the vector field of (A_0) in any direction has a single limit cycle in a small neighborhood of L_0, which is moreover structurally stable (Theorem 72).

The creation of a limit cycle from a closed path of a conservative system is considered in § 33. The main definitions are given in § 33.1. A conservative system is defined as a system which has an integral

invariant with positive density in the relevant region (see §33.1, Definitions 31 and 32).

However, §33 deals only with one particular case of conservative systems, namely systems defined in a doubly connected ("ring") region where all the paths are closed paths enclosing one another. In §33.1 it is proved that such a system is indeed conservative (Theorem 74).

Dynamic systems of the form

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x},$$

are known as Hamiltonian systems, and they constitute a particular case of conservative systems.

In §33.2, we consider systems which are close to the linear conservative system

$$\dot{x} = -y, \quad \dot{y} = x, \tag{B_0}$$

i.e., systems of the form

$$\begin{aligned} \dot{x} &= -y + \mu p_1(x, y) + \mu^2 p_2(x, y) + \ldots \\ \dot{y} &= x + \mu q_1(x, y) + \mu^2 q_2(x, y) + \ldots, \end{aligned} \tag{B_\mu}$$

and we establish under what conditions the path $x^2 + y^2 = \rho_1^2$ of the original system (B_0) creates a single limit cycle on passing to a sufficiently close system (B_μ) (Theorem 75).

In §33.3, similar conditions are derived for the general case of systems which are close to conservative systems (see Theorem 77). Theorem 78 (§33.4) shows that these conditions are particularly simple for systems close to Hamiltonian.

Note that the condition of analyticity of the dynamic systems introduced in Chapter XIII is not essential: similar results can be derived for non-analytical systems also.

§32. THE BEHAVIOR OF LIMIT CYCLES OF SOME DYNAMIC SYSTEMS FOLLOWING SMALL CHANGES IN THE PARAMETER

1. The succession function in the neighborhood of a closed path

Consider a dynamic system depending on a parameter μ

$$\frac{dx}{dt} = \bar{P}(x, y, \mu), \quad \frac{dy}{dt} = \bar{Q}(x, y, \mu), \tag{A_\mu}$$

where $\bar{P}(x, y, \mu)$ and $\bar{Q}(x, y, \mu)$ are analytical functions of x, y and of the parameter μ, defined for x, y from some region G in the plane (x, y) and for μ from some interval containing the point μ_0. Clearly (A_μ) may be treated as a one-parametric family of analytical dynamic systems defined in G.

Suppose that for $\mu = \mu_0$, (A_μ) has a closed path L_0. Without loss of generality, we may take $\mu_0 = 0$, i.e., L_0 is a closed path of the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{A_0}$$

where $P(x, y) = \bar{P}(x, y, 0)$, $Q(x, y) = \bar{Q}(x, y, 0)$.

Let

$$x = \varphi(t), \qquad y = \psi(t), \tag{1}$$

where $\varphi(t)$ and $\psi(t)$ are periodic functions of period τ which constitute the solution corresponding to the path L_0.

As in § 13, we define curvilinear coordinates s, n in the neighborhood of this path,

$$x = \bar{\varphi}(s, n), \qquad y = \bar{\psi}(s, n), \tag{2}$$

where the functions $\bar{\varphi}$ and $\bar{\psi}$ have the following properties:

1) $\bar{\varphi}$ and $\bar{\psi}$ are defined in the strip

$$-\infty < s < +\infty, \qquad -n^* < n < n^*, \tag{3}$$

where n^* is some positive number, and they are analytical functions in this strip.

2) $\bar{\varphi}$ and $\bar{\psi}$ are periodic functions of s with period τ.

3) $\bar{\varphi}(s, 0) \equiv \varphi(s)$, $\bar{\psi}(s, 0) \equiv \psi(s)$. $\qquad (4)$

4) The functional determinant

$$\Delta(s, n) = \frac{D(\bar{\varphi}, \bar{\psi})}{D(s, n)} = \begin{vmatrix} \bar{\varphi}'_s & \bar{\varphi}'_n \\ \bar{\psi}'_s & \bar{\psi}'_n \end{vmatrix} \tag{5}$$

does not vanish everywhere in the strip (3), i.e., it retains a constant sign in this strip.

The function $\bar{\varphi}(s, n)$ and $\bar{\psi}(s, n)$ may be taken, in particular, in the same form as in § 13

$$\bar{\varphi} = \varphi(s) + n\psi'(s), \qquad \bar{\psi} = \psi(s) - n\varphi'(s), \tag{6}$$

as for sufficiently small $n^* > 0$ these functions evidently satisfy all the conditions above. We shall see in the following, however, that in some cases functions of a different form are more convenient for $\bar{\varphi}$ and $\bar{\psi}$ (see § 33.3, (56)). From relation (4) we evidently have

$$\bar{\varphi}'_s(s, 0) = \varphi'(s), \qquad \bar{\psi}'_s(s, 0) = \psi'(s). \tag{7}$$

Let us change over to the variables s, n in all the systems (A_μ) with sufficiently small μ. Differentiating (2) with respect to t and using the equations of (A_μ), we find

$$\frac{dx}{dt} = \bar{\varphi}'_s \frac{ds}{dt} + \bar{\varphi}'_n \frac{dn}{dt} = \bar{P}(\bar{\varphi}, \bar{\psi}, \mu), \qquad \frac{dy}{dt} = \bar{\psi}'_s \frac{ds}{dt} + \bar{\psi}'_n \frac{dn}{dt} = \bar{Q}(\bar{\varphi}, \bar{\psi}, \mu).$$

Solving these equations for $\frac{ds}{dt}$, $\frac{dn}{dt}$ and eliminating t, we obtain the differential equation

$$\frac{dn}{ds} = \frac{\bar{Q}(\bar{\varphi}, \bar{\psi}, \mu)\bar{\varphi}'_s - \bar{P}(\bar{\varphi}, \bar{\psi}, \mu)\bar{\psi}'_s}{\bar{P}(\bar{\varphi}, \bar{\psi}, \mu)\bar{\psi}'_n - \bar{Q}(\bar{\varphi}, \bar{\psi}, \mu)\bar{\varphi}'_n} = R(s, n, \mu). \tag{R_μ}$$

For $n = 0$, $\mu = 0$, the denominator in the right-hand side of (R_μ) is evidently equal to

$$\bar{P}(\varphi(s), \psi(s), 0)\,\bar{\psi}'_n(s, 0) - \bar{Q}(\varphi(s), \psi(s), 0)\,\bar{\varphi}'_n(s, 0) =$$
$$= \bar{\varphi}'(s)\,\bar{\psi}'_n(s, 0) - \bar{\psi}'(s)\,\bar{\varphi}'_n(s, 0) = \Delta(s, 0).$$

By condition 4, $\Delta(s, 0) \neq 0$ for all s, in particular, for all $0 \leqslant s \leqslant \tau$. Therefore, if $n^* > 0$ and $\mu^* > 0$ are sufficiently small, the denominator in the right-hand side of equation (R_μ) does not vanish for all s, n, μ satisfying the respective inequalities

$$0 \leqslant s \leqslant \tau, \quad |n| < n^*, \quad |\mu| < \mu^*,$$

i.e., because of the periodicity in s of the functions $\bar{\varphi}$ and $\bar{\psi}$ for $-\infty < s < +\infty$, $|n| < n^*$, $|\mu| < \mu^*$. Hence it follows that $R(s, n, \mu)$ is an analytical function for $-\infty < s < +\infty$, $|n| < n^*$, $|\mu| < \mu^*$. Therefore, it can be expanded in a series in powers of n, μ in the neighborhood of any point in this region, and the coefficients of this series will be analytical functions of s. It is readily seen that

$$R(s, 0, 0) \equiv 0. \tag{8}$$

Therefore, the expansion of $R(s, n, \mu)$ in the neighborhood of the point $(s_0, 0, 0)$ (where s_0 is any fixed number) has the form

$$R(s, n, \mu) = A_{10}(s)\,n + A_{01}(s)\,\mu + A_{20}(s)\,n^2 + A_{11}(s)\,n\mu + A_{02}(s)\,\mu^2 + \ldots \tag{9}$$

Since $R(s, n, \mu)$ is periodic in s, it is readily seen that the coefficients $A_{ij}(s)$ are also periodic functions of period τ.

It follows from (8) that the function $n \equiv 0$ solves the equation (R_0). In general, however, it does not solve the equation (R_μ) for $\mu \neq 0$.

We will consider the succession function on the arc without contact l, defined by the equation $s = 0$ (for functions $\bar{\varphi}$ and $\bar{\psi}$ of the form (6), this arc is the normal to the path (1) at the point $s = 0$). The succession function is constructed as in Chapter V (§ 13.3). Let

$$n = f(s; 0, n_0, \mu) \tag{10}$$

be the solution of equation (R_μ) satisfying the initial condition

$$f(0; 0, n_0, \mu) \equiv n_0. \tag{11}$$

According to general theorems, this solution is defined in the region

$$-\eta < s < \tau + \eta, \quad |n_0| < \hat{n}, \quad |\mu| < \hat{\mu}, \tag{12}$$

where $\eta > 0$, and $\hat{n} \leqslant n^*$ and $\hat{\mu} \leqslant \mu^*$ are sufficiently small positive numbers; in this region, the solution is an analytical function of its arguments. Since $n \equiv 0$ is a solution of (R_0), we have

$$f(s; 0, 0, 0) \equiv 0$$

and the expansion of the function f in powers of n_0 and μ has the form

$$f(s; 0, n_0, \mu) = \alpha_{10}n_0 + \alpha_{01}\mu + \alpha_{20}n_0^2 + \alpha_{11}n_0\mu + \alpha_{02}\mu^2 + \ldots \tag{13}$$

Here the coefficients $\alpha_{ij} = \alpha_{ij}(s)$ are analytical functions of s in the interval $-\eta < s < \tau + \eta$.

Inserting (9) and (13) for $R(s, n, \mu)$ and n in equation (R_μ) and equating the corresponding coefficients in the right- and the left-hand sides, we obtain the following recursive differential equations for the function $\alpha_{ij}(s)$:

$$
\begin{aligned}
&\frac{d\alpha_{10}(s)}{ds} = A_{10}\cdot\alpha_{10},\\
&\frac{d\alpha_{01}(s)}{ds} = A_{10}\cdot\alpha_{01} + A_{01},\\
&\frac{d\alpha_{20}(s)}{ds} = A_{10}\cdot\alpha_{20} + A_{20}\alpha_{10}^2,\\
&\frac{d\alpha_{11}(s)}{ds} = A_{10}\alpha_{11} + 2A_{20}\alpha_{10}\alpha_{01} - A_{11}\alpha_{10},\\
&\frac{d\alpha_{02}(s)}{ds} = A_{10}\alpha_{02} + A_{20}\alpha_{01}^2 + A_{11}\alpha_{01} + A_{02},\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\tag{14}
$$

By (11) and (13), for $i = 1$, $j = 0$, we have $\alpha_{ij}(0) = 1$, and otherwise, $\alpha_{ij}(0) = 0$.

The equalities

$$\alpha_{10}(0) = 1, \quad \alpha_{ij}(0) = 0 \tag{15}$$

constitute the initial conditions for equations (14). The functions $\alpha_{ij}(s)$ therefore can be found by successively solving equations (14) with boundary conditions (15).

Note that the final expressions for the functions $\alpha_{i0}(s)$, which are the coefficients of the terms free from μ in expansion (13), are the same expressions as in Chapter X (§ 26.1). Indeed, these functions may be found by setting $\mu = 0$. But this leads us to system (A_0) and equation (R_0) considered in Chapter X.

Let the succession function of (A_μ) on the arc without contact l be $f(n_0, \mu)$. By definition, it is obtained from the function (10) for $s = \tau$, i.e.,

$$n = f(n_0, \mu) = f(\tau; 0, n_0, \mu). \tag{16}$$

This and (13) show that the series expansion of the succession function has the form

$$f(n_0, \mu) = \alpha_{10}(\tau)\, n_0 + \alpha_{01}(\tau)\, \mu + \alpha_{20}(\tau)\, n_0^2 + \alpha_{11}(\tau)\, n_0\mu + \dots \tag{17}$$

Setting

$$\alpha_{ij}(\tau) = u_{ij}, \tag{18}$$

we obtain

$$n = f(n_0, \mu) = u_{10}n_0 + u_{01}\mu + u_{20}n_0^2 + u_{11}n_0\mu + \dots \tag{19}$$

We will now find expressions for the coefficients u_{10} and u_{01} and indicate in general outline the structure of the coefficient u_{11}, whose explicit expression is highly complex. These coefficients will be needed in what follows.

We have

$$\frac{dn}{ds} = R(s, n, \mu) = \frac{\overline{Q}(\overline{\varphi}, \overline{\psi}, \mu)\,\overline{\varphi}'_s - \overline{P}(\overline{\varphi}, \overline{\psi}, \mu)\,\overline{\psi}'_s}{\overline{P}(\overline{\varphi}, \overline{\psi}, \mu)\,\overline{\psi}'_n - \overline{Q}(\overline{\varphi}, \overline{\psi}, \mu)\,\overline{\varphi}'_n}. \tag{R_μ}$$

We write $g(s, n, \mu)$ and $h(s, n, \mu)$, respectively, for the numerator and the denominator of this fraction, i.e.,

$$
\begin{aligned}
g(s, n, \mu) &= \overline{Q}(\overline{\varphi}, \overline{\psi}, \mu)\,\overline{\varphi}'_s - \overline{P}(\overline{\varphi}, \overline{\psi}, \mu)\,\overline{\psi}'_s,\\
h(s, n, \mu) &= \overline{P}(\varphi, \overline{\psi}, \mu)\,\overline{\psi}'_n - \overline{Q}(\overline{\varphi}, \overline{\psi}, \overline{\mu})\,\varphi'_n.
\end{aligned}
\tag{20}
$$

Expanding the functions $g(s, n, \mu)$ and $h(s, n, \mu)$ in powers of n and μ and seeing that $g(s, 0, 0) \equiv 0$ and $h(s, 0, 0) \neq 0$, we obtain

$$\begin{aligned} g(s, n, \mu) &= g_{10}n + g_{01}\mu + g_{20}n^2 + \ldots, \\ h(s, n, \mu) &= h_{00} + h_{10}n + h_{01}\mu + h_{20}n^2 + \ldots, \end{aligned} \tag{21}$$

where $g_{ij} = g_{ij}(s)$ and $h_{ij} = h_{ij}(s)$ are analytical functions of s, and $h_{00}(s) \neq 0$. Thus,

$$\frac{dn}{ds} = \frac{g(s, n, \mu)}{h(s, n, \mu)} = \frac{g_{10}n + g_{01}\mu + g_{20}n^2 + \ldots}{h_{00} + h_{10}n + h_{01}\mu + \ldots} \equiv A_{10}n + A_{01}\mu + A_{20}n^2 + A_{11}n\mu + \ldots.$$

i.e.,

$$g_{10}n + g_{01}\mu + g_{20}n^2 + \ldots \equiv (A_{10}n + A_{01}\mu + A_{20}n^2 + A_{11}n\mu + \ldots)(h_{00} + h_{10}n + h_{01}\mu + \ldots).$$

Equating the coefficients on the right- and left-hand sides of this identity and solving for A_{ij}, we obtain

$$A_{10} = \frac{g_{10}}{h_{00}}, \quad A_{01} = \frac{g_{01}}{h_{00}}, \quad A_{20} = \frac{g_{20} - A_{10}h_{10}}{h_{00}}, \quad A_{11} = \frac{g_{11} - A_{01}h_{10} - A_{10}h_{01}}{h_{00}}. \tag{22}$$

Let us determine the functions $g_{ij}(s)$ and $h_{ij}(s)$ entering (22). To this end, we expand the functions $\bar{P}$ and $\bar{Q}$ in powers of μ, and the functions $\bar{\varphi}$ and $\bar{\psi}$ in powers of n. This gives

$$\begin{aligned} \bar{P}(x, y, \mu) &= P(x, y) + \mu p_1(x, y) + \mu^2 p_2(x, y) + \ldots, \\ \bar{Q}(x, y, \mu) &= Q(x, y) + \mu q_1(x, y) + \mu^2 q_2(x, y) + \ldots, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \bar{\varphi}(s, n) &= \varphi(s) + n\beta_1(s) + n^2\beta_2(s) + \ldots, \\ \bar{\psi}(s, n) &= \psi(s) + n\gamma_1(s) + n^2\gamma_2(s) + \ldots, \end{aligned} \tag{24}$$

where p_i, q_i, β_i, and γ_i are analytical functions of the respective arguments.

Expanding the functions $\bar{P}(\bar{\varphi}(s, n), \bar{\psi}(s, n), \mu)$ and $\bar{Q}(\bar{\varphi}(s, n), \bar{\psi}(s, n), \mu)$ in Maclaurin's series in powers of μ and n and using (23) and (24), we find

$$\begin{aligned} \bar{P}(\bar{\varphi}(s, n), \bar{\psi}(s, n), \mu) &= \bar{P}(\bar{\varphi}(s, 0), \bar{\psi}(s, 0), 0) + \\ &+ [\bar{P}'_x(\bar{\varphi}(s, 0), \bar{\psi}(s, 0), 0)\bar{\varphi}'_n(s, 0) + \bar{P}'_y(\bar{\varphi}(s, 0), \bar{\psi}(s, 0), 0)\bar{\psi}'_n(s, 0)]n + \\ &+ \bar{P}'_\mu(\bar{\varphi}(s, 0), \bar{\psi}(s, 0), 0)\mu + \frac{\partial^2 \bar{P}(\bar{\varphi}(s, 0), \bar{\psi}(s, 0), 0)}{\partial n\, \partial \mu} n\mu + \ldots = \\ &= \varphi'(s) + [P'_x(\varphi(s), \psi(s))\beta_1(s) + P'_y(\varphi(s), \psi(s))\gamma_1(s)]n + p_1(\varphi(s), \psi(s))\mu + \\ &+ [p'_{1x}(\varphi(s), \psi(s))\beta_1(s) + p'_{1y}(\varphi(s), \psi(s))\gamma_1(s)]n\mu + \ldots \end{aligned} \tag{25}$$

and similarly

$$\begin{aligned} \bar{Q}(\bar{\varphi}(s, n), \bar{\psi}(s, n), \mu) &= \\ &= \psi'(s) + [Q'_x(\varphi(s), \psi(s))\beta_1(s) + Q'_y(\varphi(s), \psi(s))\gamma_1(s)]n + q_1(\varphi(s), \psi(s))\mu + \\ &+ [q'_{1x}(\varphi(s), \psi(s))\beta_1(s) + q'_{1y}(\varphi(s), \psi(s))\gamma_1(s)]n\mu + \ldots \end{aligned} \tag{26}$$

Further, inserting (25) and (26) in (20) and using (24), we find

$$\begin{aligned} g(s, n, \mu) = \{&[Q'_x(\varphi, \psi)\beta_1 + Q'_y(\varphi, \psi)\gamma_1]\varphi' - \\ &- [P'_x(\varphi, \psi)\beta_1 + P'_y(\varphi, \psi)\gamma_1]\psi' + \psi'\beta'_1 - \varphi'\gamma'_1\}n + (q_1\varphi' - p_1\psi')\mu + \\ &+ [(q'_{1x}\beta_1 + q'_{1y}\gamma_1)\varphi' - (p'_{1x}\beta_1 + p'_{1y}\gamma_1)\psi' + (q_1\beta'_1 - p_1\gamma'_1)]n\mu + \ldots, \end{aligned} \tag{27}$$

$$h(s, n, \mu) = \varphi'\gamma_1 - \psi'\beta_1 + (P_x'(\varphi, \psi)\beta_1 + P_y'(\varphi, \psi)\gamma_1)\gamma_1 - \\ - (Q_x'(\varphi, \psi)\beta_1 + Q_y'(\varphi, \psi)\gamma_1)\beta_1 + 2(\varphi'\gamma_2 - \psi'\beta_2)n + (p_1\gamma_1 - q_1\beta_1)\mu + \ldots \tag{28}$$

This and (21) give

$$\left.\begin{aligned} g_{10} &= [Q_x'(\varphi, \psi)\beta_1 - Q_y'(\varphi, \psi)\gamma_1]\varphi' - \\ &\qquad - [P_x'(\varphi, \psi)\beta_1 + P_y'(\varphi, \psi)\gamma_1]\psi' + \psi'\beta_1' - \varphi'\gamma_1', \\ g_{01} &= q_1\varphi' - p_1\psi'; \\ g_{11} &= (q_{1x}'\beta_1 + q_{1y}'\gamma_1)\varphi' - (p_{1x}'\beta_1 + p_{1y}'\gamma_1)\psi' + q_1\beta_1' - p_1\gamma_1'. \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} h_{00} &= \varphi'\gamma_1 - \psi'\beta_1 = \bar{\varphi}_s'(s, 0)\bar{\psi}_n'(s, 0) - \bar{\psi}_s'(s, 0)\bar{\varphi}_n'(s, 0) = \Delta(s, 0), \\ h_{10} &= [P_x'(\varphi, \psi)\beta_1 - P_y'(\varphi, \psi)\gamma_1]\gamma_1 - \\ &\qquad - [Q_x'(\varphi, \psi)\beta_1 + Q_y'(\varphi, \psi)\gamma_1]\beta_1 + 2(\varphi'\gamma_2 - \psi'\beta_2), \\ h_{01} &= p_1\gamma_1 - q_1\beta_1 \end{aligned}\right\} \tag{30}$$

(by (5)).

The expression for g_{10} can be simplified. Indeed, using the relations

$$\varphi''(s) \equiv P_x'(\varphi, \psi)\varphi'(s) + P_y'(\varphi, \psi)\psi'(s),$$
$$\psi''(s) \equiv Q_x'(\varphi, \psi)\varphi'(s) + Q_y'(\varphi, \psi)\psi'(s),$$

obtained by differentiating the identity

$$\varphi'(s) = P(\varphi(s), \psi(s)), \quad \psi'(s) = Q(\varphi(s), \psi(s)),$$

we can verify directly that

$$g_{10} = [P_x'(\varphi, \psi) + Q_y'(\varphi, \psi)]h_{00}(s) - h_{00}'(s) \tag{31}$$

(compare equations (26) – (28), § 13.3).

Using (22), (29), (30), (31), we find

$$A_{10}(s) = P_x'(\varphi(s), \psi(s)) + Q_y'(\varphi(s), \psi(s)) - \frac{d}{ds}\ln h_{00}(s) = \\ = P_x'(\varphi, \psi) + Q_y'(\varphi, \psi) - \frac{d}{ds}\ln\Delta(s, 0), \tag{32}$$

$$A_{01}(s) = \frac{q_1(\varphi(s), \psi(s))\varphi'(s) - p_1(\varphi(s), \psi(s))\psi'(s)}{\Delta(s, 0)}. \tag{33}$$

The last equations, together with equations (14), initial conditions (15), and relations (18), enable us to find the coefficients u_{10} and u_{01} of expansion (19). The expression for the coefficient u_{10} is identical to the expression for the derivative of the succession function obtained in Chapter V (§ 13.3, (30)), indeed:

$$u_{10} = \alpha_{10}(\tau) = e^{\int_0^\tau [P_x'(\varphi(s), \psi(s)) + Q_y'(\varphi(s), \psi(s))]ds} \tag{34}$$

Integrating the second equation in (14) with initial conditions (15) and using (32) and (33), we find

$$\alpha_{01}(s) = e^{\int_0^s [P_x'(\varphi(s), \psi(s)) + Q_y'(\varphi(s), \psi(s))]ds} \frac{1}{\Delta(s, 0)} \int_0^s [q_1(\varphi, \psi)\varphi'(s) - \\ - p_1(\varphi, \psi)\psi'(s)] e^{-\int_0^s (P_x' + Q_y')ds} ds. \tag{35}$$

Since $\Delta(\tau, 0) = \Delta(0, 0)$, we obtain

$$u_{01} = \alpha_{01}(\tau) =$$
$$= \frac{1}{\Delta(0, 0)} e^{\int\limits_0^\tau (P'_x(\varphi, \psi) + Q'_y(\varphi, \psi))\,ds} \int\limits_0^\tau e^{-\int\limits_0^s (P'_x + Q'_y)\,ds} (q_1(\varphi, \psi)\varphi'(s) - p_1(\varphi, \psi)\psi'(s))\,ds. \quad (36)$$

Let us now proceed with the determination of u_{11}. The corresponding expression is much more complex than either (34) or (36). Integrating the fourth equation in (14) with the initial condition $\alpha_{11}(0) = 0$, we obtain

$$\alpha_{11}(s) = e^{\int\limits_0^s A_{10}(s)\,ds} \int\limits_0^s e^{-\int\limits_0^s A_{10}(s)\,ds} (2A_{20}\alpha_{10}\alpha_{01} + A_{11}\alpha_{10})\,ds.$$

Hence it follows that

$$u_{11} = \alpha_{11}(\tau) = e^{\int\limits_0^\tau A_{10}(s)\,ds} \int\limits_0^\tau e^{-\int\limits_0^s A_{10}(s)\,ds} (2A_{20}\alpha_{10}\alpha_{01} + A_{11}\alpha_{10})\,ds. \quad (37)$$

But $e^{\int\limits_0^s A_{10}(s)\,ds} = \alpha_{10}(s)$ and $e^{\int\limits_0^\tau A_{10}(s)\,ds} = \alpha_{10}(\tau) = u_{10}$. Therefore,

$$u_{11} = u_{10} \int\limits_0^\tau (A_{11} + 2A_{20}\alpha_{01})\,ds. \quad (38)$$

Inserting the expression for A_{11} from (22), we obtain

$$u_{11} = u_{10} \int\limits_0^\tau \left(\frac{g_{11} - A_{01}h_{10} - A_{10}h_{01}}{h_{00}} + 2A_{20}\alpha_{01} \right) ds. \quad (39)$$

By (29), the expression for g_{11} has the form

$$g_{11} = (q'_{1x}\beta_1 + q'_{1y}\gamma_1)\varphi'(s) - (p'_{1x}\beta_1 + p'_{1y}\gamma_1)\psi'(s) + q_1\beta'_1 - p_1\gamma'_1, \quad (40)$$

where the functions p_1 and q_1 and their derivatives are evaluated at the point $(\varphi(s), \psi(s))$. Using the obvious relations

$$p'_{1y}(\varphi(s), \psi(s))\psi'(s) = \frac{dp_1(\varphi(s), \psi(s))}{ds} - p'_{1x}(\varphi(s), \psi(s))\varphi'(s)$$

and

$$q'_{1x}(\varphi, \psi)\varphi' = \frac{dq_1(\varphi, \psi)}{ds} - q'_{1y}(\varphi, \psi)\psi'(s)$$

we rewrite the expression for g_{11} in the form

$$g_{11}(s) = [p'_{1x}(\varphi(s), \psi(s)) + q'_{1y}(\varphi(s), \psi(s))](\varphi'\gamma_1 - \psi'\beta_1) + (q_1\beta_1 - p_1\gamma_1)'. \quad (41)$$

By (30), $\varphi'\gamma_1 - \psi'\beta_1 = h_{00}$. Therefore, inserting (41) in (39), we obtain

$$u_{11} = u_{10} \int\limits_0^\tau [p'_{1x}(\varphi, \psi) + q'_{1y}(\varphi, \psi)]\,ds + I_0, \quad (42)$$

where

$$I_0 = u_{10} \int_0^{\tau} \left[\frac{(q_1\beta_1 - p_1\gamma_1)\nu' - A_{01}h_{10} - A_{10}h_{01}}{h_{00}} + 2A_{20}\alpha_{01} \right] ds. \tag{43}$$

Expressions (33) for A_{01}, (30) for h_{01}, and (35) for α_{01} show that if

$$p_1(\varphi(s), \psi(s)) \equiv q_1(\varphi(s), \psi(s)) \equiv 0, \tag{44}$$

i.e., if the functions $p_1(x, y)$ and $q_1(x, y)$ vanish on a closed path L_0, we have $I_0 = 0$. However, the integral

$$\int_0^{\tau} [p'_{1x}(\varphi(s), \psi(s)) + q'_{1y}(\varphi(s), \psi(s))]\, ds \tag{45}$$

need not vanish in this case. Indeed, let $F(x, y)$ be an analytical function satisfying the following conditions:

(a) $$F(\varphi(s), \psi(s)) \equiv 0,$$

(b) $$F'_x(\varphi(s), \psi(s))^2 + F'_y(\varphi(s), \psi(s))^2 \neq 0$$

(the proof of the existence of this function in some neighborhood of the path L_0 is conducted precisely in the same way as the proof of Lemma 1, § 15.1). Then, if

$$p_1(x, y) = F(x, y)F'_x(x, y), \quad q_1(x, y) = F(x, y)F'_y(x, y), \tag{46}$$

the integral in (45) does not vanish.

2. Statement of the problem

We are interested in the number of limit cycles of (A_μ) located in a sufficiently small neighborhood of the path L_0 for sufficiently small $\mu \neq 0$, i.e., the number of limit cycles created from a closed path L_0 on passing from $\mu = 0$ to close values of μ.

We define the function

$$d(n_0, \mu) = f(n_0, \mu) - n_0. \tag{47}$$

Our problem is clearly equivalent to the determination of the number of sufficiently small real roots of the function $d(n_0, \mu)$ for sufficiently small $\mu \neq 0$.

The creation of limit cycles from a closed path of (A_0) on passing to modified systems was considered in Chapter X. However, in Chapter X we dealt with all the possible modified systems sufficiently close to (A_0), i.e., the treatment was carried out in a sufficiently small neighborhood of the point (A_0) in the space of all dynamic systems R. For the case when the closed path L_0 is a closed k-tuple limit cycle of (A_0), we established that the maximum number of limit cycles created from L_0 on passing to other systems of this neighborhood is k (Theorem 42, § 27.1).

In the present section we will consider the creation of limit cycles from a closed path L_0 of (A_0) in a more restricted sense. The modified systems are confined to the systems (A_μ) corresponding to sufficiently

small values of the parameter μ, and we no longer consider any possible system sufficiently close to (A_0). In geometrical terms, this means that we are no longer dealing with the entire neighborhood of the point (A_0) in the space R, but only with some curve l_μ in this neighborhood which passes through the point (A_0). We will consider the creation (or disappearance) of limit cycles in the neighborhood of the path L_0 of (A_0) for motion along the curve l_μ in the space R.

We will derive the sufficient conditions for creation (or disappearance) of limit cycles in a number of simplest cases. These conditions evidently coincide with the sufficient conditions of appearance or disappearance of sufficiently small real roots of the equation

$$d(n_0, \mu) = 0 \tag{48}$$

on passing from $\mu = 0$ to sufficiently small finite μ.

First note that if the closed path L_0 is a simple limit cycle, i.e., when

$$u_{10} = e^{\int_0^\tau [P'_x(\varphi(s),\ \psi(s)) + Q'_y(\varphi(s),\ \psi(s))]ds} \neq 1 \tag{49}$$

(see (34)), there exists $n^* > 0$ and $\mu^* > 0$ which satisfy the following condition: if $|\mu| < \mu^*$, equation (48) has one and only one solution $n_0 = n_0(\mu)$, such that $|n_0(\mu)| < \mu^*$.

This follows from the theorem of structural stability of a simple limit cycle (see § 14, remark to Theorem 18). This also follows directly from the theorem of implicit functions (§ 1.2, Theorem 3 and remark to Theorem 4). Indeed, we see from (47), (19), and (49) that if L_0 is a simple limit cycle of (A_0), then

$$d(0, 0) = 0, \quad d'_{n_0}(0, 0) \neq 0,$$

so that the theorem of implicit functions is applicable.

However, if $u_{10} = 1$, we have

$$d(0, 0) = 0, \quad d'_{n_0}(0, 0) = 0$$

and the conditions of the theorem of implicit functions are not met.

Suppose that in this case there exists $k \geqslant 2$ such that

$$d(0, 0) = d'_{n_0}(0, 0) = \ldots = d^{(k-1)}_{n_0^{k-1}}(0, 0), \quad d^{(k)}_{n_0^k}(0, 0) \neq 0. \tag{50}$$

It is readily seen that we then can find $n^* > 0$ and $\mu^* > 0$ with the following property: for all μ, $|\mu| < \mu^*$ the equation

$$d(n_0, \mu) = 0$$

has at most k real roots which are smaller than n^* in absolute magnitude. Indeed, let n^* and μ^* be such that if $|n_0| < n^*$, $|\mu| < \mu^*$, we have

$$d^{(k)}_{n_0^k}(n_0, \mu) \neq 0. \tag{51}$$

Then, if $|\mu| < \mu^*$ and the function $d(n_0, \mu)$ has $k + 1$ roots in the interval $(-n^*, +n^*)$, its first derivative with respect to n_0 has at least k roots

in this interval, the second derivative has at least $k-1$ roots, etc., and the k-th derivative has at least one root, which contradicts (51). Thus, for small n_0 and μ, equation (48) cannot have more than k real roots. The question of existence of these roots, if any, requires special analysis, however. The analysis is conveniently carried out with the aid of *Newton's diagram or polygon*, which is described in the next section.

3. Newton's polygon and solution of the equation $F(w, z) = 0$

A significant point in our analysis of the roots of a function is the assumption that the variables are complex-valued.

We will therefore consider an analytical function $F(w, z)$, where w and z are complex variables. The results will then be applied to the case of real w and z.

Let $F(0, 0) = 0$. The expansion of $F(w, z)$ in powers of w and z around the point $(0, 0)$ has the form

$$F(w, z) = u_{10}w + u_{01}z + u_{20}w^2 + \ldots = \sum_{\substack{i^2+j^2>0 \\ i\geq 0,\ j\geq 0}} u_{ij}w^i z^j. \tag{52}$$

We introduce one further assumption regarding the function $F(w, z)$, namely that at least one of the coefficients u_{i0} and at least one of the coefficients u_{0j} do not vanish, i.e., the expansion (52) contains at least one term without z and at least one term without w. This assumption does not constitute a fundamental restriction of our problem (the problem of solution of the equation $F(w, z) = 0$ for w). Indeed, for all $u_{i0} = 0$ $(i = 1, 2, \ldots)$, the function $F(w, z)$ may be written in the form

$$F(w, z) = zF_1(w, z)$$

and the problem reduces to the analysis of the equation $F_1(w, z) = 0$. The situation is similar when all $u_{0j} = 0$ $(j > 0)$.

The following analysis is based on a number of theorems from the theory of analytical functions, which are partly given here without proof.

Theorem 68 (the theorem of implicit functions). Let $F(w, z)$ be an analytical function in the neighborhood of $(0, 0)$ and let

$$F(0, 0) = 0, \quad F'_w(0, 0) \neq 0.$$

There exist $\delta > 0$ and $\varepsilon > 0$ such that for every z, $|z| < \delta$, the equation $F(w, z) = 0$ has one and only one root $w = f(z)$ satisfying the inequality $|f(z)| < \varepsilon$. The function $f(z)$ can be expanded in positive integral powers of z, and the series will be convergent for $|z| < \delta$, i.e., it will be a single-valued analytical function of z which vanishes at $z = 0$.

Theorem 68 is the theorem of implicit functions in the case of complex variables. Its proof can be found in /22/, Ch. IV, p. 354. The proof uses a majorant series. Theorem 68 is also a direct consequence of the following theorem.

Theorem 69 (Weierstrass's preparatory theorem). Let $F(w, z)$ *be an analytical function in the neighborhood of* $(0, 0)$ *satisfying the conditions*

$$F(0, 0) = 0, \quad \frac{\partial F(0, 0)}{\partial w} = 0, \ \ldots \quad \frac{\partial^{k-1} F(0, 0)}{\partial w^{k-1}} = 0; \quad \frac{\partial^k F(0, 0)}{\partial w^k} \neq 0. \tag{53}$$

Then in some neighborhood $|w| < \varepsilon$, $|z| < \delta$ *of* $(0, 0)$, $F(w, z)$ *may be represented in the form*

$$F(w, z) = [w^k + A_1(z) w^{k-1} + \ldots + A_k(z)] \Phi(w, z), \tag{54}$$

where $\Phi(w, z)$ *is an analytical function which does not vanish in this neighborhood, and* $A_1(z), A_2(z), \ldots, A_k(z)$ *are analytical functions for* $|z| < \delta$.

The proof of Theorem 69 can be found in /22/, Ch.IV, p.352. It follows from Theorem 69 that in a sufficiently small neighborhood of the point $(0, 0)$, the equation

$$F(w, z) = 0 \tag{55}$$

is equivalent to the equation

$$w^k + A_1(z) w^{k-1} + \ldots + A_{k-1}(z) w + A_k(z) = 0, \tag{56}$$

whose left-hand side is a polynomial in w. Weierstrass's preparatory theorem thus reduces the local investigation of the general case of an implicit function $w(z)$ defined by equation (55) to the case of an implicit function defined by an algebraic equation in w (which, in general, is not an equation in z).

Note that relations (53) and (54) and the condition $\Phi(w, z) \neq 0$ lead directly to the equalities

$$A_1(0) = 0, \quad A_2(0) = 0, \ \ldots, \quad A_k(0) = 0. \tag{57}$$

Theorem 70. Let $F(w, z)$ *be an analytical function in the neighborhood of* $(0, 0)$ *which satisfies conditions (53). There exist* $\varepsilon > 0$ *and* $\delta > 0$ *such that for every* z, $|z| < \delta$, *the equation*

$$F(w, z) = 0$$

has precisely k *roots (either different or coinciding)* $w_1, w_2, \ldots, w_k$, *which are smaller than* ε *in absolute magnitude. Moreover, if* $z \to 0$, *each of the roots* w_i *also goes to zero.*

Proof. The validity of Theorem 70 follows directly from the previous theorem and from the theorem of the continuous dependence of the roots of a polynomial on its coefficients (see /29/, §73). Indeed, Theorem 69 shows that the roots of the equation $F(w, z) = 0$ coincide with the roots of equation (56). By (57), equation (56) has k roots at $z = 0$, which are all zero. Therefore, the roots of equation (56) go to zero for $z \to 0$.

Remark. The roots $w_1, w_2, \ldots, w_k$ of equation (55) depend on z. Suppose that for some $z = z_0$ all these roots are different. It is readily seen that in this case the roots $w_1, w_2, \ldots, w_k$ can be defined in the neighborhood of the point z_0 so that they are analytical functions of z in this neighborhood (to ensure all this, it suffices to require that the roots $w_i(z)$ vary continuously with the variation of z).

In what follows, we will only consider the case when the roots $w_1, w_2, \ldots, w_k$ are different for every $z \neq 0$ in some neighborhood U of the point O (this will be so if the discriminant $D(z)$ of equation (56), being an analytical function of z, does not vanish identically). Then equation (55) naturally determines near every point $z \neq 0$ of this neighborhood k analytical functions

$$w_1(z), w_2(z), \ldots, w_k(z) \tag{58}$$

which are the single-valued branches of the implicit function w. However, in the neighborhood U of the point O, these functions, in general, are not single-valued analytical functions. Indeed, it can be shown that if the point z moves along some Jordan curve around the point O, the value of the function $w_i(z)$ in general will have changed when we return to the initial point z. The function (58) (see /35/, Ch. XIII, 1) forms one or several nonintersecting systems with the following property: if the point z travels once along the curve Γ around the point O, the functions of each of these systems undergo a cyclic permutation. These systems are known as the cyclic systems of solutions of equation (55), and the point O is the branching point of the function $w(z)$. Clearly, for $z \to 0$, all the functions $w_i(z) \to 0$.

If the discriminant $D(z) \equiv 0$, the situation is more complicated. This case, however, will not be required in the following, and it is not considered here.

We now have to consider the form of the solutions $w_i(z)$ of (55) in the neighborhood of O.

If $k = 1$, Theorem 68 shows that there exists a unique solution, which may be written as a series in integral powers of z, i.e., as a series

$$w = \gamma_1 z + \gamma_2 z^2 + \gamma_3 z^3 + \ldots.$$

which converges for sufficiently small z.

For $k > 1$, each of the k solutions which exist by Theorem 70 and the remark to the theorem can no longer be represented as a series in integral powers of z. To establish the natural form of the solutions in this case, let us first consider some simple examples. Take equation (55) in the form

$$u_{01} + u_{20}w^2 + u_{03}z^3 = 0.$$

where none of the coefficients vanish. Then

$$w = \sqrt{-\frac{u_{01}}{u_{20}}}\, z^{1/2} \sqrt{1 + \frac{u_{03}}{u_{20}} z^2},$$

where $\sqrt{-\frac{u_{01}}{u_{20}}}$ and $\sqrt{1 + \frac{u_{03}}{u_{20}} z^2}$ are to be regarded as one (fixed) value of each of these roots. For small z, the second root can be series-expanded in integral powers of z. The solution w therefore has the form

$$w = \alpha_1 z^{1/2} + \alpha_2 z^{3/2} + \ldots,$$

where $\alpha_1 \neq 0$. The last series gives both solutions for w, since $z^{1/2}$ is a two-valued function of z.

If equation (55) has the form

$$u_{01}z + u_{30}w^3 + u_{02}z^3 = 0,$$

its solutions can be expressed by the series

$$w = \beta_1 z^{1/3} + \beta_2 z^{4/3} + \ldots,$$

where $\beta_1 \neq 0$, and this notation gives all the three solutions of the equation, since $z^{1/3}$ is a three-valued function of z.

In the above examples, the solutions of equation (55) are expressed by series in fractional (rational) positive powers of z. This suggests that the same representation will obtain in the general case also. We will therefore seek solutions of (55) in the form

$$w = \gamma z^{\alpha} + \gamma_1 z^{\alpha_1} + \gamma_2 z^{\alpha_2} + \ldots, \tag{59}$$

where $\gamma \neq 0$, and α and α_i are positive rational numbers, $\alpha < \alpha_1 < \alpha_2 < \ldots$ Let us first establish what power exponent α will ensure convergence of the power series (59) to a solution of equation (55) for small z.

Let u_{k0} be the first nonzero coefficient of the form u_{i0} in expansion (52), and u_{0l} the first nonzero coefficient of the form u_{0j} (these coefficients exist by assumption).

If (59) is a solution of equation (55), we have

$$F(\gamma z^{\alpha} + \gamma_1 z^{\alpha} + \ldots, z) \equiv 0, \tag{60}$$

i.e., inserting for w in the series (52) its expansion from (59), we obtain identically zero. This enables us to determine the numbers α and γ. As long as α remains unknown, we cannot identify exactly the lowest-order terms in series (52) obtained after substitution from (59). We can nevertheless isolate a finite number of terms which definitely include the lowest-order terms.

These are primarily terms of the form

$$u_{0l} z^l \text{ and } u_{k0} \gamma^k z^{\alpha k} \tag{61}$$

and also the terms

$$u_{ij_i} \gamma^i z^{\alpha i + j_i}, \tag{62}$$

where $1 \leqslant i \leqslant k-1$, and the index j_i satisfies the inequalities $1 \leqslant j_i < l-1$ and is moreover the smallest index j for which the coefficient u_{ij} with fixed i does not vanish.

It is readily seen that the terms of lowest order in z must be contained among the terms (61) and (62).

If identity (60) is satisfied, terms of the lowest order in z mutually cancel. Since none of the coefficients u_{ij} in (61) and (62) is zero, we conclude that at least two such terms should be of the same order, which, however, should not exceed the order of the remaining terms. Thus, the numbers

$$k\alpha,\ (k-1)\alpha + j_{k-1},\ (k-2)\alpha + j_{k-2},\ \ldots,\ \alpha + j_1,\ l \tag{63}$$

include at least two equal numbers which are not greater than all the other numbers. Suppose that $j_k = 0$ and $j_0 = l$, i.e., the numbers in (63) have the form $i\alpha + j_i$, $i = 0, 1, 2, \ldots, k$. Then α should satisfy at least one of the linear equations

$$i_1\alpha + j_{i_1} = i_2\alpha + j_{i_2}, \tag{64}$$

where $i_1 \neq i_2$, and for any $i = 0, 1, 2, \ldots, k$ we should have

$$i\alpha + j_i \geqslant i_1\alpha + j_{i_1} \qquad (= i_2\alpha + j_{i_2}). \tag{65}$$

These values of α will be called feasible. There may be several feasible values. To find all the feasible values, we will use a geometrical method known as the method of Newton's polygon.

We introduce rectangular coordinates i, j in the plane and assign to every term in (61) and (62) a point A_i on the plane with the coordinates (i, j_i) $(i = 0, 1, 2, \ldots, k)$. The terms in (61) evidently correspond to points of the form $A_0(0, l)$ and $A_k(k, 0)$ which lie on the coordinate axes: the other points A_i fall in the first quadrant, and their abscissas and ordinates do not exceed $k-1$ and $l-1$, respectively.

Suppose that for some α relation (64) is satisfied, and $i_1 \neq i_2$. Then

$$\alpha = -\frac{j_{i_2} - j_{i_1}}{i_2 - i_1},$$

i.e., α is the slope factor of the straight line through the points $A_{i_1}(i_1, j_{i_1})$ and $A_{i_2}(i_2, j_{i_2})$.

The equation of this straight line is

$$j - j_{i_1} = -\alpha(i - i_1) \text{ or } j - j_{i_1} - \alpha(i - i_1) = 0. \tag{66}$$

Clearly, for the points (i, j) which lie not lower than the line (66), we have $j - j_{i_1} + \alpha(i - i_1) \geqslant 0$, i.e.,

$$j + \alpha_i \geqslant j_{i_1} + \alpha_{i_1}, \tag{67}$$

and for the points below this line, we have

$$j + \alpha_i < j_{i_1} + \alpha i_i. \tag{68}$$

Condition (65) signifies that each of the points $A_i\,(i, j_i)$, $i = 0, 1, 2, \ldots, k$, lies either on the line (66) or above this line. Hence it follows that every feasible α is in a one-to-one correspondence to a straight line passing at least through two points A_i, which has a negative slope factor and is so located that none of the points A_i lies below this line. We will refer to these lines as feasible lines. α is equal to minus the slope factor of the corresponding line.

Thus, in order to find all the feasible values of α, we should find all the feasible lines. This can be accomplished in the following way. Let s be a moving line which initially coincides with the axis i. We turn this line clockwise around the point $A_k\,(k, 0)$ until it passes through one of the points A_i, the point $A_{k_1}\,(k_1, j_{k_1})$ say. The line $A_kA_{k_1}$ obtained in this way will be designated s_1. This is evidently a feasible line. The line s_1 may pass through more than two points A_i. Then, let A_{k_1} be the leftmost of these points (i.e., the one with the least abscissa).

If A_{k_1} does not coincide with the point $A_0\,(0, l)$, the moving line will be further rotated in the clockwise direction around the point A_{k_1} until it passes through some point $A_{k_2}\,(k_2, j_{k_2})$ (if there are several such points, A_{k_2} is chosen as the leftmost). The resulting curve is designated s_2, and the process is continued until the moving line passes through the

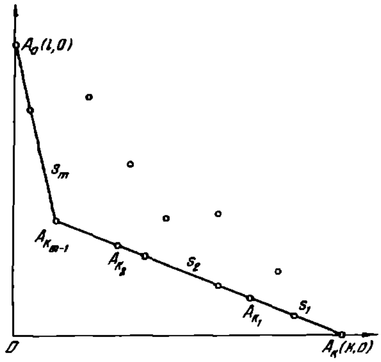

FIGURE 169

point A_0 (0, l). The result is a straight line designated s_m (Figure 169). In particular cases, m may be equal to 1.

The convex polygonal line $A_k A_{k_1} A_{k_2} \ldots$ $\ldots A_{k_{m-1}} A_0$ obtained in this construction is known as Newton's polygon (or Newton's diagram). It is clear from the construction that each side of Newton's polygon is a segment of a feasible line. It is readily seen, however, that the converse is also true: every feasible line contains a segment which is one of the sides of Newton's polygon. We have thus established that all the feasible values of α are equal to minus the slope factors of the sides of Newton's polygon.

Now let the α in expansion (59) be one of the feasible values of the index. We will determine the corresponding value of the coefficient γ. To this end, we shall make use of the condition that the lowest terms in the left-hand side of the identity (60) should mutually cancel. To fix ideas, let α be the slope factor of the side $A_{k_1}A_{k_2}$ of Newton's polygon and let $A_{l_1}, A_{l_2}, \ldots, A_{l_s}$, $k_1 > l_1 > l_2 > \ldots > l_s > k_2$, be those among the points A_i which lie on this side. The lowest terms of the corresponding expression will cancel out if the coefficient of the lowest degree of z (equal to $k_1\alpha + j_{k_1} = l_1\alpha + j_{l_1} = \ldots = k_2\alpha + j_{k_2}$) is zero, i.e., if

$$g(\gamma) = u_{k_1 j_{k_1}}\gamma^{k_1} + u_{l_1 j_{l_1}}\gamma^{l_1} + \ldots + u_{l_s j_{l_s}}\gamma^{l_s} + u_{k_2 j_{k_2}}\gamma^{k_2} = 0. \tag{69}$$

Since by assumption $\gamma \neq 0$, we are only interested in the nonzero roots of equation (69), i.e., the roots of the polynomial

$$h(\gamma) = u_{k_1 j_{k_1}}\gamma^{k_1-k_2} + u_{l_1 j_{l_1}}\gamma^{l_1-k_2} + \ldots + u_{k_2 j_{k_2}}. \tag{70}$$

The number of these roots (counting each root according to its multiplicity) is $k_1 - k_2$. Therefore, the number of values of the coefficient γ corresponding to each side of Newton's polygon is equal to the number of units accommodated by the projection of this side onto the abscissa axis. The values of the coefficient may be real or complex, and some of them may be equal to one another. Since the length of the projection of the entire Newton's polygon is k units, we conclude that the first coefficient γ of the series (59) has k values (not all of which are necessarily different).

We have established the necessary conditions to be satisfied by α and γ when $w = \gamma z^{\alpha} + \gamma_1 z^{\alpha_1} + \ldots$ is a solution of the equation $F(w, z) = 0$. These conditions can be stated as follows: α should be equal to the absolute value of the slope factor of any of the sides of Newton's polygon, and γ — for a fixed α — should be equal to one of the nonzero roots of equation (69).

Let us consider the case when $\gamma \neq 0$ is a simple root of (69). We will show that the above conditions are also sufficient for the equation $F(w, z) = 0$ to have a solution of the form (59) with the given α and γ.

Let $\alpha = \frac{p}{q}$, where p and q are irreducible numbers, be a feasible value of the index corresponding, as before, to the side $A_{k_1}A_{k_2}$ of Newton's polygon, and $\gamma \neq 0$ one of the roots of equation (69). The equalities

$$k_1\alpha + j_{k_1} = l_1\alpha + j_{l_1} = \ldots = l_s\alpha + j_{l_s} = k_2\alpha + j_{k_2}$$

clearly show that each of the numbers $k_1 - k_2,\ l_1 - k_2,\ \ldots,\ l_s - k_2$ is a multiple of q, i.e., $h(\gamma)$ is a polynomial in γ^q of degree $\frac{k_1 - k_2}{q}$.

Now let $\gamma \neq 0$ be a simple root of the polynomial $g(\gamma)$ (and hence of $h(\gamma)$). Then

$$g(\gamma) = 0, \qquad g'(\gamma) \neq 0. \tag{71}$$

To investigate the equation $F(w, z) = 0$ for $z \neq 0$, we substitute

$$w = vz^{\alpha} = vz^{p/q}, \tag{72}$$

where $z^{1/q}$ is one of the q possible values of this q-valued function. Using (52) and (69), we readily see that

$$F(w, z) = F(vz^{p/q}, z) = z^{k_1\alpha + j_{k_1}}[g(v) + z^{\lambda/q}\varphi(v, z^{1/q})], \tag{73}$$

where λ is a positive integer, and $\varphi(v, z^{1/q})$ is a power series in v and $z^{1/q}$, which is a priori known to converge for all z and $vz^{1/q} = w$ of sufficiently small magnitude and, therefore, converges for all sufficiently small $|z|$, if v takes its values from a bounded region. The equation $F(w, z) = 0$ is equivalent for $z \neq 0$ to the equation

$$g(v) + z^{\lambda/q}\varphi(v, z^{1/q}) = 0.$$

Substituting

$$z^{1/q} = \zeta, \tag{74}$$

we obtain the equation

$$\Phi(v, \zeta) = g(v) + \zeta^{\lambda}\varphi(v, \zeta) = 0. \tag{75}$$

The coefficients of the power series representing the function $\varphi(v, \zeta)$ and therefore the function itself are independent of the particular value of the q-valued function $z^{1/q}$ used in the substitution (72).

By (71),

$$\Phi(\gamma, 0) = 0, \qquad \Phi_v'(\gamma, 0) \neq 0.$$

Therefore, by Theorem 68 (the theorem of implicit functions), equation (75) has a unique solution in a sufficiently small neighborhood of the point $\zeta = 0$, which reduces to γ for $\zeta = 0$. This solution has the form

$$v = \gamma + \gamma_1\zeta + \gamma_2\zeta^2 + \ldots \tag{76}$$

The corresponding solution is

$$w = vz^{p/q} = \gamma z^{p/q} + \gamma_1 z^{(p+1)/q} + \gamma_2 z^{(p+2)/q} + \ldots \tag{77}$$

We have thus established that if $\alpha = p/q$ is a feasible value of the index, and γ is a simple root of equation (69) corresponding to this α, the equation $F(w, z) = 0$ has a solution of the form (59).

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be all the feasible values of the index α for the equation $F(w, z) = 0$, and let $\alpha_i = p_i/q_i$, where p_i and q_i are irreducible numbers. Consider the case when for any α_i $(i = 1, 2, \ldots, m)$, all the non-zero roots of the corresponding equation (69) are simple. Let the number of these simple roots be r_i; the roots are designated $\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{ir_i}$. We have seen before that if k is the number defined by (53), then

$$r_1 + r_2 + \ldots + r_n = k. \tag{78}$$

As we have just established, to every pair of numbers α_i, γ_{ij} $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, s_i)$ corresponds to a solution

$$w = \gamma_{ij} z^{p_i/q_i} + \gamma_{ij_1} z^{(p_i+1)/q_i} + \ldots \tag{79}$$

Suppose that for a fixed i, the same value of the root $\sqrt[q_i]{z} = z^{1/q_i}$ is taken in (79) for all j, $j = 1, 2, \ldots, r_i$. It follows from (78) that there exist precisely k solutions of the form (79). We will show that if $z \neq 0$ is sufficiently small in absolute magnitude, the values of the solution (79) are all different. Indeed, if this is not so, there exist two solutions of the form (79), w_1 and w_2, say, which take on identical values on a sequence of points $z_l \to 0$. If these solutions correspond to different α, they have different orders of smallness relative to z; therefore $\lim \frac{w_1(z_l)}{w_2(z_l)}$ is either 0 or ∞, and the equality $w_1(z_l) = w_2(z_l)$ breaks down for large l. If the solutions w_1 and w_2 correspond to the same α, they differ in the first coefficients γ. Therefore $\lim \frac{w_1(z_l)}{w_2(z_l)} \neq 1$ and the equality $w_1(z_l) = w_2(z_l)$ again breaks down. Our proposition is thus proved. It follows that all the k solutions (79) of the equation $F(w, z) = 0$ which go to zero for $z \to 0$ are different. But then by Theorem 70, the functions (79) exhaust all the solutions of the equation $F(w, z) = 0$ which go to zero for $z \to 0$.

We will now show how to find the cyclic systems of solutions mentioned in the remark to Theorem 70. In the process we will derive a different form of solutions (79), which will be useful at a later stage. As before, we assume that for every α_i, $i = 1, 2, \ldots, m$, all the non-zero roots of equation (69) are simple. It suffices to consider one of the feasible values α_i. We will designate it α, and let

$$\alpha = p/q, \quad (p, q) = 1.$$

Let

$$h(\gamma) = u_{k_1 j_{k_1}} \gamma^{k_1 - k_2} + u_{l_1 j_{l_1}} \gamma^{l_1 - k_2} + \ldots + u_{k_2 j_{k_2}} \tag{70}$$

be the polynomial h corresponding to the given α.

We have seen before that $h(\gamma)$ is a polynomial of degree $d = \frac{k_1 - k_2}{q}$ in γ^q.
Let

$$\Gamma = \gamma^q \tag{80}$$

and

$$H(\Gamma) = u_{k_1 j_{k_1}} \Gamma^{\frac{k_1 - k_2}{q}} + u_{l_1 j_{l_1}} \Gamma^{\frac{l_1 - k_2}{q}} + \ldots + u_{k_2 j_{k_2}} \tag{81}$$

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$ be the roots of the polynomial $H(\Gamma)$. Since

$$h(\gamma) = H(\gamma^q) \tag{82}$$

and all the roots of the polynomial $h(\gamma)$ are simple, the numbers $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$ are all different.

Let

$$\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{iq} \tag{83}$$

be all the q-th degree roots of Γ_i. By (80) and (82), these numbers are roots of the polynomial $h(\gamma)$. Thus, every root Γ_i of the polynomial $H(\Gamma)$ corresponds to a sequence (83) of roots of the polynomial $h(\gamma)$. We know from algebra that the numbers (83) can be expressed in the form

$$\gamma_{i1}, \gamma_{i1}\varepsilon, \gamma_{i1}\varepsilon^2, \ldots, \gamma_{i1}\varepsilon^{q-1}, \tag{84}$$

where γ_{i1} is one of the q-th degree roots of Γ_i (any of the q roots can be chosen, as long as it is kept fixed), and ε is the fundamental q-th degree root of 1.

Let ζ_0 be one of the values of $\sqrt[q]{z}$ (also quite arbitrary, provided it is fixed). As we have seen above, the sequence (83) of the numbers γ_{ij} corresponds to the sequence

$$w_{i1}, w_{i2}, \ldots, w_{iq} \tag{85}$$

of solutions of the equation $F(w, z) = 0$, where

$$w_{ij} = \zeta_0^p(\gamma_{ij} + \gamma_{ij1}\zeta_0 + \gamma_{ij2}\zeta_0^2 + \ldots) \quad (j = 1, 2, \ldots, q). \tag{86}$$

If the ζ_0 in the right-hand side of (86) is replaced with some other value of the q-th degree root of z, we also obtain a solution of the equation $F(w, z) = 0$. Therefore, if we keep j fixed, setting for convenience $j = 1$, and let the ζ_0 in (86) run over all the values of the q-th degree root of z, i.e., over the numbers

$$\zeta_0, \zeta_0\varepsilon, \zeta_0\varepsilon^2, \ldots, \zeta_0\varepsilon^{q-1}, \tag{87}$$

we obtain a set of q solutions

$$\widetilde{w}_{i1}, \widetilde{w}_{i2}, \ldots, \widetilde{w}_{iq}, \tag{88}$$

where

$$\widetilde{w}_{il} = (\zeta_0\varepsilon^l)^p[\gamma_{i1} + \gamma_{i11}\zeta_0\varepsilon^l + \gamma_{i12}(\zeta_0\varepsilon^l)^2 + \ldots] \quad (l = 0, 1, 2, \ldots, q-1) \tag{89}$$

or

$$\widetilde{w}_{il} = \zeta_0^p[\gamma_{i1}\varepsilon^{lp} + \gamma_{i11}\varepsilon^{p(l+1)}\zeta_0 + \ldots] \quad (l = 0, 1, 2, \ldots, q-1). \tag{90}$$

Since ε is the fundamental q-th degree root of 1, and $(p, q) = 1$, the numbers ε^{lp} $(l = 0, 1, 2, \ldots, q-1)$ constitute the set of all the roots of q-th degree of 1. Therefore, the numbers $\gamma_{i1}\varepsilon^{lp}$ $(l = 0, 1, \ldots, q-1)$ coincide with the numbers (84) and hence with the numbers (83). Since every solution of the form (79) is determined by its first coefficient, it thus follows that solutions (88) coincide with solutions (85).

The structure of the solutions (88), determined by (89), readily shows that the functions (88) constitute a cyclic system of solutions. Indeed, as the point z moves around the origin tracing a simple closed curve, the numbers $\zeta_0\varepsilon^l$, which are the q-th degree roots of z, undergo a cyclic permutation.

Let us summarize our results. Let $\alpha = p/q$ be a feasible value of the index (obtained by construction of Newton's polygon). To obtain the solutions of the equation $F(w, z) = 0$ corresponding to this α, we have to find the roots $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$ of the equation $H(\Gamma) = 0$. Let γ_{i1} be one of the q-th degree roots of Γ_i $(i = 1, 2, \ldots, d)$. The corresponding solution is

$$w_{i1} = \zeta_0^p(\gamma_{i1} + \gamma_{i11}\zeta_0 + \ldots), \tag{86}$$

where ζ_0 is any fixed value of $\sqrt[q]{z}$. Making ζ_0 run over all the values of this root, we obtain q solutions

$$w_{i1}, w_{i2}, \ldots, w_{iq},$$

which constitute a cyclic system. Since $i = 1, 2, \ldots, d$, we obtain a total of d cyclic systems of solutions, corresponding to the feasible value $\alpha = p/q$. We recall that $d = \frac{k_1 - k_2}{q}$, where $k_1 - k_2$ is the degree of the polynomial (70).

In our analysis of the creation of limit cycles from a multiple limit cycle, the function $F(w, z)$ will be identified with $d(n_0, \mu)$ (see (47)), i.e., with a real-valued function of real variables, and we will be concerned with the existence or absence of real roots of this function.

The following lemma will be of considerable importance in this connection.

Lemma 1. If all the coefficients of the series expansion of the function $F(w, z)$ *are real, and* γ *is a real simple non-zero root of the equation* $g(\gamma) = 0$ *(see* (69)*) or, equivalently, of the function* $h(\gamma)$, *all the coefficients* $\gamma_1, \gamma_2, \ldots$ *of the corresponding solution* (77)

$$w = \gamma z^{p/q} + \gamma_1 z^{(p+1)/q} + \gamma_2 z^{(p+2)/q} + \ldots$$

of the equation $F(w, z) = 0$ *are real.*

Proof. The numbers γ and γ_i $(i = 1, 2, \ldots)$ are the coefficients of the power-series expansion of the function v (see (76)) satisfying equation (75). Since the series of the function $F(w, z)$ has real coefficients, the series of the function $\Phi(v, z)$ also has real coefficients. When solving the equation $\Phi(v, z) = 0$ by the method of indeterminate coefficients, each successive coefficient γ_{i+1} is a polynomial in the coefficients of the function Φ and the coefficients $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_i$ (see /11/, Vol. II, Sec. 450). But then all the coefficients γ_i are real. Q.E.D.

4. The behavior of limit cycles of some dynamic systems following small changes in the parameter

In this subsection we will apply the previous results in order to establish what happens to a limit cycle of a dynamic system depending on a parameter following a small change in the parameter. We will

only consider systems of two particular types, one of which includes systems obtained from some fixed system by a rotation of the vector field. The methods used, however, can be applied to other types of systems depending on a parameter. As throughout this chapter, we will again consider only analytical dynamic systems.

We will first prove a number of lemmas, which are extensively used in the main propositions. These lemmas follow directly from the results of the previous subsection.

Consider the equation

$$F(w, z) = u_{10}w + u_{01}z + u_{20}w^2 + \ldots = 0. \tag{55}$$

Let all the coefficients u_{ij} of this equation be real and let u_{k0} be the first of the coefficients u_{i0} which does not vanish. When speaking of solutions or roots of equation (55), we will invariably mean roots of sufficiently small magnitude which correspond to sufficiently small non-zero real values of z.

Lemma 2. Let k be an even number, and $u_{01} \neq 0$. Then, if $\frac{u_{01}}{u_{k0}} < 0$ $\left(\frac{u_{01}}{u_{k0}} > 0\right)$, the equation $F(w, z) = 0$ has two different real roots for $z > 0$ $(z < 0)$, which are simple roots, and has no real roots for $z < 0$ $(z > 0)$.

Proof. Let $k = 2l$, $l \geqslant 1$. Since $u_{01} \neq 0$, Newton's polygon consists of a single segment through the points $A_{2l}(2l, 0)$ and $A_0(0, 1)$ (Figure 170), i.e., the index α may take on one value only, $\alpha = \frac{1}{2l}$. The values of the coefficients γ are obtained from equation (69), which takes on the form

$$u_{2l0}\gamma^{2l} + u_{01} = 0. \tag{91}$$

Therefore, the equation $F(w, z) = 0$ has $2l$ solutions. As we have seen before, all these solutions may be written in the form

$$w = z^{1/2l}(\gamma + \gamma_1 z^{1/2l} + \gamma_2 z^{2/2l} + \ldots), \tag{92}$$

where the coefficients γ and γ_i are the same, and $z^{1/2l}$ successively runs over all the $2l$-th degree roots of z.

If $\frac{u_{01}}{u_{2l0}} < 0$, equation (91) has two real roots. Let one of these roots be the γ in (92). Then all the coefficients γ_i, $i = 1, 2, \ldots$, are also real by Lemma 1.

If $z > 0$, there exist two real values of the function $z^{1/2l}$. These values correspond, by (92), to real values of w. The complex values of $z^{1/2l}$ correspond for sufficiently small z to values of w which are close to the complex number $z^{1/2l}\gamma$, i.e., are also complex. If $z < 0$, all the values of the function $z^{1/2l}$ are complex, and the function w may not have real values for sufficiently small z.

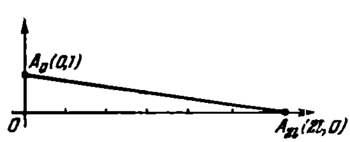

FIGURE 170

Since all the roots of equation (91) are different, all the $k = 2l$ roots of the equation

$F(w, z) = 0$ are also different for sufficiently small z, and are thus simple. We have thus proved the lemma for $\frac{u_{01}}{u_{2l0}} < 0$. The case $\frac{u_{01}}{u_{2l0}} > 0$ is reduced to the previous case by substituting $-z$ for z. This completes the proof of the lemma.

Lemma 3. If k is odd and $u_{01} \neq 0$, the equation $F(w, z) = 0$ has precisely one real root for both $z > 0$ and $z < 0$, and this root is simple.

Proof. Let $k = 2l + 1$, $l \geqslant 1$. Equation (69) has the form

$$u_{2l+1,0}\gamma^{2l+1} + u_{01} = 0,$$

and the corresponding solution of the equation $F(w, z) = 0$ is

$$w = z^{\frac{1}{2l+1}}\left(\gamma + \gamma_1 z^{\frac{1}{2l+1}} + \gamma_2 z^{\frac{2}{2l+1}} + \ldots\right).$$

The validity of the lemma is established along the same lines as in the previous proof.

Corollary. If k is odd, the equation $F(w, z) = 0$ may have more than one real solution only if $u_{01} = 0$.

Lemma 4. If k is odd and

$$u_{01} = 0, \quad u_{11} \neq 0, \tag{93}$$

the equation $F(w, z) = 0$ has either three real roots for $z > 0$ and one real root for $z < 0$, or three real roots for $z < 0$ and one real root for $z > 0$. All the roots of the equation, and in particular its real roots, are simple.

Proof. Let $k = 2l + 1$, $l \geqslant 1$ and let u_{0m} be the first of the coefficients u_{0j} which does not vanish. Since $u_{01} = 0$, we have $m \geqslant 2$.

It is readily seen that in this particular case Newton's polygon is made up of two segments, one through the points $A_0(0, m)$ and $A_1(1, 1)$ and the other through the points $A_1(1, 1)$ and $A_{2l+1}(2l+1, 0)$ (Figure 171). Consequently, α may have two values, namely $\alpha_1 = m - 1$ and $\alpha_2 = \frac{1}{2l}$. The equation for γ corresponding to $\alpha_1 = m - 1$ is $u_{11}\gamma + u_{0m} = 0$; this equation has a real root. The corresponding solution

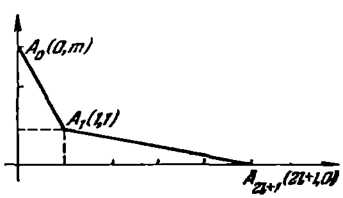

FIGURE 171

$$w = z(\gamma + \gamma_1 z + \gamma_2 z^2 + \ldots) \tag{94}$$

is an analytical function of z. By Lemma 1, all the γ_i ($i = 1, 2, \ldots$) are real and, therefore, for any real z, the solution w is real.

The equation for γ corresponding to $\alpha_2 = \frac{1}{2l}$ is

$$u_{2l+1\,0}\gamma^{2l+1} + u_{11}\gamma = 0 \tag{95}$$

(see (69)), and there are $2l$ solutions of the form

$$w = z^{\frac{1}{2l}}\left(\gamma + \gamma_1 z^{\frac{1}{2l}} + \gamma_2 z^{\frac{2}{2l}} + \ldots\right). \tag{96}$$

Consider the case $\frac{u_{11}}{u_{2l+1\,0}} < 0$. Equation (96) has a real root and, as in our proof of Lemma 2, we can show that there are two real solutions of the form (96) for $z > 0$ and no such solutions for $z < 0$. Thus, if $\frac{u_{11}}{u_{2l+1,0}} < 0$, the equation $F(w, z) = 0$ has three real solutions for $z > 0$ (one of the form (94) and two of the form (96')), and only one real solution (of the form (94)) for $z < 0$.

The case $\frac{u_{11}}{u_{2l+1\,0}} > 0$ is reduced to the previous case by substituting $-z$ for z. In this case, the equation $F(w, z) = 0$ has one real solution for $z > 0$, and three real solutions for $z < 0$. Since all the roots of equation (95) are different, all the roots of the equation $F(w, z) = 0$ (and in particular, its real roots) are also different and hence simple. This completes the proof of the lemma.

We can now proceed with the main propositions of this subsection. We will first establish what happens to a limit cycle of a dynamic system when the vector field is rotated. Theorems 71 and 72 provide the answers to this question.

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be the starting analytical system.

The system (A_μ) depending on a parameter is chosen in the form

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) - \mu Q(x, y) = \bar{P}(x, y, \mu), \\ \frac{dy}{dt} &= Q(x, y) + \mu P(x, y) = \bar{Q}(x, y, \mu), \end{aligned} \tag{A_μ}$$

and the vector field of this system is obtained from the vector field of (A) by rotating through an angle equal to $\tan^{-1}\mu$ (see end of § 3).

We will use the same notation as in § 32.2. Then, by (23), the functions p_i and q_i corresponding to (A_μ) have the form

$$\begin{aligned} &p_1(x, y) = -Q(x, y), \quad q_1(x, y) = P(x, y), \\ &p_i(x, y) \equiv q_i(x, y) \equiv 0 \quad \text{for} \quad i \geqslant 2. \end{aligned} \tag{97}$$

Suppose that system (A), or equivalently (A_0), has a limit cycle L_0.

Theorem 71. If L_0 is a limit cycle of even multiplicity of system (A_0), there exist $\varepsilon > 0$ and $\mu_0 > 0$ with the following property: either for every $\mu > 0$, $|\mu| < \mu_0$, (A_μ) has precisely two limit cycles in $U_\varepsilon(L_0)$, which are moreover structurally stable, and has no limit cycles whatsoever in $U_\varepsilon(L_0)$ for every $\mu < 0$, $|\mu| < \mu_0$, or conversely, for every $\mu > 0$, $|\mu| < \mu_0$, (A_μ) has no limit cycles in $U_\varepsilon(L_0)$, whereas for $\mu < 0$, $|\mu| < \mu_0$, (A_μ) has precisely two limit cycles in $U_\varepsilon(L_0)$, which are structurally stable. The number ε may be chosen as small as desired.

Proof. Let $n = f(n_0, \mu)$ be the succession function constructed for (A_μ) on some normal to the limit cycle L_0 (see (16)), and $d(n_0, \mu) = f(n_0, \mu) - n_0$ (see (47)).

As in § 32.1, we assume that the limit cycle L_0 of the system (A_0) corresponds to $n_0 = 0$. The series expansion of the function $d(n_0, \mu)$ near the point (0, 0), by (19) and (47), has the form

$$d(n_0, \mu) = (u_{10} - 1)\, n_0 + u_{01}\mu + u_{20}n_0^2 + \ldots \tag{98}$$

In particular,

$$d(n_0, 0) = (u_{10} - 1) n_0 + u_{20} n_0^2 + u_{30} n_0^3 + \dots \tag{99}$$

By assumption, L_0 is a limit cycle of even multiplicity of (A_0). This and Definition 28 (§ 26.2) show that $u_{10} - 1 = 0$ and that the first non-zero coefficient in expansion (99) is a coefficient before an even power of n_0. Let this coefficient be $u_{2l,0}$, where $l \geqslant 1$. Thus,

$$d(n_0, \mu) = u_{2l,0} n_0^{2l} + u_{01}\mu + \dots \tag{100}$$

The coefficient u_{01} may be computed from (36), § 32.1. Let

$$x = \varphi(t), \quad y = \psi(t),$$

where φ and ψ are periodic functions of period τ, be the solution of (A_0) corresponding to the limit cycle L_0. Then

$$P(\varphi(s), \psi(s)) \equiv \varphi'(s), \quad Q(\varphi(s), \psi(s)) \equiv \psi'(s).$$

By these relations and (97), we have

$$q_1(\varphi(s), \psi(s)) \equiv \varphi'(s), \quad p_1(\varphi(s), \psi(s)) \equiv -\psi'(s).$$

Inserting the last expressions in (36), we obtain

$$u_{01} = \frac{1}{\Delta(0, 0)} e^{\int_0^\tau (P'_x + Q'_y)\, ds} \int_0^\tau e^{-\int_0^s (P'_x + Q'_y)\, ds} [\varphi'(s)^2 + \psi'(s)^2]\, ds.$$

Hence it follows that $u_{01} \neq 0$. But then, in virtue of (100), the conditions of Lemma 2 are satisfied for the function $d(n_0, \mu)$ if it is considered as $F(w, z)$. By this lemma, (A_μ), for sufficiently small μ, has precisely two limit cycles in a sufficiently small neighborhood of the path L_0. Since by the same Lemma 2 the real roots of the equation $d(n_0, \mu) = 0$ to which these cycles correspond are simple roots, the corresponding limit cycles are structurally stable (see § 13.3, (31), and also § 13.3, Definition 18, and § 14, Theorem 18). Q.E.D.

The proposition of Theorem 71 may be formulated in the following graphic form:

When the vector field of a dynamic system is rotated in one direction, a limit cycle of even multiplicity decomposes into two structurally stable cycles, and when the field is rotated in the opposite direction, the limit cycle disappears.

Let us now consider the case when the dynamic system (A) has a limit cycle L_0 of odd multiplicity.

Theorem 72. ***If L_0 is a limit cycle of odd multiplicity of dynamic system (A_0), there exist $\varepsilon > 0$, $\mu_0 > 0$ such that for all $\mu \neq 0$, $|\mu| < \mu_0$, (A_μ) has a single limit cycle in $U_\varepsilon(L_0)$, which is moreover structurally stable. The number ε may be taken as small as desired.***

Proof. The first non-zero coefficient of n_0^k $(k = 1, 2, \dots)$ in expansion (98) is a coefficient before an odd power of n_0, e.g., the coefficient before n_0^{2l+1}, where $l \geqslant 0$. The coefficient u_{01} has the same form as in the previous

theorem, i.e., it does not vanish. The proposition of Theorem 72 then follows directly from Lemma 3. Q.E.D.

Remark 1. If L_0 is of multiplicity 1, i.e., a simple limit cycle of (A), then it is a structurally stable path (Theorem 18, §14). Therefore, not only the system (A_μ) obtained by rotation of the vector field, but any system sufficiently close to (A) will have precisely one limit cycle in a sufficiently small neighborhood of L_0. In case of higher multiplicities of the cycle L_0, there always exist systems as close as desired to (A), which have more than one limit cycle near L_0. Theorem 72 shows that in a small rotation of the vector field, a multiple cycle of odd multiplicity behaves like a simple cycle, i.e., the modified system has one and only one limit cycle in a sufficiently small neighborhood of the original cycle, which is moreover structurally stable.

Remark 2. Theorems 71 and 72 and our proof of these theorems remain valid if (A_μ) is taken in the form

$$\frac{dx}{dt}=P(x, y)-\mu f(x, y)\,Q(x, y), \qquad \frac{dy}{dt}=Q(x, y)+\mu f(x, y)\,P(x, y),$$

where $f(x, y)$ is a function which retains a constant sign at all points of the limit cycle L_0.

In conclusion of this section, we will prove another theorem, which is no longer related to rotations of the vector fields.

Let L_0 be a limit cycle of (A), $x=\varphi(t)$, $y=\psi(t)$ the solution corresponding to the path L_0, $\tau>0$ the period of this solution. Let further $F(x, y)$ be an analytical function defined in the same region as system (A), which satisfies the following conditions:

(a) $F(\varphi(s),\ \psi(s))\equiv 0$,

(b) $[F'_x(\varphi(s),\ \psi(s))]^2+[F'_y(\varphi(s),\ \psi(s))]^2\neq 0$ (see end of §33.1).

Theorem 73. Let

$$\begin{aligned}\frac{dx}{dt}&=P(x,\ y)+\mu p_1(x,\ y)+\mu^2 p_2(x,\ y)+\ldots=\bar{P}(x,\ y,\ \mu),\\ \frac{dy}{dt}&=Q(x,\ y)+\mu q_1(x,\ y)+\mu^2 q_2(x,\ y)+\ldots=\bar{Q}(x,\ y,\ \mu)\end{aligned} \qquad (A_\mu)$$

be a dynamic system, and let the functions $p_1(x, y)$ and $q_1(x, y)$ have the form

$$p_1(x,\ y)=F(x,\ y)\,F'_x(x,\ y), \qquad q_1(x,\ y)=F(x,\ y)\,F'_y(x,\ y), \qquad (101)$$

where F is a function satisfying conditions (a) and (b). Then if L_0 is a limit cycle of odd multiplicity of system (A), there exist $\mu_0>0$ and $\varepsilon>0$ which satisfy the following condition: for all μ, $|\mu|<\mu_0$, having the same sign, (A_μ) has precisely three limit cycles in $U_\varepsilon(L_0)$, and these cycles are structurally stable; for all μ, $|\mu|<\mu_0$, of the opposite sign, (A_μ) has precisely one limit cycle in $U_\varepsilon(L_0)$, which is also structurally stable.

Proof. The expansion of $d(n_0, \mu)$ in the neighborhood of (0, 0) has the form (98):

$$d(n_0,\ \mu)=(u_{10}-1)\,n_0+u_{01}\mu+u_{11}n\mu+u_{20}u_0^2+\ldots,$$

where u_{10} is computed from (34). The coefficient u_{01} is computed from (36). By (101) and condition (a), it follows that $u_{01}=0$. The coefficient u_{11} is computed from (42) and, as we have seen at the end of §33.1, $u_{11}\neq 0$ under

the conditions of our theorem. Thus, the conditions of Lemma 4 are satisfied for the function $d(n_0, \mu)$ if it is considered as $F(w, z)$. The proof of the theorem follows directly from this lemma. Q.E.D.

§33. CREATION OF A LIMIT CYCLE FROM A CLOSED PATH OF A CONSERVATIVE SYSTEM

1. The integral invariant and conservative system. Statement of the problem. The method of the small parameter

The concept of the integral invariant was introduced by Poincaré (see /24/, Sec. 235, p. 5). We will give here a definition of the integral invariant for a dynamic system of second order. In the general case — for a system of n-th order — this concept is defined analogously. As in the previous section, we assume that all the relevant systems are analytical.

Let

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{A}$$

be a dynamic system defined in G. For simplicity, we assume that G is made up of whole paths of system (A). Let D be a closed subregion of G, $M_0(x_0, y_0) \in D$ any point in this subregion. Consider the path $x = \varphi(t; x_0, y_0)$, $y = \psi(t; x_0, y_0)$ of system (A) which for $t = 0$ passes through the point M_0. Let the point with the coordinates $\varphi(t; x_0, y_0)$, $\psi(t; x_0, y_0)$ be designated $M(t, M_0)$. The set of all points $M(t, M_0)$ obtained when M_0 runs over all the points of the subregion D for a fixed t, i.e., the set of point $\{M(t, M_0); M_0 \in D\}$, is designated D_t. D_t is clearly homomorphic to D and is contained in G. D_t is obtained from D, so to say, by a translation over a distance t (in time) along the paths.

Let $\rho(x, y)$ be an analytical function in G which is not identically zero in this region.

Definition 31. The integral

$$\iint_{(D)} \rho(x, y)\, dx\, dy \tag{1}$$

is called the integral invariant of a dynamic system (A) ***if in every bounded closed subregion*** $D \subset G$ ***and for every*** t, ***we have***

$$\iint_{(D_t)} \rho(x, y)\, dx\, dy = \iint_{(D)} \rho(x, y)\, dx\, dy. \tag{2}$$

If (1) ***is an integral invariant, the function*** $\rho(x, y)$ ***is called the density of the integral invariant.***

The integral invariant readily lends itself to a hydrodynamic interpretation. Let (A) be regarded as a system of equations describing the velocity of steady-state motion of some two-dimensional "fluid" filling the region G, which contains neither sources nor sinks. Let $\rho(x, y)$ be the density of the

fluid at the point (x, y). The integral (1) then expresses the mass of fluid filling the region D, and equality (2) signifies that the fluid mass is conserved as the fluid particles move along their streamlines in a time t to fill the region D_t. For an incompressible fluid, $\rho(x, y) = \text{const}$, and the integral invariant is the area of the region D.

Let us derive the condition to be satisfied by $\rho(x, y)$ for (1) to be an integral invariant. Assuming a fixed D for the time being, we will use the notation

$$J(t) = \iint_{(D_t)} \rho(x, y)\, dx\, dy. \tag{3}$$

Then $\iint_{(D)} \rho(x, y)\, dx\, dy = J(0)$, and for equality (2) to hold true for any t, it is necessary and sufficient that

$$J'(t) \equiv 0$$

(the differentiability of $J(t)$ will be proved below). To compute $J'(t)$, we represent $J(t)$ as an integral over D. The substitution of variables

$$x = \varphi(t; x_0, y_0), \quad y = \psi(t; x_0, y_0) \tag{4}$$

in (3) maps D onto D_t. We thus obtain

$$J(t) = \iint_{(D_t)} \rho(x, y)\, dx\, dy = \iint_{(D)} \rho(\varphi(t; x_0, y_0), \psi(t; x_0, y_0))\Delta(t; x_0, y_0)\, dx_0\, dy_0, \tag{5}$$

where

$$\Delta(t; x_0, y_0) = \frac{D(\varphi(t; x_0, y_0), \psi(t; x_0, y_0))}{D(x_0, y_0)}$$

is the Jacobian of mapping (4). This Jacobian has been computed in QT, where it is shown (QT, § 3.5, Lemma 6) that

$$\Delta(t; x_0, y_0) = e^{\int_0^t [P'_x(\varphi, \psi) + Q'_y(\varphi, \psi)]\, dt}. \tag{6}$$

It follows from the last equality that $\Delta(t; x_0, y_0) \neq 0$, i.e., (4) is a regular mapping, and

$$\frac{d\Delta(t; x_0, y_0)}{dt} = \Delta(t; x_0, y_0)[P'_x(\varphi, \psi) + Q'_y(\varphi, \psi)]. \tag{7}$$

By (5), $J(t) = \iint_{(D)} \rho\Delta\, dx_0\, dy_0$. Since the region of integration in this integral is independent of t, and the integrand is differentiable, $J(t)$ is differentiable and

$$J'(t) = \iint_{(D)} \frac{d}{dt}[\rho(\varphi(t; x_0, y_0), \psi(t; x_0, y_0))\Delta(t; x_0, y_0)]\, dx_0\, dy_0. \tag{8}$$

Let us compute the integrand in (8). Using (7) and the obvious relations

$$\frac{d\varphi(t; x_0, y_0)}{dt} = P(\varphi, \psi), \quad \frac{d\psi(t; x_0, y_0)}{dt} = Q(\varphi, \psi),$$

we find

$$\frac{d}{dt}[\rho\Delta] = \left[\frac{\partial\rho}{\partial x}\frac{d\varphi}{dt} + \frac{\partial\rho}{\partial y}\frac{d\psi}{dt}\right]\Delta + \rho\Delta\,[P'_x(\varphi, \psi) + Q'_y(\varphi, \psi)] =$$
$$= \Delta\left[\frac{\partial\rho(\varphi, \psi)}{\partial x}P(\varphi, \psi) + \frac{\partial\rho(\varphi, \psi)}{\partial y}Q(\varphi, \psi) + \rho\,(P'_x(\varphi, \psi) + Q'_y(\varphi, \psi))\right]. \qquad (9)$$

For $t = 0$, $\varphi(t; x_0, y_0) = x_0$, $\psi(t; x_0, y_0) = y_0$, and $\Delta(t; x_0, y_0) = 1$. Therefore, by (8) and (9),

$$J'(0) = \iint_{(D)}\left[\frac{\partial\rho(x_0, y_0)}{\partial x}P(x_0, y_0) + \frac{\partial\rho(x_0, y_0)}{\partial y}Q(x_0, y_0) + \right.$$
$$\left. + \rho(x_0, y_0)\,(P'_x(x_0, y_0) + Q'_y(x_0, y_0))\right]dx_0\,dy_0. \qquad (10)$$

Since $J'(t) \equiv 0$, we have $J'(0) = 0$. The last equality should hold for *every* region D, so that the integrand in (10) must identically vanish, i.e., at any point (x, y) of G we should have

$$\frac{\partial\rho}{\partial x}P + \frac{\partial\rho}{\partial y}Q + \rho\,(P'_x + Q'_y) \equiv 0. \qquad (11)$$

On the other hand, it follows from (8) and (9) that if (11) is satisfied, $J'(t) \equiv 0$ for any D. We have thus established that identity (11) is the necessary and sufficient condition for (1) to be an integral invariant.

Remark. Condition (11) can be written in the form $\frac{\partial(\rho P)}{\partial x} + \frac{\partial(\rho Q)}{\partial y} \equiv 0$. For an n-th order system $\frac{dx_i}{dt} = P_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots n$, the function $\rho(x_1, x_2, \ldots, x_n)$ is the density of the integral invariant if and only if the following identity is satisfied (see /30/):

$$\sum_{i=1}^{n}\frac{\partial(\rho P_i)}{\partial x_i} \equiv 0.$$

Let G, as before, be a region consisting of whole paths of (A).

Definition 32. System (A) *is said to be conservative in* G *if it has an integral invariant with positive density in this region.*

Consider a system (A) which is conservative in G. Let $\iint_{(D)}\rho(x, y)\,dx\,dy$, where $\rho(x, y) > 0$, be its integral invariant. The paths of (A) then coincide with the paths of the system

$$\frac{dx}{dt} = \rho(x, y)\,P(x, y) = \hat{P}, \quad \frac{dy}{dt} = \rho(x, y)\,Q(x, y) = \hat{Q}. \qquad (\hat{A})$$

By (11), we have in G

$$\frac{\partial\hat{Q}}{\partial y} = -\frac{\partial\hat{P}}{\partial x}. \qquad (12)$$

Therefore, if G is a simply connected region, a single-valued function $H(x, y)$ exists in G which satisfies the equalities

$$\hat{P} = -\frac{\partial H(x, y)}{\partial y}, \quad \hat{Q} = \frac{\partial H(x, y)}{\partial x}, \qquad (13)$$

and $(\hat{A})$ may be written in the form

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x}. \qquad (14)$$

Definition 33. A dynamic system of the form

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x},$$

where $H(x, y)$ is a single-valued function defined in G is called a Hamiltonian system in G.

From Definitions 32 and 33 and condition (11) it follows that a Hamiltonian system is a particular case of a conservative system. The density ρ of the integral invariant of a Hamiltonian system may be chosen as the number 1; the integral invariant of the system is then the area of the region. A Hamiltonian system has a general integral $H(x, y) = C$ (see QT, § 1.13).

We have thus established that if (A) is a conservative system in a simply connected region G, and $\rho(x, y)$ is the density of its integral invariant, $(\hat{A})$ is a Hamiltonian system in G. $\rho(x, y)$ is the integrating factor of the differential equation $Q(x, y)\,dx - P(x, y)\,dy = 0$ corresponding to system (A). It is readily seen that if P and Q are analytical functions, $H(x, y)$ is also an analytical function.

If G is not simply connected, relation (12) is not enough to establish the existence in G of a single-valued function $H(x, y)$ which satisfies conditions (13). In this case we can only say that such a function exists in any simply connected subregion of G.

In this section we will only consider the case of G which is an "annular" region completely filled with closed paths of system (A) enclosing one another. We will show that in this case (A) is a conservative system in G, i.e., it has an integral invariant with positive density $\rho(x, y)$ in this region, and that the system

$$\frac{dx}{dt} = \rho P, \quad \frac{dy}{dt} = \rho Q \qquad (\hat{A})$$

is a Hamiltonian system

Theorem 74. Let (A) be an analytical system, and G a closed annular region completely filled with concentric closed paths of system (A). There exists an analytical function $\rho(x, y)$ defined in G which satisfies the condition

$$\frac{\partial \rho}{\partial x} P + \frac{\partial \rho}{\partial y} Q + \rho (P'_x + Q'_y) \equiv 0, \qquad (11)$$

i.e., system (A) is conservative in G, and the system

$$\frac{dx}{dt} = \rho P, \quad \frac{dy}{dt} = \rho Q \qquad (\hat{A})$$

is Hamiltonian.

Proof. Consider an analytical arc without contact l contained in G, which connects the points of two closed boundary paths (see Figure 172; the existence of this arc is proved in QT, § 19.5, Lemma 6). Let $x = f(s)$, $y = g(s)$, $a \leqslant s \leqslant b$ be the parametric equations of the arc l (f and g are analytical functions). As we know, every closed path of G has precisely one common point with l. Let $\tau(s)$ be the period of the closed path which crosses the arc l at a point corresponding to the value s of the parameter.

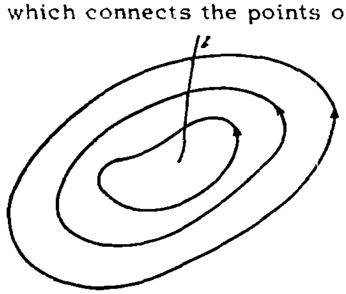

FIGURE 172

Let $x = \varphi(t; x_0, y_0)$, $y = \psi(t; x_0, y_0)$ be the solution of system (A) satisfying the initial

conditions $\varphi(0; x_0, y_0) = x_0$, $\psi(0; x_0, y_0) = y_0$. Consider the mapping T, defined by the equalities

$$x = \Phi(t, s), \quad y = \Psi(t, s), \tag{15}$$

where

$$\Phi(t, s) = \varphi(t; f(s), g(s)), \quad \Psi(t, s) = \psi(t; f(s), g(s)). \tag{16}$$

The functions Φ and Ψ are analytical, and T maps the region R of the plane (t, s) defined by the inequalities

$$a \leqslant s \leqslant b, \; 0 \leqslant t \leqslant \tau(s), \; (R)$$

onto the region G. The mapping T is regular at all the interior points of R (see QT, §3.5, Lemma 8), but it is not one-to-one, since every point of the arc l corresponding to the value s $(a \leqslant s \leqslant b)$ of the parameter is the image of two points $(0, s)$ and $(\tau(s), s)$ belonging to the boundary of the region under this mapping. This follows from the obvious relations

$$\Phi(0, s) = \Phi(\tau(s), s), \quad \Psi(0, s) = \Psi(\tau(s), s). \tag{17}$$

Let

$$r(t, s) = e^{-\int\limits_0^t [P'_x(\Phi(t, s)\Psi(t, s)) + Q'_y(\Phi(t, s), \Psi(t, s))]\,dt}. \tag{18}$$

The function $r(t, s)$ is defined in region R of the plane (t, s). In virtue of relations (15), it may be considered as a function of x, y defined in G. Let the corresponding function be $\rho(x, y)$.

We thus take

$$\rho(x, y) = r(t(x, y), s(x, y)), \tag{19}$$

where $t = t(x, y)$ and $s = s(x, y)$ are the functions defined by (15).

We will first show that the function $\rho(x, y)$ defined by (19) is single-valued in G. If the point (x, y) does not belong to the arc without contact l, it corresponds precisely to one point (t, s) of the region R and therefore to a definite value of the function r. Let now $M(x, y)$ be a point which belongs to the arc l and corresponds to the value s of the parameter. In this case, as we have seen before, the point $M(x, y)$ is the image of two points of the region R under the mapping, namely the points $(0, s)$ and $(\tau(s), s)$. To establish the single-valuedness of $\rho(x, y)$, it suffices to show that $r(0, s) = r(\tau(s), s)$. But $r(0, s) = 1$ by (18). On the other hand,

$$\int\limits_0^{\tau(s)} [P'_x(\varphi(t; f, g), \psi(t; f, g)) + Q'_y(\varphi(t; f, g), \psi(t; f, g))]\,dt$$

is equal, apart from the factor $\tau(s)$, to the characteristic index of the closed path L_M passing through the point M (see §13.3, Definition 17). Since the path L_M belongs to the family of the concentric closed paths filling the region G, it is not a limit cycle and its characteristic index, together with the last integral, vanishes (see §13.3, (31) and §12.3). But then, by (18), $r(\tau(s), s) = 1 = r(0, s)$. We thus established that the function $\rho(x, y)$ is uniquely defined by (19) in G.

We will now show that this function fulfills the proposition of the theorem, i.e., satisfies identity (11). By (18) and (19) we have

$$\begin{aligned}\frac{\partial\rho}{\partial x}&=\frac{\partial r}{\partial t}\frac{\partial t}{\partial x}+\frac{\partial r}{\partial s}\frac{\partial s}{\partial x}=-\rho\,(P'_x+Q'_y)\frac{\partial t}{\partial x}+\frac{\partial r}{\partial s}\frac{\partial s}{\partial x},\\ \frac{\partial\rho}{\partial y}&=\frac{\partial r}{\partial t}\frac{\partial t}{\partial y}+\frac{\partial r}{\partial s}\frac{\partial s}{\partial y}=-\rho\,(P'_x+Q'_y)\frac{\partial t}{\partial y}+\frac{\partial r}{\partial s}\frac{\partial s}{\partial y}.\end{aligned}\tag{20}$$

Multiplying the equalities in (20) by $P(x, y)$ and $Q(x, y)$, respectively, adding them up, and remembering that

$$P(x, y)=\frac{\partial x}{\partial t},\qquad Q(x, y)=\frac{\partial y}{\partial t},$$

$$\frac{\partial t}{\partial x}P+\frac{\partial t}{\partial y}Q=\frac{\partial t}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial t}{\partial y}\frac{\partial y}{\partial t}=\frac{\partial t}{\partial t}=1,$$

$$\frac{\partial s}{\partial x}P+\frac{\partial s}{\partial y}Q=\frac{\partial s}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial s}{\partial y}\frac{\partial y}{\partial t}=\frac{\partial s}{\partial t}=0,$$

we obtain

$$\frac{\partial\rho}{\partial x}P+\frac{\partial\rho}{\partial y}Q=-\rho\,(P'_x+Q'_y),$$

i.e., relation (11).

We will now show that the system

$$\frac{dx}{dt}=\rho P=\hat{P},\qquad \frac{dy}{dt}=\rho Q=\hat{Q}\tag{$\hat{A}$}$$

is a Hamiltonian system in the ring G. Since $\rho>0$, the paths of ($\hat{A}$) coincide with the paths of (A), i.e., they are closed paths. Furthermore, in virtue of (11),

$$\frac{\partial(\rho, P)}{\partial x}+\frac{\partial(\rho, Q)}{\partial y}\equiv 0,$$

i.e.,

$$\frac{\partial\hat{Q}}{\partial y}\equiv-\frac{\partial\hat{P}}{\partial x}.\tag{21}$$

Let $M_0(x_0, y_0)$ be a fixed, and $M(x, y)$ an arbitrary point in G. Let γ be a smooth curve contained in G which goes from M_0 to M. We will show that the line integral

$$\int_{(\gamma)}\hat{Q}(x, y)\,dx-\hat{P}(x, y)\,dy\tag{22}$$

is a single-valued function of x, y. To this end, it suffices to establish that this integral is independent of the integration path γ, i.e., the integral along any closed curve Contained in G is zero. If the region enclosed by the curve C is entirely contained in G, we have $\int_{(C)}\hat{Q}\,dx-\hat{P}\,dy=0$ by Green's formula and relation (21). Let now C be the closed path $x=\hat{\varphi}(t)$, $y=\hat{\psi}(t)$ of system ($\hat{A}$) which passes through the point M_0 (this path encloses the inner boundary curve of G, i.e., it is not homotopic to zero in G). Let τ be the period of the functions $\hat{\varphi}$ and $\hat{\psi}$. Then

$$\begin{aligned}\int_{(C)}\hat{Q}\,dx-\hat{P}\,dy&=\int_0^{\tau}[\hat{Q}(\hat{\varphi}(t), \hat{\psi}(t))\;\hat{\varphi}'(t)-\hat{P}(\hat{\varphi}(t), \hat{\psi}(t))\;\hat{\psi}'(t)]\,dt=\\&=\int_0^{\tau}[\hat{\psi}'(t)\,\hat{\varphi}'(t)-\hat{\varphi}'(t)\,\hat{\psi}'(t)]\,dt=0.\end{aligned}$$

It follows from the last relation and from (21) that the line integral vanishes along any closed curve C which belongs to G. But then the integral (22) is independent of the integration path γ, i.e., it is a function of the point $M(x, y)$. Let this function be $H(x, y)$, i.e., we take

$$H(x, y)=\int_{(\gamma)} \hat{Q}(x, y)\, dx-\hat{P}(x, y)\, dy.$$

The function $H(x, y)$ defined in this way clearly satisfies the relations $\hat{P}=-\frac{\partial H}{\partial y}$, $Q=\frac{\partial H}{\partial x}$, i.e., (A) is a Hamiltonian system. This completes the proof of the theorem.

In what follows, a *conservative system* will be an *analytical system defined in a doubly connected region G for which all the paths contained in G are closed* (they are evidently concentric). These systems evidently constitute a comparatively restricted class of conservative systems in the sense of Definition 32. However, since we will only consider these systems, we will use the general term *conservative systems* in this restricted sense from now on. Similarly, a *Hamiltonian system* in this section will be regarded as a Hamiltonian system defined in a doubly connected region which is completely filled with concentric closed paths.

We will also consider *systems close to a conservative system*, i.e., systems of the form

$$\frac{dx}{dt}=P(x, y)+\mu p(x, y, \mu), \quad \frac{dy}{dt}=Q(x, y)+\mu q(x, y, \mu), \tag{A_μ}$$

where μ is a small real number, p and q are analytical functions of the respective arguments, and system (A_0), i.e.,

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y), \tag{A_0}$$

is conservative. We will establish the *sufficient conditions* to be met by the functions $p(x, y, \mu)$, $q(x, y, \mu)$ for a limit cycle of (A_μ) to exist in the neighborhood of one of the closed paths L_0 of (A_0). In other words, using our terminology, we will establish the sufficient conditions for the creation of a limit cycle L_μ of system (A_μ) from the path L_0 of system (A_0).

As a particular case of systems close to a conservative system, we will consider *systems close to a Hamiltonian system*,

$$\frac{dx}{dt}=-\frac{\partial H}{\partial y}+\mu p(x, y, \mu), \quad \frac{dy}{dt}=\frac{\partial H}{\partial x}+\mu q(x, y, \mu), \tag{H_μ}$$

where p and q are analytical in the doubly connected region, and (H_0) is a Hamiltonian system.

We will show that on passing from the general case of systems close to a conservative system to the case of systems close to a Hamiltonian system, the sufficient conditions for the functions p and q are markedly simplified.

The situation is particularly simple when the original conservative system is linear. In this case, we may take, without loss of generality,

$$\frac{dx}{dt}=-y, \quad \frac{dy}{dt}=x. \tag{B_0}$$

(B_0) is a Hamiltonian system and its paths are the circles

$$x^2+y^2=C. \tag{23}$$

A system close to a linear conservative system has the form

$$\frac{dx}{dt}=-y+\mu p(x, y, \mu), \quad \frac{dy}{dt}=x+\mu q(x, y, \mu). \tag{B_μ}$$

This system can be investigated either as a particular case of a system close to a Hamiltonian system, or directly by changing over to polar coordinates. Since the direct approach is very simple in this case, and at the same time provides a better insight into the general properties of a system close to a conservative system, we will first investigate (in the next subsection) a system of the form (B_μ) by changing over to polar coordinates.

The results of the present section make it possible to establish in certain cases the presence (or absence) of limit cycles in nonlinear dynamic systems. This is accomplished by the so-called **Poincaré method** or **the method of a small parameter**. According to this method, a nonlinear dynamic system is considered as a system close to a conservative system. The existence of limit cycles in the nonlinear system is established by finding the closed paths of the conservative system which create these cycles. The Poincaré method is naturally applicable only if the nonlinear system is indeed close to a conservative system, or if it can be "fitted" with a conservative system by some technique.

2. Systems close to a linear conservative system

We will consider a system of the form

$$\dot{x}=-y+\mu p(x, y, \mu), \quad \dot{y}=x+\mu q(x, y, \mu), \tag{B_μ}$$

which is close to a linear conservative system (B_0) whose paths are the circles $x^2+y^2=C$. The functions p and q are assumed to be analytical in the neighborhood of the point $(0, 0, 0)$. Systems of the form (B_μ) are often encountered in applications. Thus, for example, the equation

$$\ddot{x}+x=\mu f(x, \dot{x}),$$

which is close for small μ to the equation of the harmonic oscillator $\ddot{x}+x=0$, gives in the phase plane (x, y) (setting $y=-\dot{x}$)

$$\dot{x}=-y, \quad \dot{y}=x-\mu f(x, -y),$$

and this is evidently an equation of the form (B_μ). The functions $p(x, y, \mu)$ and $q(x, y, \mu)$ will be assumed to vanish for $x=y=0$, i.e.,

$$p(0, 0, \mu)=q(0, 0, \mu)=0. \tag{24}$$

This assumption does not constitute a fundamental restriction. Indeed, suppose that condition (24) is not true. Consider the equations

$$-y+\mu p(x, y, \mu)=0, \quad x+\mu q(x, y, \mu)=0. \tag{25}$$

At the point $x=y=\mu=0$, this system satisfies the conditions of the theorem of implicit functions, and it is therefore solvable for x and y. Let

$$x=f(\mu), \quad y=g(\mu)$$

be the solution of (25) near the point $(0, 0, 0)$. Then

$$-g(\mu)+\mu p(f(\mu), g(\mu), \mu) \equiv 0, \quad f(\mu)+\mu q(f(\mu), g(\mu), \mu) \equiv 0, \tag{26}$$

where $f(\mu)$ and $g(\mu)$ are analytical functions, and

$$f(0)=g(0)=0. \tag{27}$$

Applying to (B_μ) the substitution of variables

$$x=X+f(\mu), \quad y=Y+g(\mu),$$

we obtain

$$\begin{aligned} \frac{dX}{dt} &= -Y-g(\mu)+\mu p(X+f(\mu), Y+g(\mu), \mu)=-Y+\mu p^*(X, Y, \mu), \\ \frac{dY}{dt} &= X+f(\mu)+\mu q(X+f(\mu), Y+g(\mu), \mu)=X+\mu q^*(X, Y, \mu). \end{aligned} \tag{28}$$

By (26), $p^*(0, 0, \mu)=q^*(0, 0, \mu)=0$, i.e., condition (24) is satisfied for equations (28).

We will thus consider the system (B_μ) assuming (24). Let

$$p(x, y, \mu)=p(x, y)+\mu p_2(x, y, \mu), \quad q(x, y, \mu)=q(x, y)+\mu q_2(x, y, \mu). \tag{29}$$

By (24),

$$p(0, 0)=q(0, 0)=0, \quad p_2(0, 0, \mu)=q_2(0, 0, \mu)=0. \tag{30}$$

Using (29), we write the original system (B_μ) in the form

$$\begin{aligned} \frac{dx}{dt} &= -y+\mu p(x, y, \mu)=-y+\mu p(x, y)+\mu^2 p_2(x, y, \mu), \\ \frac{dy}{dt} &= x+\mu q(x, y, \mu)=x+\mu q(x, y)+\mu^2 q_2(x, y, \mu). \end{aligned} \tag{31}$$

Changing over to polar coordinates $x=\rho\cos\theta$, $y=\rho\sin\theta$, we readily obtain

$$\begin{aligned} \frac{d\rho}{dt} &= \mu[\cos\theta p(\rho\cos\theta, \rho\sin\theta, \mu)+\sin\theta q(\rho\cos\theta, \rho\sin\theta, \mu)], \\ \frac{d\theta}{dt} &= \frac{\rho^2+\mu[\rho\cos\theta q(\rho\cos\theta, \rho\sin\theta, \mu)-\rho\sin\theta p(\rho\cos\theta, \rho\sin\theta, \mu)]}{\rho^2}. \end{aligned} \tag{32}$$

By (24), the right-hand side of the last equation for $|\rho|<\rho^*$ (where ρ^* is a positive number) has the form

$$1+\mu F(\rho, \theta, \mu),$$

where $F(\rho, \theta, \mu)$ is an analytical function of its arguments (which need not vanish at $\rho=0$). For small μ, the last expression does not vanish, and

we may therefore change over to a single equation

$$\frac{d\rho}{d\theta}=\mu R(\rho,\theta,\mu). \tag{33}$$

where

$$R(\rho,\theta,\mu)=\frac{\cos\theta p(\rho\cos\theta,\rho\sin\theta,\mu)+\sin\theta q(\rho\cos\theta,\rho\sin\theta,\mu)}{1+\mu F(\rho,\theta,\mu)}. \tag{34}$$

$R(\rho,\theta,\mu)$ is an analytical function with a period of 2π in θ which vanishes at $\rho=0$. Therefore, $\rho\equiv 0$ for every μ is a solution of equation (33). Since this solution is independent of θ, it is defined for all θ. But then by Theorem 1 of Appendix 1, the solution of equation (33) is a priori defined for all θ, $0\leqslant\theta\leqslant 2\pi$, for all sufficiently small μ, for $|\mu|<\mu^*$, say. Let the solution corresponding to the initial conditions θ_0 and ρ_0 be

$$\rho=f(\theta;\theta_0,\rho_0,\mu),$$

where f is an analytical function of its arguments. Since for $\mu=0$ equation (14) takes the form $\frac{d\rho}{d\theta}=0$, we have $f(\theta;\theta_0,\rho_0,0)\equiv\rho_0$. Therefore,

$$f(\theta;\theta_0,\rho_0,\mu)=\rho_0+\mu f_1(\theta;\theta_0,\rho_0,\mu).$$

Let $\theta_0=0$, and the solution $f(\theta;0,\rho_0,\mu)$ will be written in the form

$$\rho=\rho_0+\mu\Psi(\theta;\rho_0,\mu). \tag{35}$$

Here $\Psi(\theta;\rho_0,\mu)$ is an analytical function in the region

$$0\leqslant\theta\leqslant 2\pi,\quad |\rho|<\rho^*,\quad |\mu|<\mu^*.$$

The condition of contact between a path of system (B_μ) and the ray $\theta=\text{const}$ at the point (x,y) has the form

$$x\dot{y}-y\dot{x}=x^2+y^2+\mu(qx-py)=0.$$

Because of the particular choice of ρ^* and μ^*, this condition is not satisfied in the relevant region, i.e., the rays $\theta=\text{const}$ are without contact with the paths of the system. It follows from (35) that the succession function on the ray $\theta=0$ has the form

$$\rho=\rho_0+\mu\Psi(2\pi;\rho_0,\mu). \tag{36}$$

The closed paths of (B_μ) correspond to those of $\rho_0\neq 0$ for which

$$\mu\Psi(2\pi;\rho_0,\mu)=0,$$

i.e., for which for $\mu\neq 0$

$$\Psi(2\pi;\rho_0,\mu)=0. \tag{37}$$

Since the succession function is an analytical function of μ, Ψ may be written in the form

$$\Psi(2\pi;\rho_0,\mu)=\Psi(2\pi;\rho_0,0)+\mu\Psi'_\mu(2\pi;\rho_0,0)+\dots \tag{38}$$

The continuity of Ψ shows that if the equation $\Psi(2\pi;\,\rho_0,\,\mu) = 0$ has a root $\rho_0(\mu)$ which goes to some ρ_1 for $\mu \to 0$ ($|\rho_1| < \rho^*$), then

$$\Psi(2\pi;\,\rho_1,\,0) = 0. \tag{39}$$

Suppose that for some ρ_1, $|\rho_1| < \rho^*$, the last condition is satisfied and that, moreover,

$$\frac{\partial\Psi(2\pi;\,\rho_1,\,0)}{\partial\rho_0} \neq 0. \tag{40}$$

The theorem of implicit functions then shows that there exist $\mu^* > 0$ and $\delta > 0$ which satisfy the following condition: if $|\mu| < \mu^*$, the equation

$$\Psi(2\pi;\,\rho_0,\,\mu) = 0 \tag{41}$$

has a unique solution $\rho_0 = \rho_0(\mu)$ such that $|\rho_0(\mu) - \rho_1| < \delta$ and $\rho_0(0) = \rho_1$. This evidently implies that for a sufficiently small $\mu \neq 0$, (B_μ) has a single limit cycle in a small neighborhood of the circle $x^2 + y^2 = \rho_1^2$, and this limit cycle contracts to the particular circle for $\mu \to 0$. We can naturally say that this limit cycle of (B_μ) is "created" from the path $x^2 + y^2 = \rho_1^2$ of the original linear system.

Note that the equation $\Psi(2\pi;\,\rho_0,\,0) = 0$, alongside with ρ_1, may have other solutions satisfying condition (40). If ρ_2 is one of these solutions, the path $x^2 + y^2 = \rho_2^2$ of the original conservative system also creates a limit cycle.

Let us derive expressions for $\Psi(2\pi;\,\rho_0,\,0)$ and $\frac{\partial\Psi(2\pi;\,\rho_0,\,0)}{\partial\rho_0}$ in terms of the functions $p(x,\,y,\,\mu)$ and $q(x,\,y,\,\mu)$ entering the right-hand sides of (B_μ). To this end, we expand the right-hand side of (33) in powers of μ. The equation thus takes the form

$$\frac{d\rho}{d\theta} = \mu R_1(\rho,\,\theta) + \mu^2 R_2(\rho,\,\theta) + \ldots \tag{42}$$

Solution (35) of this equation may be written in the form

$$\rho = \rho_0 + \mu\Psi(\theta;\,\rho_0,\,0) + \mu^2\Psi'_\mu(\theta;\,\rho_0,\,0) + \ldots \tag{43}$$

Inserting the last expression in (42) and equating the right- and the left-hand sides, we obtain

$$\frac{d\Psi}{d\theta} = R_1(\rho_0,\,\theta). \tag{44}$$

From (29), (33), (34), and (42) it follows that

$$R_1(\rho_0,\,\theta) = \cos\theta\, p(\rho_0\cos\theta,\,\rho_0\sin\theta) + \sin\theta\, q(\rho_0\cos\theta,\,\rho_0\sin\theta). \tag{45}$$

On the other hand, since $f(0;\,0,\,\rho_0,\,\mu) \equiv \rho_0$, we have $\Psi(0;\,\rho_0,\,\mu) \equiv 0$. In particular,

$$\Psi(0;\,\rho_0,\,0) = 0. \tag{46}$$

Integrating (44) with the initial condition (46), using (45), and setting $\theta = 2\pi$, we obtain the following expression for $\Psi(2\pi;\,\rho_0,\,0)$:

$$\Psi(2\pi;\,\rho_0,\,0) = \int_0^{2\pi} [\cos\theta\, p(\rho_0\cos\theta,\,\rho_0\sin\theta) + \sin\theta\, q(\rho_0\cos\theta,\,\rho_0\sin\theta)]\, d\theta. \tag{47}$$

To obtain $\frac{\partial\Psi(2\pi;\rho_0,0)}{\partial\rho_0}$, we differentiate the last equality with respect to ρ_0,

$$\frac{\partial\Psi(2\pi;\rho_0,0)}{\partial\rho_0}=\int_0^{2\pi}(p'_x\cos^2\theta+q'_y\sin^2\theta+p'_y\sin\theta\cos\theta+q'_x\sin\theta\cos\theta)\,d\theta,$$

where p'_x, p'_y, q'_x, q'_y are the values of the respective functions at the point $(\rho_0\cos\theta,\rho_0\sin\theta)$. Simple manipulations then give

$$\frac{\partial\Psi(2\pi;\rho_0,0)}{\partial\rho_0}=\int_0^{2\pi}(p'_x+q'_y)\,d\theta+\int_0^{2\pi}[(-p'_x\sin\theta+p'_y\cos\theta)\sin\theta+(q'_x\sin\theta-q'_y\cos\theta)\cos\theta]\,d\theta. \tag{48}$$

The relations

$$\frac{dp(\rho_0\cos\theta,\rho_0\sin\theta)}{d\theta}=(-p'_x\sin\theta+p'_y\cos\theta)\,\rho_0,$$
$$\frac{dq(\rho_0\cos\theta,\rho_0\sin\theta)}{d\theta}=(-q'_x\sin\theta+q'_y\cos\theta)\,\rho_0$$

show that the second integral on the right in (48) is equal to

$$\frac{1}{\rho_0}\int_0^{2\pi}\left[\frac{dp(\rho_0\cos\theta,\rho_0\sin\theta)}{d\theta}\sin\theta-\frac{dq(\rho_0\cos\theta,\rho_0\sin\theta)}{d\theta}\cos\theta\right]d\theta.$$

Integration by parts, remembering that $p(\rho_0\cos\theta,\rho_0\sin\theta)$ and $q(\rho_0\cos\theta,\rho_0\sin\theta)$ are periodic functions of θ with a period of 2π, establishes that the last integral is equal to

$$-\frac{1}{\rho_0}\int_0^{2\pi}[p(\rho_0\cos\theta,\rho_0\sin\theta)\cos\theta+q(\rho_0\cos\theta,\rho_0\sin\theta)\sin\theta]\,d\theta,$$

and by (47) it is thus equal to

$$-\frac{1}{\rho_0}\Psi(2\pi;\rho_0,0).$$

Therefore

$$\frac{\partial\Psi(2\pi;\rho_0,0)}{\partial\rho_0}=\int_0^{2\pi}[p'_x(\rho_0\cos\theta,\rho_0\sin\theta)+q'_y(\rho_0\cos\theta,\rho_0\sin\theta)]\,d\theta-\frac{1}{\rho_0}\Psi(2\pi;\rho_0,0). \tag{49}$$

By (47) and (49), the conditions

$$\Psi(2\pi;\rho_1,0)=0 \text{ and } \frac{\partial\Psi(2\pi;\rho_1,0)}{\partial\rho_0}\neq 0 \tag{50}$$

are equivalent to the conditions

$$\begin{gathered}\int_0^{2\pi}[p(\rho_1\cos\theta,\rho_1\sin\theta)\cos\theta+q(\rho_1\cos\theta,\rho_1\sin\theta)\sin\theta]\,d\theta=0,\\ \int_0^{2\pi}[p'_x(\rho_1\cos\theta,\rho_1\sin\theta)+q'_y(\rho_1\cos\theta,\rho_1\sin\theta)]\,d\theta\neq 0.\end{gathered} \tag{51}$$

The above results thus lead to the following theorem.

Theorem 75. If for some ρ_1, $|\rho_1| < \rho^*$, conditions (50), *or equivalently* (51), *are satisfied, the transformation from system* (B_0) *to a sufficiently close system* (B_μ) *creates one and only one limit cycle of* (B_μ) *from the path* $\rho = \rho_1$ *of the original linear system* (B_0) *in a sufficiently small neighborhood of this path.*

Remark. If for ρ_1, $\Psi(2\pi; \rho_1, 0) = 0$ and $\frac{\partial\Psi(2\pi; \rho_1, 0)}{\partial\rho_0} = 0$, i.e., only the first condition in (50) is satisfied, it is no longer certain that a limit cycle is created from the path $\rho = \rho_1$.

Theorem 75 is a local theorem in the sense that it deals with the creation of a limit cycle in the neighborhood of one path of (B_0). We will prove now another theorem, which also relates to systems close to linear conservative systems but is nevertheless a global theorem. Let a and b be some positive numbers, $a < b$.

Theorem 76. If the equation $\Psi(2\pi; \rho_0, 0) = 0$ *has precisely s solutions* $\rho_0 = \rho_i$, $i = 1, 2, \ldots, s$, *in the segment* $[a, b]$, *each of these solutions satisfying the conditions* $a < \rho_i < b$, $\frac{\partial\Psi(2\pi; \rho_i, 0)}{\partial\rho_0} \neq 0$, *then for a sufficiently small* $\mu \neq 0$, (B_μ) *has precisely s closed paths in the ring* $a \leqslant \rho \leqslant b$.

Proof. By Theorem 75, there exist $\mu^* > 0$, $\delta > 0$ such that if $0 < |\mu| < \mu^*$, the equation $\Psi(2\pi; \rho_0, \mu) = 0$ has precisely one root in each of the intervals $(\rho_i - \delta, \rho_i + \delta)$, $i = 1, 2, \ldots, s$. Let these intervals be designated I_i. We may assume that no two of these I_i intersect and that they are all contained inside (a, b): these requirements are satisfied if δ is sufficiently small. Let Γ be the set $[a, b] \setminus \bigcup_{i=1}^{s} I_i$ (i.e., the complement of the union of the sets I_i to the segment $[a, b]$).

The proof is conducted by reductio ad absurdum. Suppose that the theorem is not true. Then there exists a sequence μ_i, $i = 1, 2, 3, \ldots$, such that $\mu_i \neq 0$, $|\mu_i| < \mu^*$, $\lim \mu_i = 0$, and (B_{μ_i}) has more than s closed paths in the ring $a \leqslant \rho \leqslant b$, i.e., the equation $\Psi(2\pi; \rho_0, \mu_i) = 0$ has more than s roots satisfying the condition $a \leqslant \rho_0 \leqslant b$. Then there exists at least one root of this equation which belongs to the set Γ. We designate this root $\rho_0^{(i)}$.

Thus, $\rho_0^{(i)} \in \Gamma$, $\Psi(2\pi; \rho_0^{(i)}, \mu_i) = 0$.

By passing to an appropriate subsequence, if necessary, we can always ensure convergence of the numerical sequence $\rho_0^{(i)}$. Let the sequence converge and let $\lim\limits_{i\to\infty} \rho_0^{(i)} = \rho_0^*$. Evidently, $\rho_0^* \in \Gamma$ and $\Psi(2\pi; \rho_0^{(i)}, 0) = 0$. But then ρ_0^* is a root of the equation $\Psi(2\pi; \rho_0, 0) = 0$ which does not coincide with any of the roots ρ_i, $i = 1, 2, \ldots, s$, in contradiction to the conditions of the theorem. This contradiction proves the theorem.

3. The general case of a system close to a conservative system

Let (A_0) be a conservative system in some doubly connected region G. We will consider a close system of the form

$$\frac{dx}{dt} = P(x, y) + \mu p(x, y, \mu) = P(x, y, \mu),$$
$$\frac{dy}{dt} = Q(x, y) + \mu q(x, y, \mu) = Q(x, y, \mu), \qquad (A_\mu)$$

where $p(x, y, \mu)$, $q(x, y, \mu)$ are analytical functions in G.

In § 13.1, in order to define the succession function, we introduced curvilinear coordinates in the neighborhood of the closed path L_0 in which L_0 was described by the equation $n = 0$.

We will now show that if (A_0) is conservative in G, and L_0 is some closed path of (A_0), $L_0 \subset G$, we can introduce curvilinear coordinates s, n in some neighborhood of the path L_0 so that the paths of (A_0) will be described by the equation $n = \text{const}$.

This is obvious from geometrical considerations. A formal proof can be formulated as follows. Let $x = \varphi(t)$, $y = \psi(t)$ be the equations of the path L_0, and τ the period of the functions φ and ψ. We first define in the neighborhood of L_0 curvilinear coordinates s and m using the equations

$$x = \varphi(s) + m \cdot \dot{\psi}(s), \quad y = \psi(s) - m \cdot \dot{\varphi}(s) \tag{52}$$

(see § 13.1, (6)). In these coordinates, (A_μ) is described by the differential equation

$$\frac{dm}{ds} = R^*(s; m, \mu). \tag{53}$$

Let the solution of this equation satisfying the initial condition $m = m_0$ for $s = 0$ be

$$m = f^*(s; m_0, \mu) \tag{54}$$

(see § 32.1, (R_μ), (10)).

Since for $\mu = 0$ all the paths of (A_μ) in the neighborhood of L_0 are closed, $f^*(s; m_0, 0)$ is a periodic function of s with a period of τ for all m_0, $|m_0| < m^*$, where m^* is a sufficiently small positive number.

We now define the coordinates s, n, taking

$$s = s, \quad m = f^*(s; n, 0). \tag{55}$$

Since L_0 is a closed path, we have $f^*(s; 0, 0) \equiv 0$. By § 32,

$$\left[\frac{\partial f^*(s, m_0, 0)}{\partial m_0}\right]_{m_0=0} = e^{\int_0^s A_{10}^*(s)\, ds} \neq 0$$

(see § 32.1, (9), (13), (14), (15)). Thus, for every s, $f^*(s, 0, 0) = 0$. $\frac{\partial f^*(s, 0, 0)}{\partial n} \neq 0$. The equation $f^*(s, n, 0) = m$ is thus uniquely solvable for n in the neighborhood of the point $n = 0$ when m is sufficiently small in absolute value, i.e., equations (55) constitute a one-to-one transformation of coordinates. The relationship between the curvilinear coordinates s, n and the Cartesian coordinates x, y, in virtue of (52) and (55), is expressed by the equalities

$$x = \varphi(s) + f^*(s, n, 0)\dot{\psi}(s) = \bar{\varphi}(s, n), \quad y = \psi(s) - f^*(s, n, 0)\dot{\varphi}(s) = \bar{\psi}(s, n). \tag{56}$$

It follows from the definition of $f^*(s, m_0, 0)$ that in the coordinates s, n the curves $n = \text{const}$ coincide with the closed paths of (A_0). Equations (56) for a fixed n therefore constitute parametric equations of these paths. In particular, $n = 0$ corresponds to the path L_0 of the original system (A_0).

The parameter s coincides with the time t along L_0. For other closed paths of (A_0), however, the parameter s in general does not coincide with

time, and the periods of motion along these paths in general are different and are not equal to τ. It is readily verified, however, that in the particular case of a linear system (A_0), s always coincides with t and the period of motion is constant for all the closed paths of (A_0). The right-hand sides of (56) have the same period τ in the variable s for all n.

It can be directly verified that the functions $\bar{\varphi}(s, n)$ and $\bar{\psi}(s, n)$ defined by (56) possess the properties 1 – 4 listed at the beginning of § 32 (see § 32.1, (2) – (5)). We will therefore use the results of this section, retaining the same notation.

In the coordinates s, n, (A_μ) is expressed by the differential equation

$$\frac{dn}{ds} = R(s, n, \mu), \tag{57}$$

where $R(s, n, \mu)$ is an analytical function in the region

$$-\infty < s < +\infty, \quad |n| < n^*, \quad |\mu| < \mu^*$$

(n^* and μ^* are sufficiently small positive numbers; see § 32.1, (R_μ)).

The coordinates s, n are chosen so that for $\mu = 0$, the functions n = const are solutions of the equation (57). Therefore $R(s, n, 0) \equiv 0$. But then all the terms in the expansion of $R(s, n, \mu)$ in powers of n, μ in the neighborhood of the point $n = 0$, $\mu = 0$ contain the factor μ, i.e., this expansion has the form

$$R(s, n, \mu) = A_{01}(s)\mu + A_{11}(s)\mu n + A_{02}(s)\mu^2 + \ldots \tag{58}$$

As in § 32.1, we write

$$n = f(s; 0, n_0, \mu) \tag{59}$$

for the solution of equation (57) corresponding to the initial condition $n = n_0$ for $s = 0$. The succession function of (A_μ) on the arc without contact l described by the equation $s = 0$ we designate $f(n_0, \mu)$. Clearly,

$$f(n_0, \mu) = f(\tau; 0, n_0, \mu). \tag{60}$$

Let

$$d(n_0, \mu) = f(n_0, \mu) - n_0. \tag{61}$$

From the particular choice of our coordinates, $f(s; 0, n_0, 0) \equiv n_0$. Therefore $d(n_0, 0) = f(n_0, 0) - n_0 \equiv 0$, and the series expansion of $d(n_0, \mu)$ near the point $(0, 0)$ has the form

$$d(n_0, \mu) = u_{01}\mu + u_{11}n_0\mu + u_{02}\mu^2 + \ldots, \tag{62}$$

so that

$$d(n_0, \mu) = \mu \cdot d_1(n_0, \mu), \tag{63}$$

where

$$d_1(n_0, \mu) = u_{01} + u_{11} + n_0 u_{02}\mu + \cdot \tag{64}$$

As we have noted before, the function $d(n_0, \mu)$ and therefore $d_1(n_0, \mu)$ are a priori defined for all sufficiently small n_0, μ.

$n_0 = 0$ corresponds to the closed path L_0 of (A_0). Let $d_1(0, 0) = 0$. Then, if $|n_0|$ and μ are sufficiently small and $\mu \neq 0$, we have $d(n_0, \mu) \neq 0$. This implies

that the system (A_μ) sufficiently close to (A_0) has no closed paths in the neighborhood of the path L_0 of (A_0). Thus the equality

$$d_1(0,0)=0$$

is the *necessary* condition for the system (A_μ) sufficiently close to (A_0) to have closed paths in a sufficiently small neighborhood of L_0.

The question of the number of closed paths is equivalent to the question of the number of the roots of the equation $d_1(n_0,\mu)=0$ which are sufficiently close to zero when $\mu\neq 0$ is sufficiently small. By the theorem of implicit functions, a *sufficient* condition for the existence of only *one* such root is

$$d_1(0,0)=0,\quad d'_{1n_0}(0,0)\neq 0. \tag{65}$$

Theorem 77. Let L_0 be a closed path of system (A_0) corresponding to $n_0=0$, and let the following conditions be satisfied:

$$d'_\mu(0,0)=0,\quad d''_{\mu n_0}(0,0)\neq 0. \tag{66}$$

Then there exist $\varepsilon>0$ and $\delta>0$ such that

(a) for any μ, $|\mu|<\delta$, (A_μ) has one and only one closed path L_μ in $U_\varepsilon(L_0)$ which contracts to L_0 for $\mu\to 0$;

(b) this path is a structurally stable limit cycle, which is stable when $\mu\cdot d''_{\mu n_0}(0,0)<0$ and unstable when $\mu\cdot d''_{\mu n_0}(0,0)>0$.

Proof. By (63),

$$d'_\mu(0,0)=d_1(0,0),\quad d''_{\mu n_0}(0,0)=d'_{1n_0}(0,0). \tag{67}$$

The last relations show that conditions (66) are equivalent to conditions (65) and proposition (a) follows directly from the theorem of implicit functions.

Let us now prove proposition (b). Let the closed path L_μ correspond to $n_0=h(\mu)$ $(\mu\neq 0,\ |\mu|<\delta)$. This value of the parameter satisfies the equation $d_1(n_0,\mu)=0$, i.e., $d_1(h(\mu),\mu)=0$. $h(\mu)$ is an analytical function, and since $d_1(0,0)=0$, we have $h(0)=0$, i.e., the expansion of $h(\mu)$ in powers of μ has the form

$$h(\mu)=\alpha_1\mu+\alpha_2\mu^2+\dots \tag{68}$$

To prove structural stability of the cycle L_μ and to establish its stability characteristics, we have to compute $d'_{n_0}(n_0,\mu)$ for $n_0=h(\mu)$. From conditions (60), using (64), (65), and (67), we obtain

$$u_{01}=0,\ u_{11}=d'_{1n_0}(0,0)=d''_{\mu n_0}(0,0)\neq 0. \tag{69}$$

Differentiation of (62) with respect to n_0 gives

$$d'_{n_0}(n_0,\mu)=\mu u_{11}+u_{12}\mu^2+2u_{21}n_0\mu+\dots,$$

whence

$$d'_{n_0}(h(\mu),\mu)=\mu u_{11}+u_{12}\mu^2+2u_{21}h(\mu)\mu+\dots$$

or, by (68) and (69),

$$d_{n_0}(h(\mu),\mu)=\mu\cdot d''_{\mu n_0}(0,0)+o(\mu).$$

The last equality shows that for sufficiently small $\mu \neq 0$, the derivative $d'_{n_0}(h(\mu), \mu)$ does not vanish and its sign is equal to the sign of $\mu \cdot d''_{\mu n_0}(0, 0)$. This signifies, in virtue of the results of Chapter V (see §12.3), that proposition (b) is satisfied. The proof of the theorem is complete.

4. Systems close to a Hamiltonian system

If the original system is Hamiltonian, the conditions of the creation of a limit cycle from a closed path of the system takes on a particularly simple form. We will derive these conditions using Theorem 77 and the relations of §32.

Let the system close to a Hamiltonian system be given in the form

$$\begin{aligned}\frac{dx}{dt} &= -\frac{\partial H}{\partial y} + \mu p_1(x, y) + \mu^2 p_2(x, y) + \ldots .\\ \frac{dy}{dt} &= \frac{\partial H}{\partial x} + \mu q_1(x, y) + \mu^2 q_2(x, y) + \ldots\end{aligned} \tag{B_μ}$$

The equations of the closed paths of the original system (B_0) have the form $H(x, y) = C$. (B_μ) is considered in a doubly connected region G filled with the paths $H(x, y) = C$, where $C_1 < C < C_2$. Let L_0 be one of these paths, $x = \varphi(t)$, $y = \psi(t)$ the motion corresponding to this path, τ the period of the functions φ and ψ. In the neighborhood of L_0, we introduce curvilinear coordinates s, n described at the beginning of §33.3 (see (56)), in which the paths of the dynamic system (B_0) are described by the equations $n = \text{const}$. Let the path L_0 be described by the equation $n = \bar{n}_0$. The expansion of $d(n_0, \mu)$ around the point $n = \bar{n}_0$, $\mu = 0$ then has the form

$$d(n_0, \mu) = u_{01}\mu + u_{11}(n_0 - \bar{n}_0)\mu + u_{02}\mu^2 + \ldots \tag{70}$$

(see (59)–(62)). In this case, the numbers $d'_\mu(0, 0)$, $d''_{\mu n_0}(0, 0)$ of Theorem 77 evidently should be replaced with $d'_\mu(\bar{n}_0, 0)$ $d''_{\mu n_0}(\bar{n}_0, 0)$.

Since $d'_\mu(n_0, 0) = u_{01}$, equation (36), §32.1, can be used to compute $d'_\mu(\bar{n}_0, 0)$, replacing $\Delta(0, 0)$ with $\Delta(0, \bar{n}_0)$. Since for a Hamiltonian system

$$P'_x(x, y) + Q'_y(x, y) = -\frac{\partial^2 H}{\partial y\, \partial x} + \frac{\partial^2 H}{\partial x\, \partial y} \equiv 0,$$

we find

$$d'_\mu(\bar{n}_0, 0) = \frac{1}{\Delta(0, \bar{n}_0)} \int_0^\tau [q_1(\varphi(s), \psi(s))\varphi'(s) - p_1(\varphi(s), \psi(s))\psi'(s)]\, ds \tag{71}$$

or, changing over to a line integral,

$$d'_\mu(\bar{n}_0, 0) = \frac{1}{\Delta(0, \bar{n}_0)} J(\bar{n}_0), \tag{72}$$

where

$$J(\bar{n}_0) = \int_{(L_0)} q_1(x, y)\, dx - p_1(x, y)\, dy. \tag{73}$$

Let G_0 be the region enclosed inside the curve L_0. If (B_μ) is defined everywhere in G_0, using Green's function we can change over from the

line integral to a double integral. The condition $d'_{\mu}(\bar{n}_0, 0) = 0$ is therefore equivalent to the condition

$$\iint_{G_0} [p'_{1x}(x, y) + q'_{1y}(x, y)]\, dx\, dy = 0. \tag{74}$$

If (B_μ) is not defined everywhere in G_0, the condition $d'_{\mu}(\bar{n}_0, 0) = 0$ is equivalent to the condition

$$J(\bar{n}_0) = \int_{(L_0)} q_1\, dx - p_1\, dy = 0. \tag{75}$$

Let us now consider the number $d''_{\mu n_0}(\bar{n}_0, 0)$. Since the number $\bar{n}_0$ is in no way distinguished from the other numbers n_0, $d''_{\mu n_0}(\bar{n}_0, 0)$ can be obtained by differentiating (72) with respect to $\bar{n}_0$. Thus,

$$d''_{\mu n_0}(\bar{n}_0, 0) = \frac{1}{\Delta(0, \bar{n}_0)} \frac{dJ(\bar{n}_0)}{d\bar{n}_0} + J(\bar{n}_0) \frac{d}{d\bar{n}_0}\left(\frac{1}{\Delta(0, \bar{n}_0)}\right). \tag{76}$$

Let us compute $\frac{dJ(\bar{n}_0)}{d\bar{n}_0}$. To fix ideas, we assume that as t increases, the closed path L_0 is traced in the positive direction. We have

$$\frac{dJ(\bar{n}_0)}{d\bar{n}_0} = \lim_{h \to 0} \frac{J(\bar{n}_0 + h) - J(\bar{n}_0)}{h}. \tag{77}$$

Let L_{0h} be the closed path of (B_0) which corresponds to the value $\bar{n}_0 + h$ of the parameter, G_h the region bounded by the paths L_0 and L_{0h}. The sign of h is chosen so that the path L_0 is enclosed inside L_{0h} (Figure 173). Then $J(\bar{n}_0 + h) - J(\bar{n}_0)$ is the line integral (73) taken along the boundary of G_h, and by Green's formula

$$J(\bar{n}_0 + h) - J(\bar{n}_0) = -\iint_{G_h} [p'_{1x}(x, y) + q'_{1y}(x, y)]\, dx\, dy. \tag{78}$$

Changing over to the curvilinear coordinates s, n in the last integral according to the relations $x = \bar{\varphi}(s, n)$, $y = \bar{\psi}(s, n)$, we obtain

$$J(\bar{n}_0 + h) - J(\bar{n}_0) = -\iint_{G_h} [p'_{1x}(\bar{\varphi}, \bar{\psi}) + q'_{1y}(\bar{\varphi}, \bar{\psi})]\, |\Delta(s, n)|\, ds\, dn, \tag{79}$$

where $\Delta(s, n)$ is the Jacobian $\frac{D(\bar{\varphi}, \bar{\psi})}{D(s, n)}$. This Jacobian, by assumption, retains a constant sign near the path L_0; let $\Delta(s, n) > 0$.

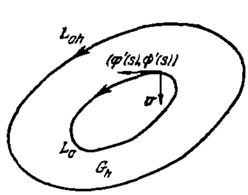

FIGURE 173

Because of the particular choice of the curvilinear coordinates s, n, the expansion of (57) in powers of μ and $n-\bar{n}_0$ contains no terms with μ (see (58)). In particular, $A_{10}(s)\equiv 0$. In our case, however, by § 32.1, (32),

$$A_{10}(s)=P'_x(\varphi,\ \psi)+Q'_y(\varphi,\ \psi)-\frac{d}{ds}\ln(\Delta(s,\ \bar{n}_0)),$$

and $P'_x+Q'_y\equiv 0$. Therefore $\frac{d}{ds}\ln(\Delta(s,\ \bar{n}_0))\equiv 0$, i.e., $\Delta(s,\ \bar{n}_0)$ for every $\bar{n}_0$ is independent of s, and for every n we have

$$\Delta(s,\ n)=\Delta(0,\ n). \tag{80}$$

By assumption, the path L_0 is traced in the positive direction as t increases, and

$$\Delta(s,\ \bar{n}_0)=\begin{vmatrix}\varphi'_s(s,\ \bar{n}_0) & \psi'_s(s,\ \bar{n}_0)\\ \varphi'_n(s,\ \bar{n}_0) & \psi'_n(s,\ \bar{n}_0)\end{vmatrix}=\begin{vmatrix}\varphi'(s) & \psi'(s)\\ \bar{\varphi}'_n(s,\ \bar{n}_0) & \bar{\psi}'_n(s,\ \bar{n}_0)\end{vmatrix}>0.$$

Therefore, the vector $\mathbf{v}(\varphi'_n(s,\ \bar{n}_0),\ \psi'_n(s,\ \bar{n}_0))$ points inside the curve L_0 (Figure 173). On the other hand, the direction of this vector corresponds to increasing n_0 on the curve $x=\bar{\varphi}(s,\ n_0)$, $y=\bar{\psi}(s,\ n_0)$, where s is constant. Therefore, since L_0 is enclosed inside L_{0h}, we have $h<0$. Changing over from a double integral in (79) to two successive integrations and seeing that $h<0$ and $\Delta(s,\ n)=\Delta(0,\ n)>0$, we obtain

$$J(\bar{n}_0+h)-J(\bar{n}_0)=-\int\limits_{\bar{n}_0+h}^{\bar{n}_0}dn\int\limits_0^{\tau}[p'_{1x}(\bar{\varphi},\ \bar{\psi})+q'_{1y}(\bar{\varphi},\ \bar{\psi})]\,\Delta(0,\ n)\,ds,$$

where $\bar{\varphi}=\bar{\varphi}(s,\ n)$, $\bar{\psi}=\bar{\psi}(s,\ n)$. Hence,

$$\frac{dJ(\bar{n}_0)}{d\bar{n}_0}=\int\limits_0^{\tau}[p'_{1x}(\varphi(s),\ \psi(s))+q'_{1y}(\varphi(s),\ \psi(s))]\,\Delta(0,\ \bar{n}_0)\,ds. \tag{81}$$

From (76) and (81) it follows that if $J(\bar{n}_0)=0$, we have

$$d''_{\mu n_0}(\bar{n}_0,\ 0)=\int\limits_0^{\tau}[p'_{1x}(\varphi(s),\ \psi(s))+q'_{1y}(\varphi(s),\ \psi(s))]\,ds. \tag{82}$$

The above considerations and Theorem 77 evidently lead to the following theorem, first formulated by Pontryagin (see /31/).

Theorem 78. Let L_0 be a closed path of the Hamiltonian system $\frac{dx}{dt}=-\frac{\partial H}{\partial y}$, $\frac{dy}{dt}=\frac{\partial H}{\partial x}$,

$$x=\varphi(t),\qquad y=\psi(t)$$

the motion corresponding to this path, τ the period of the functions φ and ψ, G_0 the region enclosed inside the path L_0, and

$$\begin{aligned}\frac{dx}{dt}&=-\frac{\partial H}{\partial y}+\mu p_1(x,\ y)+\mu^2 p_2(x,\ y)+\ldots,\\ \frac{dy}{dt}&=\frac{\partial H}{\partial x}+\mu q_1(x,\ y)+\mu^2 q_2(x,\ y)+\ldots\end{aligned} \tag{B_μ}$$

a system close to a Hamiltonian system (μ is a small parameter). Then, if

$$\iint_{G_0} [p'_{1x}(x, y) + q'_{1y}(x, y)]\, dx\, dy = 0, \tag{74}$$

and

$$I = \int_0^{\tau} [p'_{1x}(\varphi(s), \psi(s)) + q'_{1y}(\varphi(s), \psi(s))]\, ds \neq 0, \tag{75}$$

there exist $\varepsilon > 0$ and $\delta > 0$ such that

(a) for every μ, $|\mu| < \delta$, (B_μ) has one and only one closed path L_μ in $U_\varepsilon(L_0)$, and this L_μ contracts to L_0 for $\mu \to 0$;

(b) this path is a structurally stable limit cycle, which is stable for $\mu I < 0$ and unstable for $\mu I > 0$.

Remark 1. If (B_μ) is defined only in the neighborhood of the path L_0, and not everywhere in G_0, condition (74) should be replaced with the equality

$$\int_0^{\tau} [q_1(\varphi(s), \psi(s))\varphi'(s) - p_1(\varphi(s), \psi(s))\psi'(s)]\, ds = 0$$

(see (75)).

Remark 2. It is readily seen that Theorem 75 (see § 33.2), relating to systems which are close to the linear conservative system $\frac{dx}{dt} = -y$, $\frac{dy}{dt} = x$, is a particular case of Theorem 78. Indeed, the equation of a path of this linear system has the form $x = \rho_1 \cos t$, $y = \rho_1 \sin t$. Inserting these functions for φ and ψ in (74) and (75), we obtain relations (51) of § 33.2.

In conclusion of this section, we wish to comment on the creation of limit cycles from a focus or a center. We have repeatedly noted the deep analogy between the investigation of a dynamic system in the neighborhood of a closed path and the investigation in the neighborhood of a focus or a center. This analogy also extends to the creation of limit cycles.

Suppose that the original dynamic system (A_0) has at the origin an equilibrium state with pure imaginary characteristic numbers (i.e., a multiple focus or center). Consider the modified system

$$\frac{dx}{dt} = P + \mu p_1 + \mu^2 p_2 + \ldots, \quad \frac{dy}{dt} = Q + \mu q_1 + \mu^2 q_2 + \ldots \tag{A_μ}$$

Changing over to polar coordinates ρ, θ and replacing the system with a single equation, we obtain a differential equation analogous to (57) in § 33.3:

$$\frac{d\rho}{d\theta} = R(\theta, \rho, \mu) = R_{10}\rho + R_{01}\mu + R_{20}\rho^2 + \ldots, \tag{R_μ}$$

where the coefficients R_{ij} are periodic functions of θ with a period of 2π, and the series in the right-hand side of the last equation converges for all sufficiently small ρ and μ (see § 24.4, (54)).

As in § 33.3, we seek a solution $f(\theta; 0, \rho_0, \mu)$ of (R_μ) which satisfies the condition $f(0; 0, \rho_0, \mu) \equiv \rho_0$ in the form of a series

$$\rho = u_{10}\rho + u_{01}\mu + u_{20}\rho^2 + u_{11}\rho_\mu + \ldots,$$

where the coefficients u_{ij} are functions of θ satisfying recursive formulas analogous to equations (14) (§ 32.1).

Considering the succession function $\rho = f(2\pi; 0, \rho_0, \mu)$ and the function $d(\rho_0, \mu) = f(2\pi; 0, \rho_0, \mu) - \rho_0$, we can use Newton's diagram to derive the sufficient conditions of the creation of limit cycles from a multiple focus or center, as in §32.4.

Since the succession function constructed in the neighborhood of a focus or a center has a number of specific properties (see §24.1, Lemmas 1, 2, 5), the specificity is also extended to the function $d(\rho_0, \mu)$. We will not consider this aspect of the matter, however.

Note that the function $\Psi(\theta, \rho_0, \mu)$ constructed in §33.2 in connection with a system close to the linear dynamic system (31) is defined for all sufficiently small ρ_0 and μ. Therefore, the same function can be applied to investigate the creation of limit cycles from equilibrium states of the type of a center, as well as from closed paths of the original conservative system.

Chapter XIV

THE APPLICATION OF THE THEORY OF BIFURCATIONS TO THE INVESTIGATION OF PARTICULAR DYNAMIC SYSTEMS

INTRODUCTION

In this chapter, we consider some examples of dynamic systems containing parameters. Almost all systems of this kind arose in connection with particular physical or engineering topics. The main problem which is encountered in applications is to establish the partition of the space of parameters into regions corresponding to identical qualitative structures of the dynamic system.

When the parameters assume values from each of these regions, the system retains the same qualitative structure and, in general, remains structurally stable (or, in any case, "relatively structurally stable," i.e., structurally stable relative to the space of the dynamic systems spanned by the range of variation of the parameters). The points in the parameter space which lie on the boundary of two regions correspond to structurally unstable systems, which — with the exception of isolated points — are systems of the first degree of structural instability.

The aim of the present chapter is to familiarize the reader with certain techniques of the theory of bifurcations which enable us to obtain information regarding the qualitative structure of dynamic systems and to analyze the changes in qualitative structure of systems following a change in parameters.

In the application of these techniques, it is very important to be able to establish the qualitative structure of the dynamic system at least for some particular values of the parameters. The examples discussed in this chapter are therefore solved by the application of certain particular techniques developed in QT (the isocline configuration, Dulac's criterion, the topographic system, etc.).

In broad outline, the qualitative techniques based on the theory of bifurcations can be described as follows:

1) If for some values of the parameters the system has an equilibrium state with $\Delta > 0$ (i.e., a node or a focus), we first have to establish the existence (or the absence) of numerical values of the parameters for which the equilibrium state changes its stability, i.e., to find the values of the parameters for which the system has an equilibrium state with pure imaginary characteristic roots. The procedure described in §25.3 enables us to detect in this case (if the equilibrium state is not a center) the creation of a limit cycle and thus to identify the range of parameter values for which the dynamic system a priori has a limit cycle.

2) If the equilibrium states cannot be determined by elementary techniques, we search for parameter values at which the equilibrium states have maximum multiplicity.

Investigation of the possible character of the equilibrium states for parameter values close to the values which correspond to maximum multiplicity enables us to establish in certain cases all the alternatives regarding the number and the type of the equilibrium states.

3) If the qualitative structure has been established by some technique in two different points of the parameter space, the transition from one of these points to another will enable us to establish, say, the existence of a saddle-to-saddle separatrix and hence the possibility of creation of limit cycles.

The largest difficulties in qualitative investigation of dynamic systems are encountered when we attempt to prove the absence or the presence of limit cycles created from a "condensation of paths." A complete analysis is therefore often impossible.

§34. EXAMPLES

Example 12 (the creation of a limit cycle from a multiple focus). Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 6xy + \mu(4-2x-2y-2x^2+2xy+3y^2),\\ \frac{dy}{dt} &= 4-2x-2y-2x^2+2xy+3y^2-6\mu xy,\end{aligned} \tag{1}$$

where $\mu > 0$ is sufficiently small.

The vector field of this system is obtained from the vector field of the system

$$\frac{dx}{dt} = 6xy, \qquad \frac{dy}{dt} = 4-2x-2y-2x^2+2xy+3y^2 \tag{2}$$

by rotation through an angle $\tan^{-1}\mu$ (see §3.2). It is obvious that the ordinate axis $x = 0$ is a path of system (2). System (2) has two equilibrium states, the points $A\ (1, 0)$ and $B\ (-2, 0)$. They are both simple equilibrium states ($\Delta \neq 0$). The characteristic equation of the equilibrium state $A\ (1, 0)$ is $\lambda^2 + 36 = 0$, i.e., $A\ (1, 0)$ is an equilibrium state with pure imaginary characteristic numbers. Since (2) is an analytical system, A is either a multiple focus or a center. The characteristic equation of the equilibrium state $B\ (-2, 0)$ is $\lambda^2 + 6\lambda + 72 = 0$. Its roots are $\lambda_{1,2} = -3 \pm 3i\sqrt{7}$, i.e., $B\ (-2, 0)$ is a simple stable focus of system (2).

System (1) has its equilibrium states at the same points $A\ (1, 0)$ and $B\ (-2, 0)$ as system (2) (see footnote on p. 210). Clearly, if μ is sufficiently small, $B\ (-2, 0)$ is a simple stable focus of system (1) also.

To investigate the equilibrium state $A\ (1, 0)$, we will use the results of Chapter IX. Moving the origin to the point A and reverting to the previous notation, we obtain from (1)

$$\begin{aligned}\frac{dx}{dt} &= -6\mu x + 6y - 2\mu x^2 + (6+2\mu)xy + 3\mu y^2,\\ \frac{dy}{dt} &= -6x - 6\mu y - 2x^2 + (2-6\mu)xy + 3y^2,\end{aligned} \tag{3}$$

and for $\mu=0$ this system reduces to

$$\frac{dx}{dt}=6y+6xy, \qquad \frac{dy}{dt}=-6x-2x^2+2xy+3y^2. \tag{4}$$

For system (3), $\sigma(\mu)=P'_x+Q'_y=-12\mu$, $\sigma'(0)=-12<0$. The value of α_3 for (4) is computed using equation (76), §24.4; it is found to be $\frac{\pi}{72}$, i.e., $\alpha_3>0$.* The table in §25.3 thus shows that the point $A\,(1,0)$ is an unstable multiple focus of system (2) and — for a small positive μ — a stable focus of system (1) containing an unstable limit cycle in its neighborhood.

Let us now investigate the behavior of the paths of system (1) at infinity, following the scheme described in QT, §13.2. Applying the transformation $x=\frac{1}{z}$, $y=\frac{u}{z}$, we establish that the "ends" of the z axis are not equilibrium states of the system. Applying the transformation $x=\frac{v}{z}$, $y=\frac{1}{z}$ to system (1) and multiplying the right-hand sides of the resulting system by z, we obtain

$$\begin{aligned}\frac{dv}{dt}&=3\mu-2\mu z+(3+2\mu)v+4\mu z^2+(2-2\mu)vz+\\&\quad+(4\mu-2)v^2-4z^2v+2v^2z+2v^3=\hat{P}(v,z),\\ \frac{dz}{dt}&=-3z+(6\mu-2)vz+2z^2+2v^2z+2vz^2-4z^3=\hat{Q}(v,z).\end{aligned} \tag{5}$$

Since $\hat{Q}(v,z)$ contains a factor z, the axis $z=0$ consists of paths of system (5). To find the equilibrium states lying on this axis, we have to consider the equation

$$\hat{P}(v,0)=3\mu+(3+2\mu)v+(4\mu-2)v^2+2v^3=0. \tag{6}$$

The derivative $\hat{P}'_v(v,0)=3+2\mu+(8\mu-4)v+6v^2$ has no real roots for small μ. Therefore if μ is small and $\mu>0$, equation (6) has a single real root v_0, and it is readily seen that $v_0<0$ and $\lim\limits_{\mu\to 0}v_0=0$. System (5) thus has a single equilibrium state on the axis $z=0$, $\hat{D}\,(v_0,0)$. Consider the determinant

$$\Delta(v_0,0)=\begin{vmatrix}\hat{P}'_v(v_0,0) & \hat{P}'_z(v_0,0)\\ \hat{Q}'_v(v_0,0) & \hat{Q}'_z(v_0,0)\end{vmatrix}= \\ =[3+2\mu+(8\mu-4)v_0+6v_0^2][-3+(6\mu-2)v_0+2v_0^2].$$

When μ is sufficiently small, v_0 is small and $\Delta(v_0,0)<0$, i.e., the equilibrium state $\hat{D}\,(v_0,0)$ is a saddle point of system (5). Two of its separatrices coincide with the semiaxes $z=0$ adjoining the point $\hat{D}\,(v_0,0)$. The direction of motion along these separatrices is defined by the equation $dv/dt=\hat{P}\,(v,0)$. We see from this equation that the two separatrices lying on the axis $z=0$ are α-separatrices. Therefore, the ω-separatrices of the saddle point $\hat{D}$ lie on the two sides of the axis $z=0$, and the path configuration of system (5) near the saddle point $\hat{D}$ can be schematically represented by the diagram shown in Figure 174.

The rules for establishing the behavior at infinity (QT, §13.2) now show that the paths of system (1) near the equator are arranged as shown in Figure 175.

* The indices α_i, and in particular α_3, are introduced in Chapter IX (§24.4, (61)). In this chapter it is shown that if O is a multiple focus and $\alpha_3\neq 0$, the sign of α_3 determines the stability of the focus.

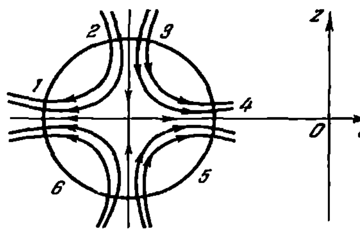

FIGURE 174

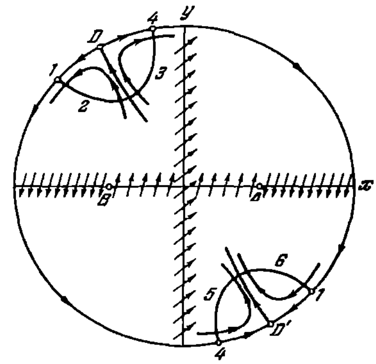

FIGURE 175

By (1), for $x=0$, $\frac{dx}{dt}=\mu(4-2y+3y^2)>0$, and $\frac{dy}{dt}=4-2y+3y^2>0$. For $y=0$, $\frac{dx}{dt}=2\mu(x+2)(1-x)$, and $\frac{dy}{dt}=2(x+2)(1-x)$. Therefore the paths of system (1) cross the coordinate axes in the directions shown in Figure 175.

We have thus established that system (1) for a small $\mu>0$ has two stable foci A (1, 0) and B (– 2, 0) and an unstable limit cycle in the neighborhood of the focus A. Using this fact, and taking into account the path configuration at infinity and the direction of motion along the paths at the points of intersection with the coordinate axes, we conclude that in the half-plane $x<0$ system (1) has a limit cycle C_1 unstable from the outside to which the separatrix of the saddle point D goes for $t\to-\infty$, and in the half-plane $x>0$ the system has a limit cycle C_2 stable from the outside to which the separatrix of the saddle point D' goes for $t\to+\infty$. Evidently, any closed path of system (1), other than C_1 and C_2, is entirely contained either in the half-plane $x<0$, and then it is enclosed inside C_1 and encloses the point B, or in the half-plane $x>0$, and then it is enclosed inside C_2 and encloses the focus A. It is readily seen that system (1) may only have a finite number of closed paths. Indeed, suppose that this is not so. Then there exists an infinite set of closed paths $\{\Gamma\}$ arranged concentrically inside a cycle, C_1 say. We can select an infinite sequence of paths Γ_1, Γ_2, Γ_3, ... with the following property: every successive path encloses its predecessor and every path of the set $\{\Gamma\}$ is enclosed at least inside one of the paths Γ_i (see QT, §16.9, Lemma 16). Let K be the topological limit of the sequence Γ_i (see QT, Appendix, §1.10). It is readily seen that either K is a closed path Γ^*, such that closed paths lie in any sufficiently small neighborhood inside Γ^* and no such paths are observed outside Γ^*, or K has the structure of a 0-limit continuum (see QT, §23.2, Theorem 70), i.e., it consists of

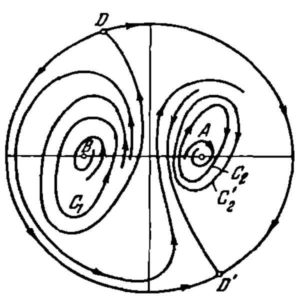

FIGURE 176

equilibrium states alternating with separatrices which go to these equilibrium states. However, since (1) is an analytical system, it cannot have a closed path Γ^* of this kind. The topological limit K cannot have the structure of the null-limit continuum either, because all the equilibrium states of system (1) are foci.

We have thus established that system (1) only has a finite number of closed paths, and there is at least one limit cycle in the half-plane $x<0$ and at least two limit cycles in the half-plane $x>0$. The exact number of the limit cycles of system (1) cannot be established. It is clear, however, that the number of limit cycles (counting each according to its multiplicity) in the half-plane $x<0$ is odd, and the number of limit cycles in the half-plane $x>0$ is even. The configuration of the paths of system (1) in a circle, which is used as a model of the Euclidean plane, is shown in Figure 176 (up to an even number of limit cycles).

Example 13 (the creation of a limit cycle from a multiple focus and a separatrix loop, see /20/).

Consider the system

$$\dot{x}=y, \quad \dot{y}=-x+\mu y+xy+x^2+y^2. \tag{7}$$

It has two equilibrium states, $O(0,0)$ and $A(1,0)$. A is a saddle point for any μ. The characteristic numbers of the equilibrium state $O(0,0)$ are $\lambda_{1,2}=\frac{\mu}{2}\pm\sqrt{\frac{\mu^2}{4}-1}$; it is therefore

1) a stable node for $\mu\leqslant-2$;
2) a stable focus for $-2<\mu<0$;
3) an unstable focus for $0<\mu<2$;
4) an unstable node for $+2\leqslant\mu$.

For $\mu=0$, O is an equilibrium state with pure imaginary characteristic numbers. To investigate its character, we will use, as in the previous example, the results of §25.3. We have $\sigma(\mu)=\mu$, $\sigma'(\mu)=1>0$. α_3 is computed from (76), §24.4, and is found to be equal to $\frac{\pi}{2}>0$. Therefore, for small $\mu>0$, and also for $\mu=0$, the point O is an unstable focus of system (7), without any limit cycles in its sufficiently small neighborhood, and for small $\mu<0$ the point O is a stable focus with an unstable cycle in its neighborhood (see table in §25.3).

To investigate the situation at infinity, we first apply the Poincaré transformation $x=\frac{v}{z}$, $y=\frac{1}{z}$. Multiplying the right-hand sides of the resulting system by z, we obtain

$$\frac{dv}{dt}=-v+z-\mu vz-v^2-v^2z-v^3, \quad \frac{dz}{dt}=-z-vz-\mu z^2+vz^2-v^2z. \tag{8}$$

For $z=0$, system (8) has a single equilibrium state $(0,0)$, which is a stable node. The second equation in (8) shows that the axis $z=0$ is made up of paths of system (8). The Poincaré transformation $x=\frac{1}{z}$, $y=\frac{u}{z}$ gives the system

$$\frac{du}{dt}=1+u-z+\mu uz-zu^2+u^2, \quad \frac{dz}{dt}=-uz^2.$$

For $z = 0$, the last system has no equilibrium states. Hence it follows that, in accordance with the rules for investigation at infinity (QT, §13.2), the paths of system (7) are arranged at infinity as shown in Figure 177.

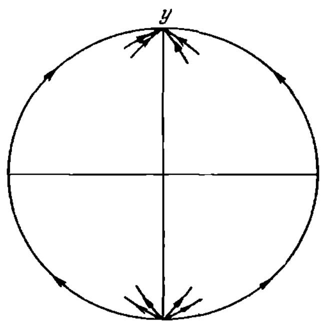

FIGURE 177

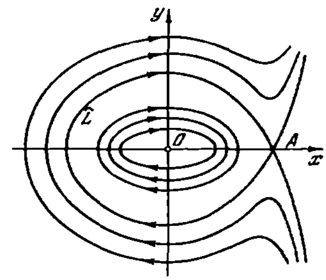

FIGURE 178

To proceed with further investigation of system (7), we will use the auxiliary system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + x^2. \tag{9}$$

Its general integral, as is readily seen, is

$$\frac{y^2}{2} = \frac{x^3}{3} - \frac{x^2}{2} + C. \tag{10}$$

System (9) has its equilibrium states at the same points O and A as system (7), $O(0, 0)$ being a focus and $A(1, 0)$ a saddle point of (9). The curves (10) and the paths of system (9) making up these curves are shown in Figure 178. One of the paths is the loop $\hat{L}$ originating and terminating at the saddle point A, which is described by the equation

$$\frac{y^2}{2} = \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{6}. \tag{11}$$

All the paths of system (9) enclosed inside the loop L_0 are closed.

Consider the contact curve of systems (7) and (9) (see QT, §12.5). Its equation is

$$(-x + \mu y + xy + x^2 + y^2)\, y - y\,(-x + x^2) = 0$$

or

$$y^2 (\mu + x + y) = 0. \tag{12}$$

It consists of the points of contact of the paths of systems (7) and (9) and the equilibrium states of these systems. The contact curve decomposes into straight lines $x + y + \mu = 0$ and $y = 0$, but the line $y = 0$ constitutes a

false contact, i.e., a path of system (7) contacting a path of system (9) at a point with $y = 0$ will inevitably cross this path from one side to another (see QT, §12.5).

Since the sum of the indices of the equilibrium states enclosed inside a closed path is 1, any closed path of system (7) encloses the equilibrium state O without enclosing A. This path therefore necessarily crosses the closed paths of system (9).

Consider the intersection of the curve (11), part of which is represented by the loop $\hat{L}$, with the contact line $x + y + \mu = 0$. Inserting $y = -x - \mu$ in (11), we obtain the equation

$$f(x) = \frac{x^3}{3} - x^2 - \mu x + \frac{1}{6} - \frac{\mu^2}{2} = 0. \tag{13}$$

Since $f'(x) = x^2 - 2x - \mu$ has the roots $x_{1,2} = 1 \pm \sqrt{1+\mu}$, $f'(x) > 0$ for $\mu < -1$ and equation (13) only has one real root. For $\mu = -1$ equation (13) has a triple root $x = 1$, i.e., the contact line $x + y + \mu = 0$ crosses the curve (11) at the point A (1, 0). Thus, for $\mu = -1$ the loop $\hat{L}$ has one point in common with the contact line (the point A). But then, as we see from the position of the line $x + y + \mu = 0$, the loop $\hat{L}$ does not cross the contact line $x + y + \mu = 0$ for $\mu < -1$. Hence the conclusion that f o r $\mu \leqslant -1$ s y s t e m (7) h a s n o c l o s e d p a t h s. Indeed, let L be a closed path of system (7). As we have seen before, L crosses the closed paths of system (9) and encloses some of these paths. This means that there exists a closed path of system (9) which has a (true) point of contact with the path L, namely the "least" path $\hat{L}_1$ having a point of contact with L. Let M be this point of contact. M lies inside the loop $\hat{L}$ on the contact curve (12), i.e., on the straight line $y = 0$. This is a contradiction, however, since all the points of this line correspond to false contacts.

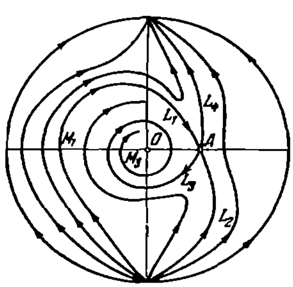

FIGURE 179. $\mu = -1 < \mu^*$.

Having thus established that system (7) has no closed paths for $\mu \leqslant -1$, we can find its topological structure. To fix ideas let $\mu = -1$ (for $\mu < -1$, the structure is the same, but for $\mu \leqslant -2$ the point O is a stable node, and not a focus). The ω-separatrices L_1 and L_2 of the saddle point A (1, 0) should extend from the unstable focus to infinity, since there are no other α-limit points in the plane. This, however, automatically fixes the behavior of the α-separatrices L_3 and L_4: one of them, L_3 say, winds onto the focus O for $t \to +\infty$, and the other, L_4, goes to the stable node at infinity. Allowing for the direction of the paths at the intersection points with the coordinate axes, we obtain the path configuration shown schematically in Figure 179.

Let us now consider the path configuration of system (7) for $\mu = 3$. In this case, O (0, 0) is an unstable node. The abscissa of the intersection point of the curve (11) with the contact line $x + y + 3 = 0$ is determined, after eliminating y, from the equation $f(x) = x^3 - 3x^2 - 9x - 13 = 0$. A standard investigation of the function $y = f(x)$ readily shows that it has a single real root x_0, and $x_0 > 3$ (the curve of $f(x)$ is shown in Figure 180).

Hence it follows that the contact line $x+y+3=0$ does not cross the loop $\hat{L}$ and, reasoning as before, we conclude that for $\mu=3$ system (7) has no closed paths either. The behavior of the separatrices of the saddle point $A(1, 0)$ of system (7) is thus fixed automatically (as in the case $\mu \leqslant -1$). Indeed, the α-separatrices L_3 and L_4 of the saddle point A may only extend to the stable node at infinity, and the closed curve formed by these separatrices should enclose the point O. Consequently, one of the ω-separatrices of the saddle point A, L_1 say, should go to the node O for $t \to -\infty$ and the other separatrix, L_2, should go to the unstable node at infinity.

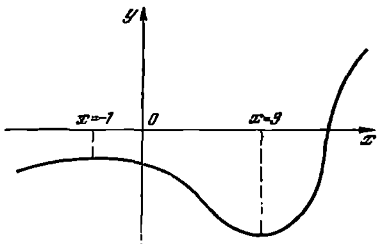

FIGURE 180

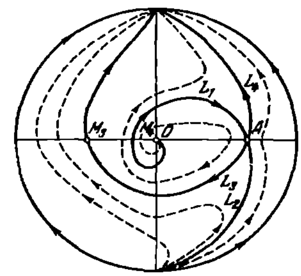

FIGURE 181

Allowing for the direction of the paths at the points of the coordinate axes and at the node O and for the direction of the separatrices at the saddle point A, we conclude that the path configuration of system (7) for $\mu=3$ can be schematically illustrated by the diagram in Figure 181. It is readily seen that for $\mu>3$ system (7) still has no closed paths and its topological structure is therefore the same as for $\mu=3$.

Let us investigate the behavior of the separatrices L_1 and L_3 as the parameter μ varies from -1 to 3. We should first note that on passing from a value μ_1 of the parameter to some value $\mu_2>\mu_1$, all the field vectors are turned in the same direction, specifically counterclockwise. This follows from the fact that by equations (7)

$$\left(\frac{dy}{dx}\right)_{\mu=\mu_2}-\left(\frac{dy}{dx}\right)_{\mu=\mu_1}=\mu_2-\mu_1>0. \tag{14}$$

Let the "first" intersection point of the separatrix L_1 (L_3) of system (7) with the negative half-axis x be $M_1(\mu)$; the abscissa of this intersection point is $x_1(\mu)$ ($M_3(\mu)$ and $x_3(\mu)$, respectively). The negative half-axis x has no contact points with the paths of the system for any μ. We may therefore apply the results of §11.1. If we take into consideration the directions in which the separatrices L_1 and L_3 cross the axis x, this lemma shows that for increasing μ, $x_3(\mu)$ decreases and $x_1(\mu)$ increases, i.e., the points $M_1(\mu)$ and $M_3(\mu)$ move in opposite directions along the axis x. In accordance with the previous results, $x_1(-1)<x_3(-1)$, and $x_1(3)>x_3(3)$ (see Figures 179 and 181). Furthermore, $x_1(\mu)$ and $x_3(\mu)$ are continuous functions (see remark to Lemma 3, §9.2). Therefore, there exists one and only one value

of the parameter, μ^*, in the interval $(-1, 3)$ such that $x_1(\mu^*) = x_3(\mu^*)$. This signifies that the separatrices L_1 and L_3 corresponding to this μ^* coincide, i.e., for $\mu = \mu^*$ the system has a separatrix which forms a loop. Let this loop separatrix be L^*. The value of the function $\sigma(x, y) = P'_x(x, y) + Q'_y(x, y)$ at the saddle point $A(1, 0)$ for $\mu = \mu^*$ is $\mu^* + 1$. Since $\mu^* > -1$, we have $\mu^* + 1 > 0$. Therefore (see Theorem 44, §29.1), the loop L^* is unstable, i.e., all the sufficiently close paths enclosed inside this loop are spirals unwinding from the loop. For $\mu = \mu^*$, the topological structure of system (7) *outside the loop* L^* is determined unambiguously. The topological structure of the system *inside the loop* L^* depends on the number of limit cycles and the stability characteristics of the equilibrium state O, which are not known. If $\mu^* < 0$, then O is a stable focus and system (7) has no closed paths inside the loop L^*, or else it has an even number of such paths (counting according to multiplicities). If, however, $\mu^* > 0$, O is an unstable focus or node and system (7) has an odd number of closed paths inside the loop L^*. Figure 182 shows the topological structure of system (7) for $\mu = \mu^*$ under the assumption that this system has no closed paths.

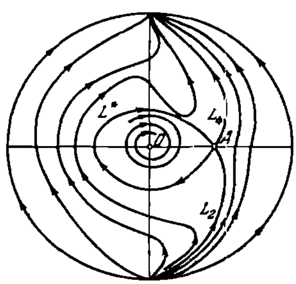

FIGURE 182. $\mu = \mu^*$.

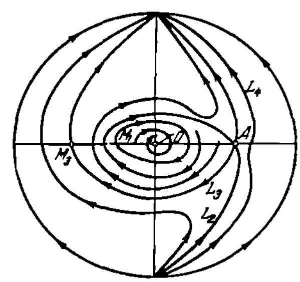

FIGURE 183. $0 > \mu > \mu^*$.

Applying the results of Chapter XI, we can trace the behavior of the separatrix loop as the parameter varies near its value μ^*. It follows from the remark to Theorem 49 (§29.3) that as the parameter μ *increases* above μ^*, the separatrix loop L^* disappears, and a single unstable limit cycle is created in its neighborhood (see Figure 183, drawn under the assumption that the system has no other limit cycles). As the parameter μ *decreases* below μ^*, the separatrix disappears without giving rise to limit cycles in its neighborhood.

To follow the changes in the topological structure of the system as the parameter varies, we require the exact number of the limit cycles of the system for every μ. The number of limit cycles may change only when the system passes through a bifurcation value of the parameter, and in such a system limit cycles are created either from a multiple focus, or from a loop of a saddle-point separatrix, or from a *condensation of paths* (see §22, Example 8), or finally from a multiple limit cycle (if any). We will nevertheless proceed with a tentative analysis *assuming that* $\mu^* < 0$ *and that system (7) for every* μ *has at most one closed*

path, which is moreover structurally stable. As a first step, let us consider what happens to a limit cycle of the system when the vector field is rotated.

Remark regarding the variation of the limit cycle following rotation of the vector field

Let

$$\frac{dx}{dt}=P(x, y, \mu), \quad \frac{dy}{dt}=Q(x, y, \mu) \tag{A_μ}$$

be a dynamic system which for $\mu=\mu_0$ has a limit cycle L_0. Suppose that L_0 is a stable limit cycle with motion in the positive direction for increasing t (Figure 184) and suppose that the field vectors rotate in the positive sense as μ increases.

Through some point M_0 of the cycle L_0 we draw an arc without contact l, choose a point M_1 on this arc, which lies inside L_0 and is sufficiently close to M_0 (Figure 184), and consider the path L_1 through the point M_1. L_1 will cross the arc l at another point M_2, which lies between M_1 and M_0.

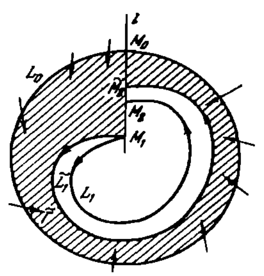

FIGURE 184

Consider the system (A_μ), where μ is sufficiently close to μ_0, $\mu>\mu_0$. The path $\widetilde{L}_1$ of (A_μ) passing through the point M_1 crosses the arc without contact l at a point $\widetilde{M}_2$, which is sufficiently close to M_2. The limit cycle L_0 of (A_{μ_0}) is a cycle without contact for (A_μ), and, as t increases, every path $\widetilde{L}$ of (A_{μ_0}) crossing the cycle L_0 will enter into the region $\widetilde{\Gamma}$ bounded by the curve L_0 and the closed curve $\widetilde{C}$ consisting of the coil $M_1\widetilde{M}_2$ of the path $\widetilde{L}_1$ and the segment $M_1\widetilde{M}_2$ of the arc l (in Figure 184, $\widetilde{\Gamma}$ is diagonally hatched). Since $\widetilde{L}$ cannot leave $\widetilde{\Gamma}$ and there are no equilibrium states in this region, $\widetilde{\Gamma}$ contains one and (by Theorem 72, §32.4) only one closed path $\widetilde{L}_0$ of system (A_μ). We thus conclude that as the vector field of system (A_{μ_0}) is rotated in the positive direction, the limit cycle L_0 contracts. Clearly, when the field is rotated in the negative direction, the limit cycle L_0 expands. A similar result is obtained for an unstable limit cycle. Applying the same reasoning to a semistable cycle (of even multiplicity) and using Theorem 71 (§32.4), we conclude that as the vector field is rotated in one direction, the cycle of even multiplicity disappears, and as the field is rotated in the opposite direction, the cycle decomposes into two cycles (a stable and an unstable one), of which one contracts and the other expands.

After this digression, we can return to system (7). Thus, let $\mu^*<0$ and let system (7) have at most one closed path, which is moreover structurally stable. For $\mu\leqslant-1$, the system has the topological structure shown in Figure 179; in particular, it has no closed paths. For μ ranging between -1 and μ^*, system (7) has no closed paths either. Indeed, as μ increases from -1 to $\mu^*<0$, a closed path can be created only from a condensation of paths, but then it is not a structurally stable path. Thus, as μ increases from $-\infty$ to μ^*, the topological structure of the system remains unchanged

(Figure 179), the points M_1 and M_3 of the separatrices L_1 and L_3 moving one toward the other along the x axis.

For $\mu = \mu^*$, the separatrices L_1 and L_3 merge forming an unstable loop. Since the equilibrium state O is stable in this case, system (7) may have no limit cycles under our assumptions, and its topological structure is correspondingly shown in Figure 182.

As μ increases above $\mu = \mu^*$, the separatrix loop creates an unstable limit cycle. The system may not have any other cycles, and its topological structure assumes the form shown in Figure 183. As μ increase from μ^* to 0, this cycle contracts and for $\mu = 0$ it "collapses" into the equilibrium state O, which changes its stability at this instant. Since for $\mu = 3$ the system has no closed paths, it may not have any closed paths for $\mu = 0$ either. Indeed, as μ decreases from 3 to 0, a closed path may be created only from a condensation of paths, but then it is not structurally stable. Thus, the topological structure of the system corresponding to $\mu > 0$ is shown in Figure 181.

As μ decreases from ∞, the system at first has no closed paths, and it is only when the system crosses the value $\mu = 0$ that an unstable limit cycle is created from the focus O, which then expands and for $\mu = \mu^*$ transforms into a separatrix loop which disappears as the system passes through the value $\mu = \mu^*$.

The reader is advised to work out for himself a similar ("tentative") analysis assuming that $\mu^* > 0$ or $\mu^* = 0$ and that the system at any time has the least possible number of closed paths.

Example 14 (the creation of a limit cycle from the loop of a saddle-node separatrix, see /34/).

Consider the system

$$\frac{dx}{dt} = y(x+\mu) + x^2 + y^2 - 1 = P(x, y, \mu), \quad \frac{dy}{dt} = -x(x+\mu) = Q(x, y, \mu). \quad (S_\mu)$$

It is readily seen that the circle

$$x^2 + y^2 - 1 = 0$$

(which we designate W) is made up of paths of (S_μ).

The equilibrium states are determined from the equations $P(x, y, \mu) = 0$, $Q(x, y, \mu) = 0$. There are two equilibrium states on the y axis, $A_\mu(0, a_\mu)$ and $B_\mu(0, b_\mu)$, where

$$a_\mu = -\frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} + 1} > 0, \quad b_\mu = -\frac{\mu}{2} - \sqrt{\frac{\mu^2}{4} + 1} < 0.$$

The other equilibrium states are obtained from the equations $x + \mu = 0$ and $x^2 + y^2 - 1 = 0$. For $|\mu| > 1$, these equations are contradictory. For $0 < |\mu| < 1$, they determine two additional equilibrium states (other than A_μ and B_μ), $C_\mu(-\mu, c_\mu)$ and $D_\mu(-\mu, -c_\mu)$, where $c_\mu = \sqrt{1 - \mu^2}$. Thus, for $0 < |\mu| < 1$, the system has four equilibrium states. For $|\mu| = 1$, the equilibrium states C_μ and D_μ merge into one, and the system has three equilibrium states A_1, B_1, and C_1 (C_1 coincides with D_1). Finally, for $\mu = 0$, the system has two equilibrium states A_0 and B_0 (which may be regarded as coinciding with C_0 and D_0, respectively). C_μ and D_μ lie on the circle W. Since $a_\mu \cdot b_\mu = -1$, then for $\mu \neq 0$, one of the points A_μ, B_μ lies inside W and the other outside the circle. The configuration of the equilibrium states of

(S_μ) for the various values of the parameter μ is shown in Figure 185 (the arrows in this figure mark the direction of motion of the equilibrium states as μ increases).

The character of the equilibrium states is determined by the usual techniques. For $\mu \neq 0$, $|\mu| \neq 1$, the points A_μ and B_μ are nodes or foci when $\mu > 0$ and saddle points when $\mu < 0$, whereas the points B_μ and C_μ are nodes or foci when $\mu < 0$ and saddle points when $\mu > 0$ (Figure 185). The equilibrium state C_μ of (S_μ) is multiple for $\mu = \pm 1$, having $\Delta = 0$, $\sigma = P'_x + Q'_y \neq 0$. To fix ideas, let $\mu = -1$. We will proceed by the method outlined in QT, §21.2, using Theorem 65 of this section. System (S_μ) for $\mu = -1$ has the form

$$\frac{dx}{dt} = y(x-1) + x^2 + y^2 - 1,\qquad \frac{dy}{dt} = -x(x-1). \tag{S_{-1}}$$

The coordinates of the equilibrium state C_{-1} are $(1, 0)$. Moving the origin to the point C_{-1}, we effect a transformation of coordinates $x = -2\bar{y}$, $y = \frac{1}{2}\bar{x} + \bar{y}$, $t = \frac{\bar{t}}{2}$, and reverting to the old notation x, y, t we obtain

$$\frac{dx}{dt} = \frac{1}{8}x^2 - \frac{5}{2}y^2,\quad \frac{dy}{dt} = y - \frac{1}{16}x^2 - \frac{3}{4}y^2.$$

Applying Theorem 65 from QT, §21.2 to this system, we find that the point C_{-1} is a saddle-node of (S_{-1}) and that the path configuration near this point can be schematically shown as in Figure 186 (this result could be foreseen beforehand, by regarding the point C_{-1} as the outcome of the merging of the node C_μ and the saddle point D_μ of (S_μ) for $\mu \to -1$). The situation is entirely analogous with regard to the point C_1 of (S_1).

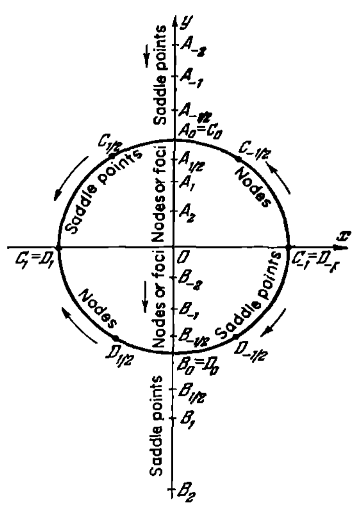

FIGURE 185

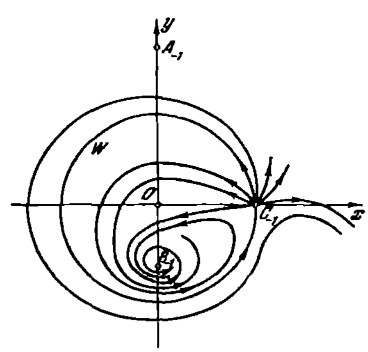

FIGURE 186

A_0 and B_0 are also saddle-nodes (of system (S_0)). We do not prove this proposition, however, since the only relevant fact for what follows is that the Poincaré index of each of these equilibrium states is 0. The latter point can be established if we note that on passing to a close system (S_μ), two equilibrium states A_μ and C_μ (B_μ and D_μ, respectively) appear in the neighborhood of the points A_0 (or B_0), the sum of their indices being zero.

As we have seen before the circle W $(x^2+y^2-1=0)$ consists of paths of (S_μ). If this curve does not contain equilibrium states, i.e., if $|\mu|>1$, W is a closed path of system (S_μ). We will now show that (S_μ) has no closed paths other than W for any value of μ.

First consider the case $\mu=0$. (S_μ) then has two equilibrium states A_0 and B_0, each with Poincaré index 0. Since the sum of the indices of the equilibrium states lying inside a closed path is 1 (QT, §11.2, Theorem 28, Corollary 1), (S_0) has no closed paths.

Let now $\mu\neq 0$. Consider an auxiliary system

$$\frac{dx}{dt}=\mu y+x^2+y^2-1,\qquad \frac{dy}{dt}=-x\mu \qquad (\tilde{S}_\mu)$$

(the two systems (S_μ) and $(\tilde{S}_\mu)$ can be considered as particular cases of the system

$$\frac{dx}{dt}=y(ax+\mu)+x^2+y^2-1,\qquad \frac{dy}{dt}=-x(ax+\mu)).$$

It is directly verified that (S_μ) has a general integral

$$(x^2+y^2-1)\,e^{\frac{2}{\mu}y}=C. \qquad (15)$$

The family of curves (15) for $\mu>0$ is shown in Figure 187a and for $\mu<0$ in Figure 187b. The contact curve of (S_μ) and $(\tilde{S}_\mu)$ has the equation $x^2(x^2+y^2-1)=0$. It can be decomposed into a straight line $x=0$ and the circle W. The contact points on the line $x=0$ are all false (see previous example, and also QT, §12.5), and the sign of the expression $x^2(x^2+y^2-1)$ is reversed only when we cross the circle W.

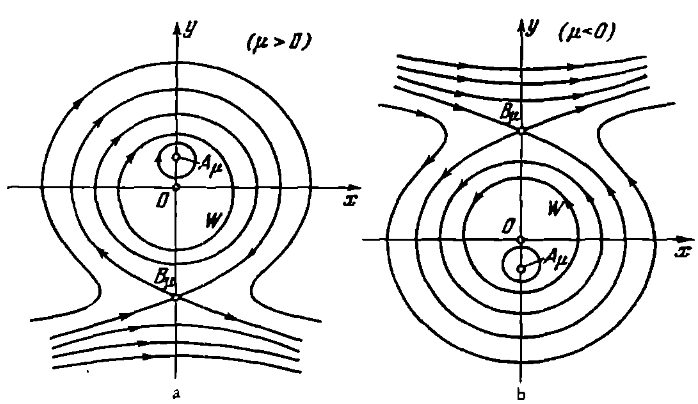

FIGURE 187. a) $\mu>0$; b) $\mu<0$.

Suppose that (S_μ) has a closed path L. Since the sum of the indices of the equilibrium states of (S_μ) enclosed inside L is 1, at least one node of (S_μ) lies inside L and at least one saddle point lies outside L. This means, however, that the path L inevitably crosses closed paths of system $(\tilde{S}_\mu)$ and consequently has a (true) contact (see previous example) with one of these closed paths. Since the (true) contact may only lie on the circle W, which is a path of $(\tilde{S}_\mu)$ for every $\mu \neq 0$, the closed path L of (S_μ) should be tangent to the circle W. This is feasible only if L coincides with W (since W is a path of (S_μ) or consists of paths of this system).

We have thus established that (S_μ) has no closed paths for $|\mu| \leqslant 1$ and has precisely one closed path — the circle W — for $|\mu| > 1$. For $\mu = -1$, (S_μ) has a saddle-node C_{-1} whose ω-separatrix L^+ forms a loop which, together with the point C_{-1}, constitute the circle W. For $t \to -\infty$, the separatrix L^+ goes to the saddle-node C_{-1}, being one of the interior paths of the node sector. At the point $C_{-1}(1, 0)$,

$$P'_x(x, y, -1) + Q'_y(x, y, -1) = 2 > 0. \tag{16}$$

Let us now consider the bifurcations which take place near the circle W as μ varies around -1. For $\mu > -1$, the system has no closed paths, and on the circle W there is a saddle point and a node near the point $(1, 0)$. As μ decreases, these points draw closer, and they merge into a saddle-node for $\mu = -1$; one of the separatrices of the saddle-node forms a loop. As μ decreases further, the saddle-node C_{-1} disappears and, by Theorems 51 and 52 (§30) and relation (16), the system should have precisely one, and at that stable, limit cycle in the neighborhood of the loop L^+. This cycle is the circle W. In this example, the limit cycle is created from a separatrix loop in the following manner: the equilibrium state C_{-1} is transformed into a regular point which forms, together with the separatrix loop, the limit cycle W. The situation is entirely analogous for the variation of the parameter μ around $\mu = 1$.

To elucidate the topological structure of the dynamic system (S_μ) on the entire plane, we should consider in more detail the character of the equilibrium states and the path configuration at infinity. This is left to the reader as an exercise. The bifurcation values of the parameter are evidently $\mu = 0, 1, -1$.

Example 15 (the creation of a limit cycle from the loop of a saddle-point separatrix and a multiple focus).

Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -(1-\beta) + y(1+\beta x) = P(x, y),\\ \frac{dy}{dt} &= -\varepsilon(x+y^2)(1+\beta x) = Q(x, y).\end{aligned} \tag{A}$$

Physical considerations (see /26/) restrict the variable x to the range

$$1 + \beta x > 0, \tag{17}$$

and the constants β and ε are constrained by the inequalities

$$0 < \beta < \tfrac{1}{3}, \quad 0 < \varepsilon < \infty. \tag{18}$$

In the half-plane defined by (17) (i.e., the half-plane to the right of the line $x = -\frac{1}{\beta}$), (A) has two equilibrium states $O_1(-1, 1)$ and $O_2(x_2, y_2)$, where

$$x_2 = \frac{1}{2} - \frac{1}{\beta} + \sqrt{\frac{1}{\beta} - \frac{3}{4}}, \quad y_2 = -\frac{1}{2} + \sqrt{\frac{1}{\beta} - \frac{3}{4}}.$$

After some manipulations, we find that $O_1(-1, 1)$ is a node or a focus, which is stable for $\varepsilon > \frac{\beta}{2(1-\beta)}$ and unstable for $\varepsilon < \frac{\beta}{2(1-\beta)}$. For $\varepsilon = \frac{\beta}{2(1-\beta)}$, O_1 is an equilibrium state with pure imaginary characteristic numbers.

The equilibrium state O_2 is a saddle point for all the relevant values of the parameters. Let us establish some properties of its separatrices.

The isoclines of vertical and horizontal inclinations are described by the respective equations

$$-(1-\beta) + y(1+\beta x) = 0 \tag{19}$$

and

$$x + y^2 = 0. \tag{20}$$

The first is the equation of a hyperbola and the second that of a parabola.

These equations partition the half-plane $1 + \beta x > 0$ into regions where $\dot{x}$ and $\dot{y}$ retain a constant sign. Let these regions be designated I, II, III, IV (Figure 188). The signs of $\dot{x}$ and $\dot{y}$, respectively, are indicated in parentheses in the figure.

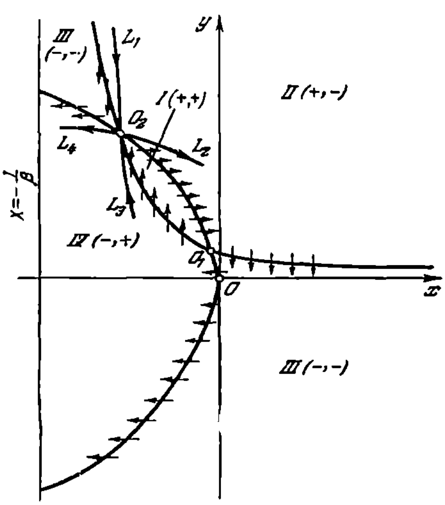

FIGURE 188.

The equation characterizing the directions of the separatrices for the saddle point O_2 has the form

$$k^2 + \frac{\beta y_2 + 2\varepsilon(1-\beta)}{1+\beta x_2} k + \varepsilon = 0. \tag{21}$$

Since $\varepsilon > 0$, $1 + \beta x_2 > 0$, $y_2 > 0$, and $0 < \beta < \frac{1}{3}$, the two roots of (21) are negative. Hence it follows that the segments of the separatrices located near the point O_2 lie in regions II and IV (Figure 188).

Seeing that two separatrices go to the saddle point for $t \to +\infty$ and the other two for $t \to -\infty$, and examining the variation of x and y in regions II and IV, we readily find that the configuration of the separatrices near the saddle point O_2 can be represented as in Figure 188.

Consider the point of the separatrix L_4 lying near O_2. Let this point correspond to the time t_0. Since this point lies in region IV, it will move to the left and up as t increases (x decreases, y increases). But then the separatrix L_4 will not cross the parabola (20), which is an isocline of the horizontal inclinations, and it should evidently cross the straight line $1 + \beta x = 0$. We similarly conclude that the separatrix L_1 goes to infinity as t increases.

The behavior of the separatrices L_2 and L_3 cannot be determined unambiguously, and a detailed analysis is required. To this end, consider an auxiliary system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\varepsilon(x + y^2). \tag{B}$$

Dividing the second equation through by the first equation and substituting $y^2 = z$, we obtain a linear equation, which can be integrated to give a general integral of system (B),

$$F(x, y) = e^{2\varepsilon x}\left(-\frac{1}{2\varepsilon} + x + y^2\right) = C. \tag{22}$$

(B) has a single equilibrium state, the origin $O(0, 0)$ (it corresponds to $C = -\frac{1}{2\varepsilon}$ in (22)). Investigation of the curves (22) by the usual elementary methods shows that they can be depicted as in Figure 189. $C = -\frac{1}{2\varepsilon}$ corresponds to the equilibrium state $O(0, 0)$. The values $-\frac{1}{2\varepsilon} < C < 0$ correspond to closed paths of system (B): as C increases these closed paths expand. $C = 0$ corresponds to the parabola $-\frac{1}{2\varepsilon} + x + y^2 = 0$, which is obtained from the parabola $y + x^2 = 0$ as a result of a shift to the right by $\frac{1}{2\varepsilon}$ (in Figure 189, this displaced parabola is shown by dashed curve; it is not a path of system (B)). Paths which are not closed correspond to $C > 0$.

Let us derive the equation of the contact curve of systems (A) and (B). Taking the derivative of the function $F(x, y)$

$$\frac{dF(x, y)}{dt} = F'_x \frac{dx}{dt} + F'_y \frac{dy}{dt} = F'_x P(x, y) + F'_y Q(x, y)$$

(see QT, §3.13) and using (A), we obtain

$$\frac{dF}{dt} = -2\varepsilon e^{2\varepsilon x}(x + y^2)(1 - \beta). \tag{23}$$

Hence it follows that the contact curve $\frac{dF}{dt} = 0$ is the parabola $x + y^2 = 0$, which is an isocline of the horizontal inclinations both for system (A) and for system (B).

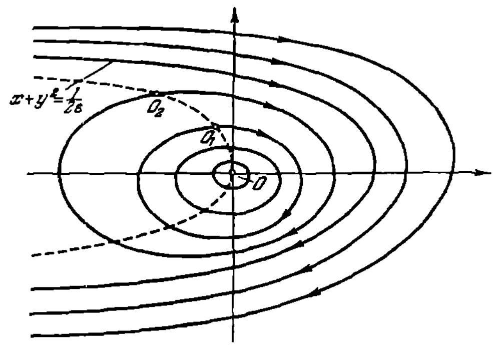

FIGURE 189

Since $\beta<\frac{1}{3}$, and $\varepsilon>0$, $\frac{dF}{dt}<0$ in regions where $x+y^2>0$, i.e., in regions II and III in Figure 188. This evidently implies that the paths of system (A) cross the closed paths of system (B) in the direction from outside to inside as t increases. In regions I and IV, on the other hand, where $x+y^2<0$, $\frac{dF}{dt}>0$ and the paths of system (A) cross the closed paths of (B) from inside to outside as t increases.

The axis x is an isocline of the vertical inclinations of (B). Each closed path of (B) crosses the positive half-axis x at a single point in the downward direction (Figure 189).

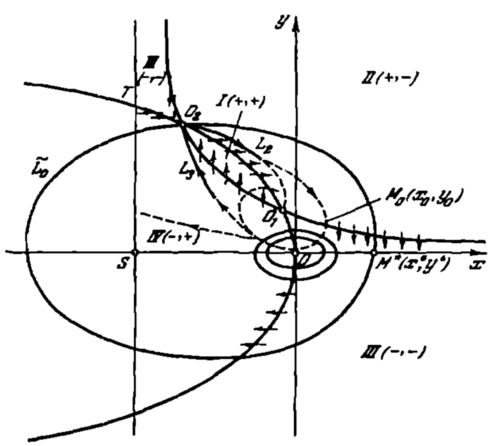

FIGURE 190. The dashed curve shows the possible behavior of the separatrix L_2.

Now consider the α-separatrix L_2 of system (A) which emerges from saddle point O_2 into II ($\dot{x}>0, \dot{y}<0$, see Figure 190). Since O_2 lies to the left

of the parabola $-\frac{1}{2\varepsilon}+x+y^2=0$, the path of system (B) passing through this point is a closed path (Figure 190). We designate this closed path $\tilde{L}_0$. At the point O_2, the separatrix L_2 enters into $\tilde{L}_0$, since at this point the inclination of L_2 is negative and the inclination of $\tilde{L}_0$ is horizontal (Figure 190; O_2 lies on the isocline of horizontal inclinations of system (B)). While in region where $x+y^2>0$, the separatrix L_2 cannot cross the curve $\tilde{L}_0$ going from inside to outside, since in this region its behavior is such that it crosses all the closed paths of (B) from outside to inside. Moreover, as t increases, the separatrix L_2 cannot move from region II into region I crossing the arc O_1O_2 of the parabola $y+x^2=0$, since in all the interior points of this arc the paths of system (A) are directed from left to right.

We should therefore consider the following alternatives for the separatrix L_2:

1) without leaving region II, L_2 goes to the equilibrium state O_1 for $t\to+\infty$;

2) remaining inside the curve C_0, L_2 crosses the isocline of the vertical inclinations (the hyperbola (19)) and then crosses the parabola (20) to enter region IV.

As t increases further, the following three possibilities should be considered for the separatrix L_2 (they are consistent with the fact that in regions I and IV the paths of (A) cross the closed paths of (B) from inside to outside):

2') L_2 goes to the equilibrium state O_1, if it is stable, or to a stable limit cycle surrounding this equilibrium state, if such a cycle exists;

2") L_2 goes to the saddle point O_2; in this case, it evidently merges with the separatrix L_3 forming a loop;

2"') L_2 crosses the straight line $1+\beta x=0$ (L_2 cannot go to infinity in the lower half-plane since in region IV $y>0$).

The behavior of the separatrix L_3, as is readily seen, is determined by the behavior of L_2.

Let us now consider the existence or otherwise of closed paths in system (A).

Every closed path of system (A) should enclose the point O_1, while leaving the point O_2 on the outside, and it may not have any points in common with other paths, in particular, not with the separatrices L_2 and L_3. Hence it follows that if the separatrix L_2 goes to the equilibrium state O_1, (A) has no closed paths. Now suppose that the separatrix L_2 crosses the hyperbola (19), and let x_0 be the abscissa of the intersection point M_0 (Figure 190). Every closed path of (A) of necessity crosses the segment O_1O_2 of the parabola (20). Making use of this fact and taking into consideration the direction of the field of system (A) in regions I, II, and IV and on the isoclines (19) and (20), we can readily show that if (A) has closed paths, they may only lie inside the region between the vertical lines $x=x_2$ and $x=x_0$.

Let $M^*(x^*, y^*)$ be the intersection point of the path $\tilde{L}_0$ of the auxiliary system (B) with the positive half-axis x. Since $x_0<x^*$ (this is clear from an examination of the behavior of L_2), we conclude from the above that *if system (A) has closed paths, they may only lie between the vertical lines $x=x_2$ and $x=x^*$.*

We will now apply Dulac's criterion (QT, §12.3). Dulac's function is taken in the form

$$\Phi(x, y)=\frac{e^{2\varepsilon x}}{1+\beta x}.$$

Computations show that

$$\frac{\partial}{\partial x}(\Phi P)+\frac{\partial}{\partial y}(\Phi Q)=\frac{-(1-\beta)e^{2\varepsilon x}}{(1+\beta x)^2}[2\varepsilon(1+\beta x)-\beta].$$

Since $\beta<\frac{1}{3}$, and $1+\beta x\neq 0$, the last expression vanishes only at the points of the straight line

$$x=\frac{\beta-2\varepsilon}{2\varepsilon\beta}.$$

By Dulac's criterion, (A) may not have closed paths in the region where $\frac{\partial}{\partial x}(\Phi P)+\frac{\partial}{\partial y}(\Phi Q)\neq 0$. Therefore, if the above straight line does not lie between the verticals $x=x_2$ and $x=x^*$, i.e., if either

$$\frac{\beta-2\varepsilon}{2\varepsilon\beta}\leqslant x_2=\frac{1}{2}-\frac{1}{\beta}+\sqrt{\frac{1}{\beta}-\frac{3}{4}}$$

or (24)

$$\frac{\beta-2\varepsilon}{2\varepsilon\beta}\geqslant x^*,$$

system (A) has no closed paths.

The first of the two conditions in (24) can be written in the form

$$\frac{1}{2\varepsilon}\leqslant\frac{1}{2}+\sqrt{\frac{1}{\beta}-\frac{3}{4}}.$$

It is satisfied for any β, $0<\beta<\frac{1}{3}$, if $\varepsilon>1$.

Let us now consider the second condition in (24). The equation of the curve $\widetilde{L}_0$ is given by (22), where C is determined from the condition that $\widetilde{L}_0$ passes through the point $O_2(x_2, y_2)$. The equation of $\widetilde{L}_0$ is thus

$$e^{2\varepsilon x}\left(x+y^2-\frac{1}{2\varepsilon}\right)=e^{2\varepsilon x_2}\left(x_2+y_2^2-\frac{1}{2\varepsilon}\right).$$

Setting $y=0$, we obtain an equation for x^*:

$$e^{2\varepsilon x^*}\left(x^*-\frac{1}{2\varepsilon}\right)=e^{2\varepsilon x_2}\left(x_2+y_2^2-\frac{1}{2\varepsilon}\right)$$

or after elementary manipulations,

$$1-2\varepsilon x^*=e^{2\varepsilon(x_2-x^*)}[1-2\varepsilon(x_2+y_2^2)]. \tag{25}$$

Since every curve (22) crosses the positive half-axis x at one point only (Figure 189), equation (25) has a single positive solution for x^*. Let

$2\varepsilon x^* = z$. Equation (25) then can be written in the form

$$1 - z - e^{-z} f(\varepsilon) = 0, \tag{26}$$

where $f(\varepsilon) = e^{2\varepsilon x_2}[1 - 2\varepsilon(x_2 + y_2^2)]$ (x_2 and y_2 are independent of ε).

For $\varepsilon = 0$, equation (26) has the form

$$1 - z - e^{-z} = 0.$$

It has a unique solution $z = 0$, and the derivative of its left-hand side at the point $z = 0$ does not vanish. Therefore, by Theorem 3 and the remark to Theorem 4 (the theorem of small increments of implicit functions, §1.2), for sufficiently small ε equation (26) has a unique solution close to zero, which goes to zero for $\varepsilon \to 0$. It is readily verified that for $\varepsilon > 0$ this solution is positive. And since equation (25) has a unique positive solution for x^*, we conclude that $2\varepsilon x^* \to 0$ for $\varepsilon \to 0$.

The second inequality in (24) can be written in the form

$$\beta - 2\varepsilon \geqslant 2\varepsilon x^* \cdot \beta.$$

This and the condition $\lim\limits_{\varepsilon \to 0} 2\varepsilon x^* = 0$ show that for any β, the second condition in (24) is satisfied as soon as $\varepsilon > 0$ is sufficiently small.

We have thus established that for every fixed β, $0 < \beta < \frac{1}{3}$, system (A) has no closed paths if ε is sufficiently small or sufficiently large. Let us now try to elucidate the topological structure of (A) in these cases.

Let

$$R = \frac{\beta}{2(1-\beta)} - \varepsilon. \tag{27}$$

As we have seen before, for $R > 0$, the equilibrium state $O_1(-1, 1)$ is an unstable focus or node, and for $R < 0$, it is a stable focus or node.

We will now show that for every β, $0 < \beta < \frac{1}{3}$, there exists a certain $\varepsilon > 0$ for which the separatrices L_2 and L_3 merge into a loop.

Indeed, if $\varepsilon = \varepsilon_1 > 0$ is sufficiently small, system (A) has no closed paths and, moreover, $R > 0$. But then O_1 is an unstable node or focus; the separatrix L_2 does not go to O_1 for $t \to +\infty$ and it either merges with the separatrix L_3 or leaves the region IV by crossing the segment ST of the line $x = -\frac{1}{\beta}$. In the former case, our proposition is proved, and in the latter case, the separatrices are arranged as shown schematically in Figure 191.

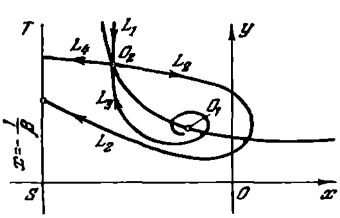

FIGURE 191. $\varepsilon = \varepsilon_1$.

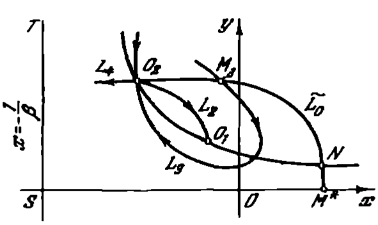

FIGURE 192. $\varepsilon = \varepsilon_2$.

We can similarly show that if $\varepsilon = \varepsilon_2 > 0$ is sufficiently large (in particular, so large that $R < 0$), the separatrices L_2 and L_3 either merge forming a loop or are arranged as shown in Figure 192.

Now suppose that the separatrices L_2 and L_3 do not merge either for $\varepsilon = \varepsilon_1$ or for $\varepsilon = \varepsilon_2$ (i.e., the configuration of the separatrices is according to Figures 191 and 192). We will show that in this case they merge for some $\varepsilon = \varepsilon_0$, $\varepsilon_1 < \varepsilon_0 < \varepsilon_2$. First note that the straight line $x = -\frac{1}{\beta}$ has no points of contact with the paths of the system (A) (this is evident directly from the equations of (A)). Moreover, since the contact curve of systems (A) and (B) is parabola (20), the arc O_2N of the path $\widetilde{L}_0$ of (B) extending between its intersection points O_2 and N with the parabola (20) (Figure 192) has no contacts with the paths of system (A) either. Therefore, by the basic properties of separatrices (§9.2, Lemma 3 and the remark to this lemma), if for some value of the parameter ε the separatrix L_2 crosses the line $x = -\frac{1}{\beta}$ (Figure 191) or the separatirx L_3 crosses the arc O_2N of the curve $\widetilde{L}_0$ (Figure 192), the behavior of these separatrices will not change for all close values of the parameters. Now suppose that the separatrices L_2 and L_3 do not merge for any ε, $\varepsilon_1 < \varepsilon < \varepsilon_2$. Let ε vary from ε_1 to ε_2. Then, as is readily seen, there exists some "first" value of the parameter $\varepsilon = \varepsilon^*$ for which the separatrix L_2 no longer crosses the line $x = -\frac{1}{\beta}$ (whereas for all ε, $\varepsilon_1 \leqslant \varepsilon < \varepsilon^*$, it does cross this line). Since by assumption the separatrices L_2 and L_3 do not form a loop, for $\varepsilon = \varepsilon^*$ the separatrix L_2 goes for $t \to +\infty$ either to O_1 or to a limit cycle encircling O_1. In either case, L_3 crosses the arc O_2N of $\widetilde{L}_0$. But then, for all close values of ε, and in particular for $\varepsilon < \varepsilon^*$, L_3 crosses the arc O_2N and consequently, L_2 may not cross the line $x = -\frac{1}{\beta}$. We have established a contradiction, which proves that for some ε_0, $\varepsilon_1 < \varepsilon_0 < \varepsilon_2$, the saddle point O_2 of (A) has a separatrix forming a loop.

Choose a fixed β and follow the changes in the topological structure of (A) in the neighborhood of the equilibrium state O_1 $(-1, 1)$ as ε is varied.

Let $\bar{\varepsilon} = \frac{\beta}{2(1-\beta)}$. For $\varepsilon = \bar{\varepsilon}$, $O_1(-1, 1)$ is an equilibrium state with pure imaginary characteristic numbers. Since at the point $O_1(-1, 1)$, $\sigma = \beta - 2\varepsilon(1-\beta)$, we conclude that for $\varepsilon = \bar{\varepsilon}$ and $\sigma = 0$, $\frac{\partial\sigma}{\partial\varepsilon} < 0$. α_3 is computed from equation (76), §24.4, which gives

$$\frac{\pi\sqrt{2}\beta^3(5-9\beta)}{8(1-3\beta)(1-\beta)^2\sqrt{\beta(1-3\beta)}} > 0.$$

Therefore, for $\varepsilon = \bar{\varepsilon}$, O_1 is an unstable focus, and when ε increases above $\bar{\varepsilon}$, the focus O_1 is transformed into a stable focus and a single unstable limit cycle is created in its neighborhood (see table in §25.3).

Thus, as ε varies from a sufficiently small ε_1 to a sufficiently large ε_2, the following bifurcations definitely occur:

1) at least for one $\varepsilon = \varepsilon_0$, the separatrices L_2 and L_3 of the saddle point O_2 merge forming a loop;

2) precisely once, for $\varepsilon = \bar{\varepsilon}$, O_1 is transformed into a multiple focus, and as ε is further increased, an unstable limit cycle is created from this focus.

In addition to these bifurcations, we may assume a priori the existence of bifurcations of still another type in this system:

3) appearance (or disappearance) of a limit cycle of multiplicity 4, and in particular of multiplicity 2, from a "condensation of paths."

Very considerable difficulties are encountered when one tries to establish the presence (or the absence) of bifurcations of this last type, and nothing certain can be said in our example regarding this point.

Let us consider one further fact relating to separatrix loops. As we know, the loop formed by a saddle-point separatrix is stable or unstable, according as the value of $P'_x + Q'_y$ at the saddle point is negative or positive (Theorem 44, §29.1). In our example, for $\varepsilon = \varepsilon_0$,

$$P'_x(x_2, y_2) + Q'_y(x_2, y_2) = y_2[\beta - 2\varepsilon_0(1 + \beta x_2)]. \tag{28}$$

But

$$y_2 = -\frac{1}{2} + \sqrt{\frac{1}{\beta} - \frac{3}{4}} \quad \text{and} \quad 1 + \beta x_2 = \frac{1-\beta}{y_2}.$$

Therefore

$$P'_x(x_2, y_2) + Q'_y(x_2, y_2) = y_2\left[\beta - \frac{2\varepsilon_0(1-\beta)}{y_2}\right]. \tag{29}$$

Let

$$\varepsilon_0 \leqslant \frac{\beta}{2(1-\beta)} = \bar{\varepsilon} \tag{30}$$

(ε_0 is the value of ε for which the separatrices L_2 and L_3 merge forming a loop; $\bar{\varepsilon}$ is the value for which O_1 is a multiple focus). For $0 < \beta < \frac{1}{3}$, it is readily seen that $y_2 > 1$. This and (29), (30) show that

$$P'_x(x_2, y_2) + Q'_y(x_2, y_2) = y_2\left[\beta - \frac{2\varepsilon_0(1-\beta)}{y_2}\right] > \beta - 2\frac{\beta(1-\beta)}{2(1-\beta)} = 0.$$

We have thus established that if the separatrices form a loop for $\varepsilon_0 \leqslant \bar{\varepsilon}$, the loop is unstable.

Let us now analyze the changes in the topological structure of (A) as ε increases from ε_1 to ε_2 assuming that two further conditions are satisfied:

(a) as ε varies from ε_1 to ε_2, no bifurcations of type 3 occur;

(b) there exists a single value ε_0 of the parameter ε at which the separatrices L_2 and L_3 merge forming a loop.

We should stress that the applicability of these conditions (or of either of them) is not known in advance, and we introduce them as an assumption in order to proceed with our analysis.

For $\varepsilon = \varepsilon_1$, the system has no closed paths and the separatrices are arranged as shown in Figure 191.

As ε increases, the topological structure of the system at first does not change. It may change only when the parameter crosses its bifurcation

value. The bifurcation values of the parameter in our case are $\bar{\varepsilon}$ and ε_0. We will show that $\bar{\varepsilon} < \varepsilon_0$. Indeed, if $\bar{\varepsilon} > \varepsilon_0$, O_1 is an unstable node or focus for $\varepsilon = \varepsilon_0$, and the system has a separatrix forming a loop, which loop, as we have seen before, is also unstable. By condition (a), no closed paths may develop in the system. Therefore, the paths enclosed inside the loop should go either to the loop or to the point O_1 for $t \to +\infty$, which is impossible since both the loop and the point O_1 are unstable.

We have thus established that $\bar{\varepsilon} < \varepsilon_0$. For $\varepsilon = \bar{\varepsilon}$, the point O_1 is a multiple unstable focus, and for all ε between ε_1 and $\bar{\varepsilon}$ the system has a constant topological structure (that shown on Figure 191).

As ε is further increased, O_1 is transformed into a stable focus, and a simple unstable limit cycle L_0 is created in its neighborhood (§25.3). The system acquires the topological structure schematically shown in Figure 193. As ε varies between the limits $\bar{\varepsilon} < \varepsilon < \varepsilon_0$, this topological structure is retained.

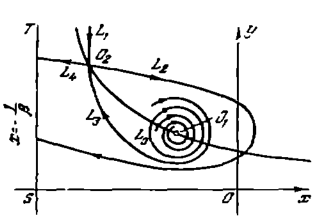

FIGURE 193. $\bar{\varepsilon} < \varepsilon < \varepsilon_0$.

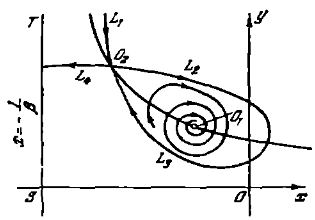

FIGURE 194. $\varepsilon = \varepsilon_0$.

For $\varepsilon = \varepsilon_0$, the separatrices of the saddle point O_2 form a loop. We will show that at this instant the limit cycle L_0 disappears (is "swallowed" by the separatrix loop), and the loop itself is unstable, i.e., the system has the structure shown in Figure 194.

Indeed, suppose that the unstable limit cycle L_0 exists for $\varepsilon = \varepsilon_0$. We know that for $\varepsilon = \varepsilon_2$ the system has no closed paths. Therefore the structurally stable limit cycle L_0 should disappear with the increase in ε. This may occur only by merging with some limit cycle created from the separatrix loop, i.e., as a result of a bifurcation of type 3, which is ruled out by condition (a). Thus, the limit cycle must disappear for $\varepsilon = \varepsilon_0$, and since the focus in this case is stable, the separatrix loop is unstable (Figure 194).

As ε is further increased, the separatrix loop breaks up. It does not produce closed paths, since eventually these paths will have to disappear, which is ruled out by condition (a). Therefore, for $\varepsilon > \varepsilon_0$, the topological structure of the system is the same as for $\varepsilon = \varepsilon_2$ (Figure 192).

We thus see that with the aid of conditions (a) and (b) we succeeded in deriving unambiguously the topological structure of the system for all $\varepsilon > 0$.

Note that if conditions (a) and (b) (or either of them) are not satisfied, the number of a priori possible topological structures of course markedly increases. Thus, if we allow bifurcations of type 3 — creation of a limit

cycle of multiplicity 2 (from a condensation of paths) or disappearance of a limit cycle — the topological structure may undergo the following changes:

1) A limit cycle of multiplicity 2 is created.

2) The limit cycle splits into two: an unstable outer cycle L' and a stable inner cycle L''.

3) The outer cycle L' is "swallowed" into the separatrix loop, and the separatrix breaks up.

4) The focus changes its stability, and an unstable limit cycle L''' is created.

5) The cycles L'' and L''' merge into a cycle of multiplicity 2, which then disappears.

There is nothing to prevent the separatrix loop and the type 3 bifurcation from occurring several times!

Example 16 (the creation of a limit cycle from a multiple focus, see /37/).

Consider the system

$$\frac{dx}{dt} = -[x + (x+\gamma)\,y + \delta] = P(x,\,y), \quad \frac{dy}{dt} = -y + (x+\gamma)\,x = Q(x,\,y) \tag{31}$$

for positive values of the parameters γ and δ.

The coordinates of the equilibrium states satisfy the equations

$$x + (x+\gamma)\,y + \delta = 0, \quad y - (x+\gamma)\,x = 0. \tag{32}$$

Eliminating y between these equations, we obtain a single equation for the abscissas of the equilibrium states:

$$F(x,\,\gamma,\,\delta) \equiv x^3 + 2\gamma x^2 + (\gamma^2+1)\,x + \delta = 0. \tag{33}$$

Since this is a cubic equation, (31) has at least one equilibrium state and at most three equilibrium states for all the relevant values of the parameters γ and δ.

To determine the exact number of equilibrium states, we will follow the general instructions outlined in the Introduction to this chapter. Specifically, we will look for those values of the parameters for which the equilibrium states have maximum multiplicity. As we shall see, local analysis near these values of the parameters will solve the problem of the existence of regions with different number of equilibrium states.

The ordinate y of the equilibrium states is obtained from the equation $y = x\,(x+\gamma)$, i.e., it is a single-valued function of the abscissa. Therefore, by Definition 15, §7.3, the multiplicity of the equilibrium state $(x,\,y)$, where x is a root of equation (33), is equal to the multiplicity of the corresponding root, i.e., the maximum multiplicity of the equilibrium state of our system may not exceed 3. Let us try to establish the values of the parameters γ and δ when equation (33) has a triple root. As we know, a triple root should simultaneously satisfy the three equations

$$\begin{aligned} &F(x,\,\gamma,\,\delta) = x^3 + 2\gamma x^2 + (\gamma^2+1)\,x + \delta = 0,\\ &F'_x(x,\,\gamma,\,\delta) = 3x^2 + 4\gamma x + (1+\gamma^2) = 0,\\ &F''_{x^2}(x,\,\gamma,\,\delta) = 6x + 4\gamma = 0. \end{aligned} \tag{34}$$

From the last equation, we have $x = -\frac{2}{3}\gamma$.

Inserting this result in the second equation and remembering that we are interested only in positive values of the parameters, we find $\gamma = +\sqrt{3}$. Hence $x = -\frac{2\sqrt{3}}{3}$, and $\delta = \frac{8}{9}\sqrt{3} > 0$.

We have thus established the existence of a single pair of positive parameters γ and δ, namely $\gamma_0 = \sqrt{3}$, $\delta_0 = \frac{8}{9}\sqrt{3}$, for which system (31) has an equilibrium state of multiplicity 3. Let this equilibrium state be $M_0(x_0, y_0)$, where $x_0 = -\frac{2\sqrt{3}}{3}$, $y_0 = -\frac{2}{3}$. For these values of the parameters, the system evidently does not have other equilibrium states.

Since $M_0(x_0, y_0)$ is a multiple equilibrium state, $\Delta(x_0, y_0) = 0$ (this also can be found by a direct computation). Furthermore,

$$\sigma(x_0, y_0) = P'_x(x_0, y_0) + Q'_y(x_0, y_0) = -2 - y_0 = -\frac{4}{3} < 0.$$

Investigating the equilibrium state M_0 by the methods described in QT (Chapter IX, §21.2, Theorem 65), we conclude that M_0 is a stable topological node. By Theorem 35, §23.2, there exist small increments for which the multiple equilibrium state M_0 decomposes into three structurally stable equilibrium states (two nodes and one saddle point) and also increments which replace M_0 with a single (stable) structurally stable node. However, the results of §23 cannot be applied to our case, since we are not dealing with just any increment to the right-hand sides of the corresponding system, but only with increments which result from changes in the particular parameters γ and δ of the system.

To investigate the possible existence of three equilibrium states, let us consider the function $F(x, \gamma, \delta)$ (see (33)) in the neighborhood of the point x_0, γ_0, δ_0. Let $x = x_0 + \xi$, $\gamma = \gamma_0 + h$, $\delta = \delta_0 + k$.

Seeing that x_0, γ_0, δ_0 satisfy equations (34), we readily find that

$$\begin{aligned} F(x, \gamma, \delta) = F(x_0 + \xi, \gamma_0 + h, \delta_0 + k) = \\ = \xi^3 + 2h\xi^2 + (4x_0h + 2\gamma_0 h + h^2)\xi + 2x_0^2 h + 2\gamma_0 x_0 h + \\ + x_0 h^2 + k = \xi^3 + A\xi^2 + B\xi + C, \end{aligned}$$

where $A = 2h$, etc.

Let $k = -2x_0^2 h - 2\gamma_0 x_0 h - x_0 h^2$, i.e., $C = 0$. The equation $F(x, \gamma, \delta) = 0$ then takes the form $\xi(\xi^2 + A\xi + B) = 0$. One of its roots is $\xi_1 = 0$, and the character of the other roots depends on the value of the discriminant $A^2 - 4B = -16x_0 h - 8\gamma_0 h = -\frac{56}{3}\sqrt{3}h$. Therefore, if $h < 0$, and k is selected as above, the equation $F(x, \gamma, \delta) = 0$ has three different real roots, one of which is x_0, and if $h > 0$, the equation has a single real root x_0 and two complex roots. Clearly, if h is sufficiently small in absolute value, k is also sufficiently small and the parameters $\gamma = \gamma_0 + h$, $\delta = \delta_0 + k$ are positive.

We have thus established that for certain positive values of the parameters γ and δ, system (31) has three equilibrium states, whereas for other values of the parameters it only has one equilibrium state.*

* Our proof is confined to the neighborhood of a triple equilibrium state. In this case, the sought results also can be obtained from a direct analysis of the cubic equation $F(x, \gamma, \delta) = 0$.

On crossing over from one region of parameter values to the other region, we evidently pass through the bifurcation values of the parameters for which the system has a multiple (double or triple) equilibrium state. Since there exists only one pair of values γ_0, δ_0 for which the system has a triple equilibrium state, whereas the values of the parameters for which the system has one structurally stable or three structurally stable equilibrium states fill whole regions in the parameter plane, we should inevitably pass through the parameter values corresponding to the double equilibrium state on crossing over from one of these regions (the region with three equilibrium states, say) into the other (that with one equilibrium state). The equation of the curve in the parameter plane corresponding to systems with multiple equilibrium states can be found without difficulty. To this end, we equate to zero the discriminant of the cubic equation (33). After some manipulations, we obtain the equation

$$\varphi(\gamma, \delta) = [27\delta - 2\gamma(\gamma^2 + 9)]^2 - 4(\gamma^2 - 3)^3 = 0, \tag{35}$$

which is equivalent to two equations

$$\delta = \frac{2\gamma(\gamma^2+9) + 2(\gamma^2-3)\sqrt{\gamma^2-3}}{27} = h(\gamma) \tag{36}$$

and

$$\delta = \frac{2\gamma(\gamma^2+9) - 2(\gamma^2-3)\sqrt{\gamma^2-3}}{27} = f(\gamma). \tag{37}$$

We are clearly interested only in positive values of the parameter γ which are greater than or equal to $\sqrt{3}$. For $\gamma = \gamma_0 = \sqrt{3}$, $h(\gamma_0) = f(\gamma_0) = \frac{8}{9}\sqrt{3} = \delta_0$, i.e., we obtain the point $M_0(\gamma_0, \delta_0)$ corresponding to a system with a triple equilibrium state. The curve (35) consisting of two branches (36) and (37) meeting at a common point M_0 is shown in Figure 195.

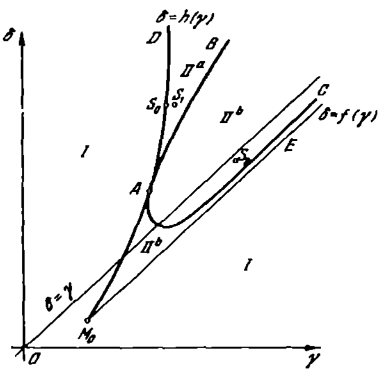

FIGURE 195

The curve 35 partitions the first quadrant of the parameter plane into two regions I and II (Figure 195). It is readily seen that the dynamic

systems corresponding to the points of region I ($\varphi(\gamma, \delta) > 0$) have one (structurally stable) equilibrium state, whereas systems corresponding to the points of region II ($\varphi(\gamma, \delta) < 0$) have three, also structurally stable, equilibrium states. Systems corresponding to the points of the curve (35) other than M_0 have one structurally stable and one double equilibrium state.

Let us now consider the character of the equilibrium states of system (31). The first question is the possible existence of equilibrium states with pure imaginary characteristic numbers (a center or a multiple focus from which a limit cycle may be created). It should be established whether such equilibrium states exist or not, and if they do, then for what values of the parameters. For such an equilibrium state,

$$\Delta = \begin{vmatrix} P'_x & P'_y \\ Q'_x & Q'_y \end{vmatrix} > 0,$$

and $\sigma = P'_x + Q'_y = 0$. For an equilibrium state (x, y) of system (31), as is readily seen,

$$\Delta = 3x^2 - 4x\gamma + \gamma^2 + 1, \tag{38}$$

$$\sigma = -(x^2 + \gamma x + 2). \tag{39}$$

Let us first establish whether an equilibrium state can exist at all for which $\Delta = 0$ and $\sigma = 0$ simultaneously.* The abscissa of an equilibrium state of system (31) for which $\sigma = 0$ simultaneously satisfies equation (33) and the equation

$$\sigma = -(x^2 + \gamma x + 2) = 0. \tag{40}$$

Eliminating γ between these equations, we obtain

$$\psi(\gamma, \delta) = 2\gamma^2 - 3\gamma\delta + \delta^2 + 2 = 0. \tag{41}$$

In the plane of the parameters γ, δ equation (41) is a curve (a hyperbola), and only dynamic systems corresponding to the points of this hyperbola may have equilibrium states with $\sigma = 0$.

Let us now determine the values of the parameters for which the dynamic system may have an equilibrium state (x, y) with $\Delta = \sigma = 0$.

From the equations

$$\Delta = 3x^2 + 4x\gamma + \gamma^2 + 1 = 0, \qquad \sigma = -(x^2 + \gamma x + 2) = 0$$

we readily find $x = \frac{5-\gamma^2}{\gamma}$. Inserting this result in the second equation, we find $\gamma = \frac{5}{\sqrt{3}}$. Hence $x = -\frac{2}{\sqrt{3}}$ and, by (33), $\delta = \frac{8}{\sqrt{3}}$.

There is thus a single point $A(\gamma_1, \delta_1)$ in the parameter plane, with $\gamma_1 = \frac{5}{\sqrt{3}}$, $\delta_1 = \frac{8}{\sqrt{3}}$, which corresponds to a system having an equilibrium state with $\Delta = 0$, $\sigma = 0$. Since the condition $\Delta = 0$ is equivalent to the existence of a multiple equilibrium state of a dynamic system, i.e., the existence of a multiple root of equation (33), the point $A(\gamma_1, \delta_1)$ lies on the

* We are thus considering, as before, equilibrium states of maximum multiplicity.

curve $\varphi(\gamma, \delta) = 0$ (see (35)) and is the only common point of this curve and the curve $\psi(\gamma, \delta) = 0$. It is readily seen that the curves $\varphi(\gamma, \delta) = 0$ and $\psi(\gamma, \delta) = 0$ touch at the point A. Moreover, it can be shown that the branch $\psi(\gamma, \delta) = 0$ of the hyperbola located in the first quadrant is entirely contained in region II (Figure 195). Therefore, the stability of an equilibrium state with $\Delta \neq 0$ can change only for those values of the parameters when system (31) has three equilibrium states.

Let us try to identify the equilibrium states with $\sigma = 0$. Since $\sigma = P'_x + Q'_y = -(y+2)$, an equilibrium state (x, y) with $\sigma = 0$ has $y = -2$. The abscissa is $x = \delta - 2\gamma$ in virtue of the first equation in (31). Moving the origin in the plane (x, y) to coincide with this equilibrium state $(\delta - 2\gamma, -2)$, i.e., using the substitution of variables

$$x = X + \delta - 2\gamma, \quad y = Y - 2,$$

we obtain a system

$$\frac{dX}{dt} = X + (\gamma - \delta) Y - XY, \quad \frac{dY}{dt} = (2\delta - 3\gamma) X - Y + X^2 + 2\gamma^2 + \delta^2 - 3\gamma\delta + 2.$$

In virtue of the condition $\sigma = 0$, relation (41) is satisfied, and the last system takes the form

$$\frac{dX}{dt} = X + (\gamma - \delta) Y - XY, \quad \frac{dY}{dt} = (2\delta - 3\gamma) X - Y + X^2. \tag{42}$$

The equilibrium state (with $\sigma = 0$) now has the coordinates $(0, 0)$. Its characteristic equation is

$$\begin{vmatrix} 1-\lambda & \gamma - \delta \\ 2\delta - 3\gamma & -1-\lambda \end{vmatrix} = \lambda^2 - [(\gamma - \delta)(2\delta - 3\gamma) + 1] = 0. \tag{43}$$

Depending on the sign of the expression in brackets, the characteristic roots are either complex conjugate numbers or real numbers of opposite sign.

It is readily seen that the curves

$$\left.\begin{aligned} \psi(\gamma, \delta) = 2\gamma^2 - 3\gamma\delta + \delta^2 + 2 = 0 \\ (\gamma - \delta)(2\delta - 3\gamma) + 1 = 0 \end{aligned}\right\} \tag{41}$$

and

have a single common point in the first quadrant, namely the point $A\left(\frac{5}{\sqrt{3}}, \frac{8}{\sqrt{3}}\right)$, and that this is their point of intersection (and not a true point of contact). The expression $(\gamma - \delta)(2\delta - 3\gamma) + 1$ therefore retains the same sign everywhere along the branch AB of the curve (41) (except the point A) and an opposite sign everywhere along the branch AC of this curve.

Consider the points $(3, 5)$ and $(9/2, 5)$ of curve (41); we readily find that the expression $(\gamma - \delta)(2\delta - 3\gamma) + 1$ is negative on the branch AB and positive on the branch AC. Therefore, by (43), if the dynamic system (31) corresponds to a point on the branch AB, its equilibrium state with $\sigma = 0$ is a multiple focus or center (we shall see later on that this is a multiple focus of multiplicity 1). If the system corresponds to a point on the branch AC, its equilibrium state with $\sigma = 0$ is a saddle point.

The branch AB of the hyperbola (41) partitions the region II into two subregions II^a and II^b (Figure 195).

Let us consider the character of the equilibrium states of systems corresponding to the points of each of these regions. First note that for any values of the parameters γ, δ in (31), infinity is absolutely unstable. Indeed, along a path of system (31)

$$\frac{d(x^2+y^2)}{dt} = 2\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = -2\,[x^2+y^2+\delta x].$$

It is readily seen that for $x^2 + y^2 > \delta^2$ the last expression is negative. Therefore all paths of the system enter into a circle of a sufficiently large radius centered at the origin as t increases.

Hence it follows, in particular, that the sum of the Poincaré indices of all the equilibrium states is + 1.

Systems corresponding to points of region I, as we have seen before, have one equilibrium state, which is moreover structurally stable. Since its Poincaré index is 1, it is either a node or a focus. It is readily seen – the infinity being absolutely unstable – that this node (or focus) is stable.

In region II, the system has three simple equilibrium states whose Poincaré indices add up to 1. Therefore one of these states is a saddle point, and each of the other two is either a node or a focus. It can be shown (we omit the proof here) that the nodes or the foci of a system corresponding to a point inside region II^b are stable. On the other hand, the points of the branch AB of curve (41) represent systems with three equilibrium states, one of which is a focus with $\sigma = 0$.

When we cross over from II^b into II^a across the branch AB, σ reverses its sign and one of the foci consequently becomes unstable. The points of region II^a therefore represent systems which have one stable and one unstable focus (or node) and one saddle point.

As we have established before, the points of the curve (35) other than M_0 correspond to systems which have one simple and one double equilibrium state. On crossing over into region I, the double equilibrium state vanishes, and on crossing over into region II it decomposes into two simple equilibrium states.

The point M_0 corresponds to a system with a triple equilibrium state.

The points of the branch AB of curve (41) represent systems with one multiple focus, and the points of the branch AC represent systems which have a saddle point with $\sigma = 0$. Finally, the point A corresponds to a double equilibrium state with $\sigma = 0$.

Let us now elucidate the position of the equilibrium states in the phase plane.

To this end, we consider the isoclines of horizontal and vertical inclinations. The isocline of the horizontal inclinations $Q(x, y) = 0$ is the parabola

$$y = x(x+\gamma), \tag{44}$$

and the isocline of the vertical inclinations $P(x, y) = 0$ is

$$y(x+\gamma) + x + \delta = 0. \tag{45}$$

For $\delta = \gamma$, this isocline decomposes into two straight lines

$$x = -\gamma \quad \text{and} \quad y = -1. \tag{46}$$

If $\gamma \neq \delta$, (45) is a hyperbola and, for every δ, the lines (46) are its asymptotes.

Figure 196 shows the isocline $Q(x, y) = 0$ and the family of the isoclines $P(x, y) = 0$ for a fixed $\gamma = \gamma_0 > \sqrt{3}$.*

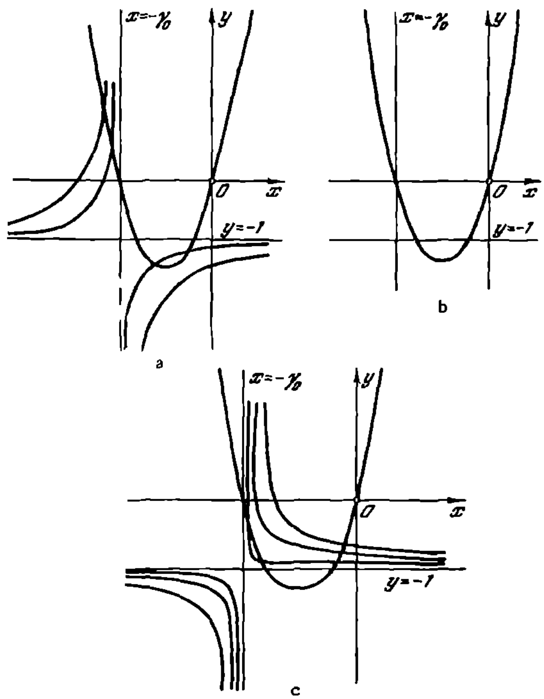

FIGURE 196. a) $\delta > \gamma_0$; b) $\delta = \gamma_0$; c) $\delta < \gamma_0$.

On the isocline of the horizontal inclinations — the parabola $y = (x + \gamma)x$ — nodes and foci alternate with saddle points (by the Poincaré theorem, §23.3, Theorem 36). Since out of the possible three equilibrium states, two are nodes or foci and one is a saddle point, the two extreme equilibrium states on the parabola are nodes or foci, and the middle state is a saddle point.

Let us now establish which of the two equilibrium states that are not a saddle point changes its stability on crossing over from II^b into II^a, i.e., which of the states has $\sigma = 0$. Since $\sigma = -y - 2$, $y + 2$ cannot retain a constant sign for this equilibrium state. Consider the equilibrium state O_1 with the least abscissa. It is readily seen that its ordinate y_1 is always greater than -1, since O_1 is the intersection point of the parabola with the branch of the hyperbola extending above the asymptote $y = -1$ (Figure 196). Thus, $y_1 + 2 > 0$, i.e., only the focus with the largest abscissa may change its stability.

* If $\gamma \leqslant 3$, $\varphi(\gamma, \delta) > 0$ (see (35)) and system (35) has one equilibrium state for all δ.

Let us now consider the existence of limit cycles for system (31).

We will first show that on passing from branch AB of curve (41) into region II^b, the multiple focus will create precisely one limit cycle, which will be unstable.

Using the same substitution of variables as before, i.e., displacing the origin to coincide with the multiple focus, we obtain

$$\frac{dX}{dt}=X+(\gamma-\delta)Y-XY, \quad \frac{dY}{dt}=(2\delta-3\gamma)X-Y+X^2. \tag{42}$$

To determine the character of the equilibrium state, let us find the value of α_3 (see §24.4, (76)). Computations show that the sign of α_3 coincides with the sign of the ratio

$$\frac{3\delta-4\gamma}{2\delta-3\gamma}. \tag{47}$$

We thus have to determine the sign of this expression at the points of the curve AB. This can be done as follows. Consider an auxiliary straight line

$$5\delta-8\gamma=0, \tag{48}$$

which joins the origin with the point $A(5/\sqrt{3}, 8/\sqrt{3})$. It is readily seen that the branch AB of curve (41) and the half-lines $3\delta-4\gamma=0$ and $2\delta-3\gamma=0$ located in the first quadrant lie on the two sides of the line (48). Expression (47) thus has the same sign on the branch AB and on the straight line (48). On the line (48) this expression is positive. Hence, $\alpha_3>0$. Then, according to the table at the end of §25.3, the multiple focus is unstable and its multiplicity is 1; on crossing over into region II^b, it becomes stable and creates a single unstable limit cycle.

Let us now isolate certain regions in the parameter plane which correspond to systems without closed paths (in particular, without limit cycles).

Consider the straight line $x+\gamma=0$. We see from the first equation in (31) that for $\delta=\gamma$ this line is an integral curve. If $\delta\neq\gamma$, it is a line without contact. Indeed, for $x=-\gamma$ and $\delta\neq\gamma$, $\frac{dx}{dt}=\delta-\gamma\neq 0$, i.e., the paths do not touch this line.

Take Dulac's function in the form

$$\Phi(x, y)=\frac{1}{x+\gamma}. \tag{49}$$

This gives

$$D(x, y)=\frac{\partial(\Phi P)}{\partial x}+\frac{\partial(\Phi Q)}{\partial y}=\frac{-x+\delta-2\gamma}{(x+\gamma)^2}. \tag{50}$$

This expression reverses its sign on the straight line

$$x=\delta-2\gamma. \tag{51}$$

Hence, by Dulac's criterion, it follows that every closed path of system (31) crosses the line (51).

For $\delta=\gamma$ the line (51) coincides with the line $x+\gamma=0$, which is an integral curve. Therefore, in this case the system has no closed paths.

For $\delta < \gamma$, the line (51) passes to the left of the line $x + \gamma = 0$, and all the equilibrium states of the system lie to the right of this line (this can be checked if we remember that for $\delta < \gamma$, the isocline $Q(x, y) = 0$ (the parabola) may only intersect the right branch of the isocline $P(x, y) = 0$ (the hyperbola), i.e., the branch which extends to the right of the line $x + \gamma = 0$; see Figure 196c). Every closed path of the system should cross the straight line (51) in virtue of Dulac's criterion. On the other hand, every closed path should enclose at least one equilibrium state. But then the closed path should intersect the line without contact $x + \gamma = 0$ at least at two points, which is impossible.

We have thus established that for $\delta \leqslant \gamma$ the system has no closed paths.

We will now show that systems corresponding to the points of region I (i.e., systems with one equilibrium state) do not have closed paths either. For $\delta \leqslant \gamma$, this has been proved above. Let $\delta > \gamma$. In this case, the equilibrium state of the system is the intersection point of the parabola $y = x(x + \gamma)$ with the branch of the hyperbola $P(x, y) = 0$ which extends to the left of the line $x + \gamma = 0$ (see Figure 196a), while the line (51) passes to the right of the line $x + \gamma = 0$. Therefore, if there exists a closed path, it should cross the integral curve $x + \gamma = 0$, which is impossible.

Let us now consider the systems corresponding to the points of the line (35), $\varphi(\gamma, \delta) = 0$, i.e., the systems with multiple equilibrium states. The line (35) consists of two branches (36) and (37) (Figure 195). It is readily seen that $\delta < \gamma$ on the branch (37) (and in particular at the point M_0). Therefore systems corresponding to the points of this branch have no closed paths. Systems corresponding to the points of the branch (36) which lie in the region $\delta > \gamma$ have one structurally stable state — a stable node or focus — and one double equilibrium state with zero Poincaré index. It is readily seen (see Figure 196a) that the line without contact $x + \gamma = 0$ passes between the focus (or the node) and the line (51) in this case. Therefore, by Dulac's criterion, no closed path enclosing a focus may exist. But then the closed path should enclose the double equilibrium state, and this is impossible because its Poincaré index is zero.

Multiple equilibrium states corresponding to the points of curve (35) can be investigated using the results of QT, §21 and §22. The point A (Figure 195) is found to represent a system with a degenerate equilibrium state, and point M_0 a system with a stable topological node of multiplicity 3 (we have indicated this before). All the other points of the curve $\varphi(x, y) = 0$ represent systems for which the double equilibrium state is a saddle-node.

It is readily seen that a system with a saddle-node cannot have paths forming a loop which goes to the saddle-node both for $t \to -\infty$ and $t \to +\infty$. This is proved in the same way as the absence of closed paths.

Now consider some point S_0 on the segment AD of curve (35) (Figure 195). The dynamic system corresponding to this point has no closed paths and no paths forming a loop, as we have seen just now. Moreover, this system has no multiple foci ($\sigma \neq 0$, since S_0 does not coincide with the point A). But then we can show that dynamic systems corresponding to points sufficiently close to S_0 do not have closed paths either (there is "nowhere" these paths can be "created" from; the rigorous proof of this proposition is left to the reader). In particular, systems corresponding to the points sufficiently close to S_0 in region II, i.e., systems with three equilibrium states, have no limit cycles and no closed contours consisting of paths.

Consider the point S_1 which is sufficiently close to S_0 in II^a and point S_2 which lies in II^b below the straight line $\gamma = \delta$. Let $\bar{S}_1$ and $\bar{S}_2$ be the dynamic systems corresponding to these points. Since neither system has closed paths and closed contours consisting of paths, their topological structure is determined unambiguously. Let us prove this fact.

I. System $\tilde{S}_1$. It has one stable node or focus, one unstable node or focus, and a saddle point. Since the infinity is unstable, the two α-separatrices of the saddle point go to a stable focus.* The unstable focus (or node) clearly lies inside the closed curve formed by the α-separatrices, one of the ω-separatrices of the saddle point goes to this unstable focus for $t \to -\infty$, and the other goes to infinity. The corresponding topological structure is shown in Figure 197.

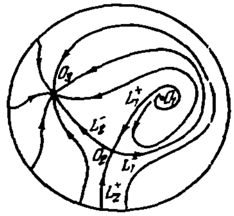

FIGURE 197

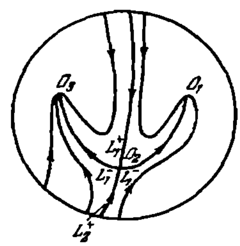

FIGURE 198

II. System $\tilde{S}_2$. It has two stable nodes or foci. The two ω-separatrices of the saddle point go to infinity for $t \to -\infty$ and separate the α-separatrices, which go to the foci for $t \to +\infty$. The resulting topological structure is shown in Figure 198.

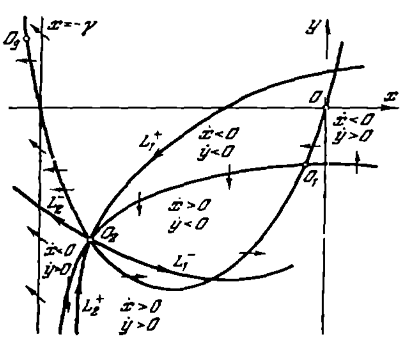

FIGURE 199

Now consider the points of region II which are not close to S_0, but for which we nevertheless have $\delta > \gamma$. To establish the possible configurations of the separatrices of the systems corresponding to these points, consider the isoclines of the horizontal and vertical inclinations, i.e., curves (32). These isoclines partition the plane into regions in which $\dot{x}$ and $\dot{y}$ retain a constant sign (Figure 199).

Since $\delta > \gamma$, one of the equilibrium states (O_3 on Figure 199) lies to the left of the line $x = -\gamma_0$, and the other two equilibrium states, O_2 and O_1, lie to the right of this line (Figure 196a). The equilibrium state O_2 is a saddle point, and O_3 is a stable focus or node. It is

* They form a closed line consisting of paths, but these paths are not a continuation of one another. When we said above that the system had no closed contours consisting of paths, we meant contours of the type of limit continua.

readily seen that the four separatrices of the saddle point O_2 enter — as t increases or decreases — into four of the previously identified regions adjoining the point O_2 (this can be shown by the same technique as that used to classify the saddle point; see QT, §7.3). The following properties can also be established without difficulty:

The separatrix located near the point O_2 in the region where $\dot{x} < 0$, $\dot{y} < 0$ is an ω-separatrix. We designate it L_1^+. As t decreases, it either crosses the isocline of horizontal inclinations and leaves the region where $\dot{x} < 0$, $\dot{y} < 0$, or it does not leave this region, going to infinity instead (i.e., it crosses the cycle without contact of the system).

The separatrix L_1^- issuing from the saddle point O_2 into the region where $\dot{x} > 0$, $\dot{y} < 0$ crosses the isocline $y = x(x + \gamma)$ with increasing t and enters into the region where $\dot{x} > 0$, $\dot{y} > 0$ (Figure 199). Indeed, if the separatrix L_1^- does not leave the region where $\dot{x} > 0$, $\dot{y} < 0$ across the parabola, it should go to the equilibrium state O_1. This is impossible, however, since the saddle point O_2 is located below the point O_1, and in the relevant region $\dot{y} < 0$, i.e., y decreases along L_1^-.

We are dealing with the case when $\delta > \gamma$, i.e., when the straight line (51), $x = \delta - 2\gamma$, passes to the right of the line without contact $x = -\gamma$. In this case, a closed path, if it exists, may not enclose the equilibrium state O_3, and it only may enclose the focus (or the node) O_1.

It follows from the above that three configurations of separatrices are a priori possible for the systems being considered (i.e., systems with three equilibrium states and $\delta > \gamma$):

1) The separatrix L_1^+ goes to infinity across a cycle without contact, and the separatrix L_1^- either goes to a stable node or focus O_1 (as for $\delta < \gamma$), or goes to a stable or semistable limit cycle encircling the equilibrium state O_1 (Figure 198).

2) The separatrix L_1^- goes to a stable node or focus O_3, forming a closed curve together with the separatrix L_2^-, which encloses the separatrix L_1^+. The separatrix L_1^+ goes to an unstable node or focus O_1 or to an unstable (or semistable) cycle encircling the equilibrium state O_1 (Figure 197).

3) The separatrix L_1^- merges with the separatrix L_1^+ to form a loop (Figure 204).

Arguing as before, we can show that if configuration 1 (configuration 2) is observed for certain values of the parameters, the same configuration is retained for all close values of the parameters.

As we have seen before, however, both configuration 1 and configuration 2 are possible for different values of the parameters.

Thus, reasoning precisely as the previous example, we will establish the existence of certain values of the parameters for which the separatrix extends between saddle points. On every continuous line joining a point of the type S_1 with a point of the type S_2 there evidently exists a point S_3 which represents a system with a saddle-to-saddle separatrix. Then it is clear that region II contains at least one continuous line Γ which passes between the straight line $\delta = \gamma$ and the branch AB of curve (41), whose points correspond to dynamic systems with a saddle-to-saddle separatrix.

We can now investigate the topological structure assuming, as before, that no closed paths form from path condensations and that all the points corresponding to systems with a separatrix loop form a non-closed line Γ (extending between the branch AB of curve (41) and the line $\delta = \gamma$ (Figure 200)).

In region I, the system has a single equilibrium state — a stable node or focus — and its topological structure is as shown in Figure 201.

Systems corresponding to the points of the segment AM_0E of the line $\varphi = 0$ other than the point M_0 have one stable node or focus and a saddle-node with a stable node region.

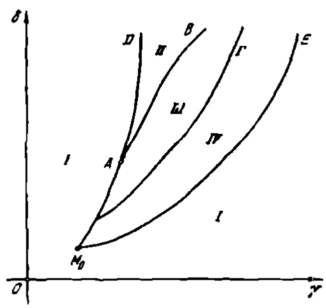

FIGURE 200.

Systems corresponding to the points of the branch AD of the curve $\varphi = 0$ (other than the point A) have a stable node or focus and a saddle-node with an unstable node region. Their topological structure is shown in Figure 202.

On passing from branch AD to region II, we obtain systems with one stable and one unstable node or focus, whose topological structure is shown in Figure 197.

The topological structure of the systems on the line AD is analogous to the structure depicted in Figure 197, but the point O_1 is a multiple unstable focus of multiplicity 1.

In region III (Figure 200), the focus changes its stability (i.e., it becomes stable), and a single stable limit cycle is created from the focus. The corresponding topological structure is shown in Figure 203.

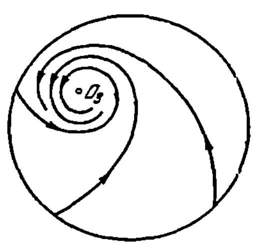

FIGURE 201

FIGURE 202

On Γ, the limit cycle vanishes, after "being swallowed up" by the loop of the separatrix which originates and terminates in the saddle point O_2. The topological structure of the system is shown in Figure 204.

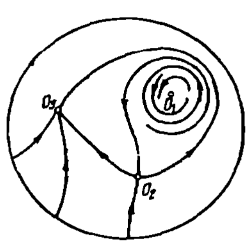

FIGURE 203.

In region IV (Figure 200), the separatrix loop breaks up and the system acquires the topological structure shown in Figure 198. The point M_0 corresponds to a dynamic system with a triple equilibrium state. It is a stable node, and the topological structure of the system is the same as in Figure 201. Finally, the dynamic system corresponding to the point A has a stable focus and a degenerate equilibrium state. Its topological structure is shown in Figure 205.

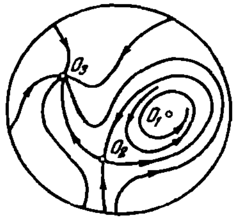

FIGURE 204.

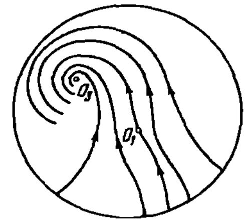

FIGURE 205.

We should emphasize that the preceding analysis was carried out using fairly arbitrary simplifying assumptions, whose validity is by no means certain.

Example 17 (the creation of a limit cycle from a closed path of a conservative system).

Consider a system

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x + \mu(\alpha + \beta x - \gamma x^2) y, \tag{52}$$

which arises, in particular, in connection with a tube generator operating in the "soft mode" (see /6/, p. 703). Only positive values of the parameters α, β, γ are meaningful, and we will therefore take in our analysis $\alpha > 0$, $\beta > 0$, $\gamma > 0$.

System (52) may be considered for small μ as being close to the Hamiltonian system

$$\frac{dx}{dt} = -y = -\frac{\partial H}{\partial y}, \quad \frac{dy}{pt} = x = \frac{\partial H}{\partial x}, \tag{53}$$

where $H(x, y) = \frac{1}{2}(x^2 + y^2)$. The paths of system (53) are circles centered at the origin.

Since (53) is a linear conservative system, we may investigate system (52) using Theorem 75 (§33.2), or the more general Theorem 78. We will use Theorem 78 here, as it also provides an indication of the stability of the created limit cycle.

Let

$$x = \rho_1 \cos t, \quad y = \rho_1 \sin t, \tag{54}$$

where $\rho_1 > 0$, be a closed path of system (53), and G_0 the circle enclosed within this path. Let us evaluate the integral

$$\iint_{G_0} [p'_{1x}(x, y) + q'_{1y}(x, y)]\, dx\, dy, \quad \text{where} \quad p_1 \equiv 0, \ q_1 = (\alpha + \beta x - \gamma x^2) y$$

(see (B_μ) in the statement of Theorem 78). We have

$$\iint_{G_0} [p'_{1x}(x, y) + q'_{1y}(x, y)]\, dx\, dy = \iint_{G_0} (\alpha + \beta x - \gamma x^2)\, dx\, dy =$$
$$= \int_0^{2\pi} \int_0^{\rho_1} (\alpha + \beta\rho\cos\theta - \gamma\rho^2\cos^2\theta)\, \rho\, d\rho\, d\theta = \frac{\pi}{4}\rho_1^2 (4\alpha - \gamma\rho_1^2).$$

Therefore, condition (74) of Theorem 78 (§33.4) is satisfied if and only if

$$\rho_1 = +\sqrt{\frac{4\alpha}{\gamma}} \tag{55}$$

(we are evidently concerned only with positive ρ_1).

Let us compute the value of l for this ρ_1 (§33.4, (75)). We have

$$l = \int_0^{2\pi} [p'_{1x}(\rho_1 \cos s, \rho_1 \sin s) + q'_{1y}(\rho_1 \cos s, \rho_1 \sin s)]\, ds =$$

$$= \int_0^{2\pi} (\alpha + \beta\rho_1 \cos s - \gamma\rho_1^2 \cos^2 s)\, ds = 2\pi\alpha - \frac{1}{2}\gamma\rho_1^2 2\pi =$$

$$= 2\pi\alpha - \frac{1}{2}\gamma \cdot 2\pi \cdot \frac{4\alpha}{\gamma} = -2\pi\alpha < 0.$$

Applying Theorem 78, we conclude that the closed path (circle)

$$x = \sqrt{\frac{4\alpha}{\gamma}} \cos t, \qquad y = \sqrt{\frac{4\alpha}{\gamma}} \sin t$$

of the linear conservative system (53) creates — on passing to system (52) — a structurally stable limit cycle which is stable for $\mu > 0$ and unstable for $\mu < 0$.

Example 18 (a system close to a linear conservative system).

Consider the system

$$\dot{x} = -y, \quad \dot{y} = x + \mu(\alpha y + \beta y^2 + \gamma y^3 + \delta y^4 - \varepsilon y^5). \tag{56}$$

This is a system of type (B_μ) (§33.2), and here

$$p(x, y, \mu) \equiv 0, \quad q(x, y, \mu) = \alpha y + \beta y^2 + \gamma y^3 + \delta y^4 - \varepsilon y^5.$$

Therefore, by (47), §33.2,

$$\psi(2\pi; \rho_0, 0) = \int_0^{2\pi} (\alpha\rho_0 \sin\theta + \beta\rho^2 \sin^2\theta + \gamma\rho^3 \sin^3\theta +$$

$$+ \delta\rho^4 \sin^4\theta - \varepsilon\rho^5 \sin^5\theta) \sin\theta\, d\theta =$$

$$= \pi\rho_0\left(\alpha + \frac{3}{4}\gamma\rho_0^2 - \frac{5}{8}\varepsilon\rho_0^4\right) \tag{57}$$

$\psi(2\pi; \rho_0, 0)$ depends only on the parameters α, γ, and ε.

In §33.2 we considered the creation of limit cycles from the paths of the system $\dot{x} = -y$, $\dot{y} = x$ corresponding to the nonzero roots of the equation $\psi(2\pi; \rho_0, 0) = 0$. The argument of that section, however, remains valid for a zero root also (provided such a root exists). $\rho_0 = 0$ corresponds to the equilibrium state $O(0, 0)$ of the system $\dot{x} = -y$, $\dot{y} = x$.

Thus, setting $\rho_0 = 0$, we are in fact dealing with the creation of a limit cycle from an equilibrium state $O(0, 0)$ of a linear conservative system. For small μ, the point $O(0, 0)$ is a focus of system (56), which is stable for $\mu\alpha < 0$ and unstable for $\mu\alpha > 0$. Therefore, the periodic solution created from the equilibrium state $O(0, 0)$ when we change over from the linear system to system (56) with a small μ is not a closed path (i.e., not a limit cycle), but rather the equilibrium state $O(0, 0)$ itself.

Let us investigate the nonzero roots of the equation

$$\psi(2\pi;\ \rho_0,\ 0) = \pi\rho_0\left(\alpha + \frac{3}{4}\gamma\rho_0^2 - \frac{5}{8}\varepsilon\rho_0^4\right) = 0, \tag{58}$$

taking $\gamma > 0$, $\varepsilon > 0$, and assuming that α may take on both positive and negative values (this corresponds to the physical conditions of the actual problem associated with this system; see /6/, Chapter IX, §10).

We should consider only the positive roots of (58). As we have seen above, for $\alpha \neq 0$, the equilibrium state $O(0,\ 0)$ of the original system may not create a limit cycle.

Setting in (58)

$$\rho^2 = r, \quad \frac{3}{4}\gamma = a, \quad \frac{5}{8}\varepsilon = b$$

and dividing through by $\pi\rho_0$, we obtain

$$\alpha + ar - br^2 = 0, \tag{59}$$

the roots of this equation being $\frac{a \pm \sqrt{a^2 + 4b\alpha}}{2b}$.

If $\alpha < 0$ and $a^2 + 4b\alpha < 0$, equation (59) has no real roots.

If, on the other hand, $\alpha < 0$ and $a^2 + 4b\alpha > 0$, the two roots of equation (59) are positive. Therefore, equation (58) has two positive roots which correspond to two limit cycles created from the closed paths of the original linear system. It is readily seen that the conditions of Theorem 76 (§33.2) are satisfied and, by this theorem, the system has no other closed paths in a sufficiently small neighborhood of O. If $\mu > 0$, the equilibrium state $O(0,\ 0)$ is a stable focus for $\alpha < 0$. Therefore, one of the limit cycles created (namely, the inner cycle) is unstable, and the outer cycle is stable.

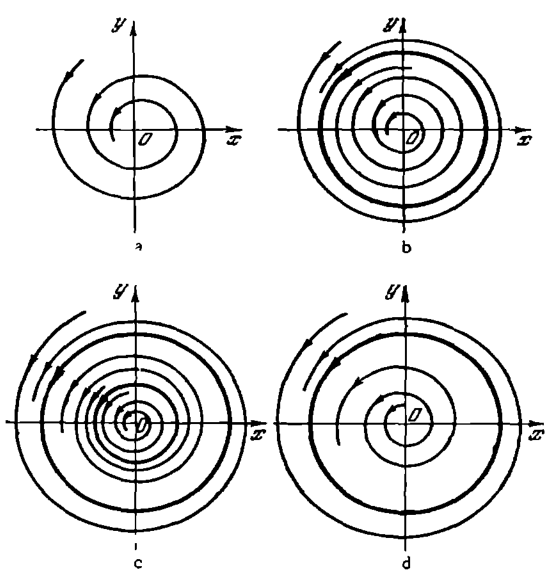

FIGURE 206

Between the (negative) α for $a^2 + 4b\alpha < 0$ and those for which $a^2 + 4b\alpha > 0$, we have one value $\alpha = \alpha^* = -\frac{a^2}{4b} < 0$ for which $a^2 + 4b\alpha = 0$. This value corresponds to a single positive root ρ_0^* of equation (37). Calculations show that $\psi'_{\rho_0}(2\pi; \rho_0^*, 0) = 0$; therefore the theory of §33.2 is inapplicable to this case (see remark to Theorem 75, §33.2). It is readily seen, however, that the closed path of system (56) corresponding to this ρ_0^* is a double limit cycle which separates into two limit cycles — a stable and an unstable one — as α increases.

As α is further increased and passes through zero, the unstable limit cycle contracts to the equilibrium state, which becomes stable for $\alpha \geqslant 0$.

Figure 206, a–d, shows the path configurations of system (56) corresponding to the cases $\alpha < \alpha^*$, $\alpha = \alpha^*$, $\alpha^* < \alpha < 0$, $\alpha \geqslant 0$. It is assumed that $\mu > 0$ is sufficiently small.

Remark. It is readily seen that the problem of the creation of a limit cycle from an equilibrium state of the type of a center cannot be solved with the aid of the sufficient conditions of §33.2. Indeed, these conditions only prove the existence of an equilibrium state $\rho = 0$ for system (B_μ).

Investigation of the nonzero roots of the equation $\psi(2\pi; \rho, \mu) = 0$, which go to zero for $\mu \to 0$, may naturally shed some light on the problem of creation of a limit cycle from a center-type equilibrium state of system (B_0). Note that the solution of the equation $\psi(2\pi; \rho, \mu) = 0$ corresponding to this limit cycle should be sought as an expansion in fractional, and not integral, powers of μ, and, in the simplest case, as an expansion in powers of $\sqrt{\mu}$.

APPENDIX

1. Theorems of the continuous dependence of the solutions of a system of differential equations on the right-hand sides and of the differentiability of solutions

Let

$$\frac{dx_k}{dt} = P_k(t, x_1, x_2, \ldots, x_n), \qquad k = 1, 2, \ldots, n, \tag{A}$$

be a system of differential equations defined in region G of the $(n+1)$-dimensional space. Let P_k be continuous functions which have in G continuous partial derivatives of first order with respect to the variables $x_1, x_2, \ldots, x_n$.

Together with (A), we consider a modified system

$$\frac{dx_k}{dt} = \widetilde{P}_k(t, x_1, \ldots, x_n) = P_k(t_1, x_1, \ldots, x_n) + p_k(t_1, x_1, x_2, \ldots, x_n), \tag{$\widetilde{A}$}$$

where $\widetilde{P}_k$ and hence the "increments" p_k are functions satisfying the same conditions as the function P_k $(k = 1, 2, \ldots, n)$. If we are dealing with a system dependent on a parameter,

$$\frac{dx_k}{dt} = P_k(t, x_1, x_2, \ldots, x_n, \mu), \tag{A_μ}$$

the transition from the value μ_0 of the parameter to another value μ corresponds to a transformation from system (A_{μ_0}) to a modified system (A_μ), and the increments are the functions

$$p_k = P_k(t, x_1, x_2, \ldots, x_n, \mu) - P_k(t, x_1, x_2, \ldots, x_n, \mu_0).$$

The situation is completely analogous for a system dependent on several parameters.

Theorem 1 (theorem of the continuous dependence of solutions on the right-hand sides of the equations). Let

$$x_k = \varphi_k(t; t_0, x_{10}, x_{20}, \ldots, x_{n0}) = \varphi_k(t), \qquad k = 1, 2, \ldots, n, \tag{1}$$

be the solution of system (A) *satisfying the initial conditions*

$$\varphi_k(t_0; t_0, x_{10}, x_{20}, \ldots, x_{n0}) = x_{k_0} \tag{2}$$

which is defined for all t from the interval (τ, T), where $\tau < t_0 < T$. Let,

further

$$x_k = \tilde{\varphi}_k (t;\, t_0,\, x_{10},\, x_{20},\, \ldots,\, x_{n0}) = \tilde{\varphi}_k (t) \tag{3}$$

be the solution of system $(\tilde{\text{A}})$ *satisfying the same initial conditions. Then, for any* τ_1 *and* τ_2, $\tau < \tau_1 < t_0 < \tau_2 < T$, *and any* $\varepsilon > 0$, *there exists* $\delta > 0$ *satisfying the following condition: if* $|p_k (t;\, x_1,\, x_2,\, \ldots,\, x_n)| < \delta$, $k = 1,\, 2,\, \ldots,\, n$, *solution* (3) *is defined for all* t, $\tau_1 \leqslant t \leqslant \tau_2$, *and for every* t *from this interval we have*

$$|\tilde{\varphi}_k (t;\, t_0,\, x_{10},\, x_{20},\, \ldots,\, x_{n0}) - \varphi_k (t;\, t_0,\, x_{10},\, x_{20},\, \ldots,\, x_{n0})| < \varepsilon. \tag{4}$$

Proof. We will only consider $t \in [t_0,\, \tau_2]$. The same reasoning can be applied to $t \in [\tau_1,\, t_0]$. Choose some fixed $\varepsilon > 0$.

Let $\overline{G}_1$ be a closed bounded region completely contained in G, which is convex in all the coordinates x_i $(i = 1, 2, \ldots, n)$ and contains the part of the integral curve (1) corresponding to $t \in [t_0,\, \tau_2]$.* Let ρ_0 be the distance of this part of the integral curve (1) to the boundary of $\overline{G}_1$ (evidently, $\rho_0 > 0$). Since $\overline{G}_1$ is a bounded closed region, the following inequalities are satisfied at every point $M\, (t;\, x_1,\, x_2,\, \ldots,\, x_n)$ of this region:

$$|p'_{kx_i} (t;\, x_1,\, x_2,\, \ldots,\, x_n)| < B, \tag{5}$$

where $B > 0$ is a constant.

Let

$$\xi_k = \tilde{\varphi}_k (t;\, t_0,\, x_{10},\, x_{20},\, \ldots,\, x_{n0}) - \varphi_k (t;\, t_0,\, x_{10},\, x_{20},\, \ldots,\, x_{n0}). \tag{6}$$

The functions ξ_k are defined for all t for which the functions $\tilde{\varphi}_k$ and φ_k are a priori defined. They are therefore a priori defined for all t sufficiently close to t_0. For $t = t_0$, $\xi_k = 0$. Clearly,

$$\begin{aligned} \frac{d\xi_k}{dt} = \tilde{P}_k (t;\, \tilde{\varphi}_1,\, \tilde{\varphi}_2,\, \ldots,\, \tilde{\varphi}_n) - P_k (t;\, \varphi_1,\, \varphi_2,\, \ldots,\, \varphi_n) = \\ = P_k (t;\, \varphi_1 + \xi_1,\, \varphi_2 + \xi_2,\, \ldots,\, \varphi_n - \xi_n) - \\ - P_k (t;\, \varphi_1,\, \varphi_2,\, \ldots,\, \varphi_n) + p_k (t;\, \tilde{\varphi}_1,\, \tilde{\varphi}_2,\, \ldots,\, \tilde{\varphi}_n) = \\ = \sum_{i=1}^{n} P'_{kx_i} (t;\, \tilde{\eta}_{k1},\, \tilde{\eta}_{k2},\, \ldots,\, \tilde{\eta}_{kn})\, \xi_i + p_k (t;\, \tilde{\varphi}_1,\, \tilde{\varphi}_2,\, \ldots,\, \tilde{\varphi}_n). \end{aligned} \tag{7}$$

where the point $(t;\, \tilde{\eta}_{k1},\, \tilde{\eta}_{k2},\, \ldots,\, \tilde{\eta}_{kn})$ lies on the segment joining the points $(t;\, \varphi_1,\, \varphi_2,\, \ldots,\, \varphi_n)$ and $(t_1;\, \tilde{\varphi}_1,\, \tilde{\varphi}_2,\, \ldots,\, \tilde{\varphi}_n)$.

Consider an auxiliary differential equation

$$\frac{dv}{dt} = nBv + \delta n, \tag{8}$$

where $v = v(t)$ is a function of t, and $\delta > 0$ is a constant. The solution of this equation satisfying the condition $v(t_0) = 0$ is

$$v = \frac{\delta}{B} [e^{nB(t-t_0)} - 1]. \tag{9}$$

* Convexity in all the coordinates x_i indicates that $\overline{G}_1$, together with any pair of its points $M\,(t,\, x_1,\, x_2,\, \ldots,\, x_n)$ and $M'\,(t,\, x'_1,\, \ldots,\, x'_n)$, also contains the entire segment MM'.

Evidently, for all t, $t_0 < t \leqslant \tau_2$,

$$0 < v(t) \leqslant \frac{\delta}{B}\left[e^{nB(\tau_2 - t_0)} - 1\right] \tag{10}$$

and $v'(t) > 0$.

Let $\varepsilon^* = \min\left\{\varepsilon, \frac{\rho_0}{n}\right\}$. For $\delta > 0$ we choose any number satisfying the relation

$$\frac{\delta}{B}\left[e^{nB(\tau_2 - t_0)} - 1\right] < \varepsilon^*. \tag{11}$$

Then, by (10), for all t, $t_0 < t \leqslant \tau_2$,

$$v(t) < \varepsilon^*. \tag{12}$$

We will now show that the δ chosen in this way satisfies the proposition of the theorem.

Let

$$|p_k(t; x_1, x_2, \ldots, x_n)| < \delta. \tag{13}$$

Suppose that the point $(t, \tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n)$ and therefore the entire segment joining it with the point $(t, \varphi_1, \varphi_2, \ldots, \varphi_n)$ are contained in $\bar{G}_1$. By (7), (5), and (13) we then have (since $(t, \tilde{\eta}_{k1}, \ldots, \tilde{\eta}_{kn})$ is a point of the relevant segment)

$$\left|\frac{d\xi_k}{dt}\right| < B\sum_{i=1}^{n}|\xi_i| + \delta. \tag{14}$$

This inequality is a priori satisfied for all t sufficiently close to t_0. For $t = t_0$, $\xi_i = 0$. Therefore, for t sufficiently close to t_0, $\left|\frac{d\xi_k}{dt}\right| < \delta$, and hence

$$\left|\frac{d\xi_k}{dt}\right| < n\delta.$$

On the other hand, for all $t \in [t_0, \tau_2]$,

$$v'(t) = nBv + \delta n \geqslant n\delta.$$

It follows from the last inequalities that for all t sufficiently close to t_0 $(t \geqslant t_0)$,

$$\left|\frac{d\xi_k}{dt}\right| < v'(t)$$

and thus

$$|\xi_k(t)| = \left|\int_{t_0}^{t} \xi_k'(t)\,dt\right| < \int_{t_0}^{t} v'(t)\,dt = v(t). \tag{15}$$

Now suppose that the proposition of the theorem is not satisfied for the given δ. Then, either solution (3) of $(\tilde{A})$ is defined only for $t \in [t_0, T^*]$, where $T^* < \tau_2$, or it is defined for all $t \in [t_0, t_2]$, but for some t in this segment at least one of the inequalities

$$|\xi_k(t)| < \varepsilon, \tag{16}$$

and hence at least one of the inequalities

$$|\xi_k(t)| < \varepsilon^* \tag{17}$$

is broken.

In the former case, there exists t, $t_0 < t \leqslant T^* < \tau_2$, for which the corresponding point of the integral curve $x_i = \tilde{\varphi}_i(t)$ lies outside $\bar{G}_1$ (this follows from QT, Appendix, §8.1, Theorem A'). Then its distance from the point $(t, \varphi_1(t), \varphi_2, \ldots, \varphi_n(t))$ in G_1 should be greater than ρ_0, i.e., $\sqrt{\xi_1(t)^2 + \ldots + \xi_n(t)^2} > \rho_0$. This is possible, however, only if for some k, $|\xi_k(t)| > \frac{\rho_0}{n} \geqslant \varepsilon^*$. Thus, in either case, for some t^*, $t_0 < t^* \leqslant \tau_2$, and at least for one k, $1 \leqslant k \leqslant n$, we have the inequality $|\xi_k(t^*)| > \varepsilon^*$, and hence, by (12),

$$|\xi_k(t^*)| > v(t). \tag{18}$$

On the other hand, by (15), for all t sufficiently close to t_0 and for all $i = 1, 2, \ldots, n$, we have

$$|\xi_i(t)| < v(t). \tag{19}$$

It follows from (18) and (19) that the interval of t values where the two solutions $\varphi_i(t)$ and $\tilde{\varphi}_i(t)$ are defined contains a point t_1, $t_0 < t_1 \leqslant \tau_2$, which satisfies the following conditions:

(a) for all t, $t_0 \leqslant t < t_1$, and for all $i = 1, 2, \ldots, n$,

$$|\xi_i(t)| < v(t); \tag{20}$$

(b) for all $i = 1, 2, \ldots, n$,

$$|\xi_i(t_1)| \leqslant v(t_1), \tag{21}$$

and at least for one of these i, $i = k$ say,

$$|\xi_k(t_1)| = v(t_1). \tag{22}$$

From (a) and (b) and inequality (12), it follows that for $t \in [t_0, t_1]$

$$\sqrt{\xi_1(t)^2 + \xi_2(t)^2 + \ldots + \xi_n(t)^2} < \sqrt{n\varepsilon^{*2}} < \sqrt{n\left(\frac{\rho_0}{n}\right)^2} < \rho_0,$$

i.e., the part of the integral curve $x_i = \tilde{\varphi}_i(t)$ corresponding to $t \in [t_0, t_1]$ lies in $\bar{G}_1$. Then, by (7) and (8) and in virtue of the assumption $|p_k| < \delta$, $k = 1, 2, \ldots, n$, we have

$$|\xi_k(t_1)| = \left|\int_{t_0}^{t_1} \xi_k'(t)\,dt\right| \leqslant \int_{t_0}^{t_1} |\xi_k'(t)|\,dt < \int_{t_0}^{t_1} \left[B\sum_{i=1}^{n} |\xi_i| + \delta\right] dt <$$

$$< \int_{t_0}^{t_1} [Bnv(t) + n\delta]\,dt = \int_{t_0}^{t_1} v'(t)\,dt = v(t_1),$$

i.e., $|\xi_k(t_1)| < v(t_1)$, which contradicts (22). This contradiction proves the theorem.

Theorem 2 (theorem of the continuous dependence of solutions on the right-hand sides and initial values). Let

$$x_k = \varphi_k(t, t_0, x_{10}, x_{20}, \ldots, x_{n0}) = \varphi_k(t) \tag{φ}$$

be the solution of system (A) *corresponding to the initial values* t_0, x_{10}, x_{20}, $\ldots$, x_{n0}, *which is defined in the interval* (τ_0, T), *and*

$$x_k = \tilde{\varphi}_k(t, \tilde{t}_0, \tilde{x}_{10}, \ldots, \tilde{x}_{n0}) = \tilde{\varphi}_k(t) \qquad (\tilde{\varphi})$$

the solution of system $(\tilde{A})$ *corresponding to the initial values* $\tilde{t}_0$, $\tilde{x}_{10}$, $\tilde{x}_{20}$, $\ldots$, $\tilde{x}_{n0}$. *Then, for any* τ_1 *and* τ_2, $\tau_0 < \tau_1 < t_0 < \tau_2 < T$, *and any* $\varepsilon > 0$, *there exist* $\delta > 0$ *and* $\eta > 0$ *which satisfy the following conditions: if*

$$|p_k(t, x_1, x_2, \ldots, x_n)| < \delta, \qquad k = 1, 2, \ldots, n,$$

and

$$|\tilde{t}_0 - t_0| < \eta \quad u \quad |\tilde{x}_{i0} - x_{i0}| < \eta, \qquad i = 1, 2, \ldots, n,$$

the solution $(\tilde{\varphi})$ *is defined for* t, $\tau_1 \leqslant t \leqslant \tau_2$, *and for every* t *from this interval*

$$|\tilde{\varphi}_k(t) - \varphi_k(t)| < \varepsilon \qquad (k = 1, 2, \ldots, n).$$

Proof of Theorem 2 is readily obtained if we change over in $(\tilde{A})$ to the variables τ and z_k defined by the equalities

$$t = \tau - t_0 + \tilde{t}_0, \quad x_k = z_k - x_{k0} + \tilde{x}_{k0},$$

and apply Theorem 1, using the compactness of the segment $[\tau_1, \tau_2]$ and the fact that the solution (φ) is defined on a segment $[\tau_1 - \sigma, \tau_2 + \sigma]$, where σ is some positive number.

Corollary 1. Let Ω be a compact set in the $(n+2)$-dimensional space t, t_0, x_{10}, x_{20}, $\ldots$, x_{n0}, with the functions $\varphi_k(t, t_0; x_{10}, x_{20}, \ldots, x_{n0})$ defined everywhere in this space. For every $\varepsilon > 0$, there exist $\delta > 0$ and $\eta > 0$ satisfying the following conditions: if $|p_k(t; x_1, x_2, \ldots, x_n)| < \delta$, $|t' - t''| < \eta$, $|t_0 - \tilde{t}_0| < \eta$, $|x_{k0} - \tilde{x}_{k0}| < \eta$ and the point $(t', t_0; x_1, \ldots, x_n) \in \Omega$, the functions $\tilde{\varphi}_k$ are defined at the point $(t'', \tilde{t}_0; \tilde{x}_{10}, \ldots, \tilde{x}_{n0})$ and

$$|\tilde{\varphi}_k(t'', \tilde{t}_0; \tilde{x}_{10}, \ldots, \tilde{x}_{n0}) - \varphi_k(t', t_0; x_{10}, \ldots, x_{n0})| < \varepsilon$$
$$(k = 1, 2, \ldots, n).$$

Corollary 1 follows from the previous theorem. It is proved again by reductio ad absurdum, and the compactness of Ω is used.

Corollary 2. For a system

$$\frac{dx_k}{dt} = P_k(t; x_1, x_2, \ldots, x_n, \mu),$$

whose right-hand sides are functions of class 1 in t, x_1, x_2, $\ldots$, x_n, and continuous functions of the parameter μ, the solutions

$$x = \varphi_k(t, t_0; x_{10}, \ldots, x_{n0}, \mu)$$

are continuous functions in all the variables for all those values of the variables for which they are defined.

Corollary 2 follows directly from Theorem 2.

Now let the functions P_k and $\tilde{P}_k$ entering the right-hand sides of (A) and ($\tilde{\mathrm{A}}$) be functions of class $r \geqslant 1$. In this case, the solutions $\varphi_k(t, t_0; x_{10}, x_{20}, \ldots, x_{n0})$, for all the values of the variables for which they are defined, have continuous (in all the variables) derivatives: (a) to order $r+1$ with respect to t; (b) to order r with respect to all the variables x_{i0}; (c) to order $r+1$ with respect to all the variables, provided it contains at least one differentiation with respect to t (see QT, Appendix, §8.3, Theorem B''). An analogous proposition applies to $\tilde{\varphi}_k$.

Theorem 3. Let the solution (φ) *of system* (A) *be defined for all* $t \subset (\tau_0, T)$, *and let* $\tau_0 < \tau_1 < t_0 < \tau_2 < T$. *Then, for every* $\varepsilon > 0$, *there exist* $\delta > 0$ *and* $\eta > 0$ *satisfying the following conditions: if* ($\tilde{\mathrm{A}}$) *is* δ*-close to rank* r *to* (A) *and if* $|\tilde{t}_0 - t_0| < \eta$, $|\tilde{x}_{i0} - x_{i0}| < \eta$ $(i = 1, 2, \ldots, n)$ *the solution* ($\tilde{\varphi}$) *is defined for all* t, $\tau_1 \leqslant t \leqslant \tau_2$, *and for these values of the variables*

$$\left| \frac{\partial^m \tilde{\varphi}_k(t, \tilde{t}_0, \tilde{x}_{10}, \ldots, \tilde{x}_{n0})}{\partial t^i \partial \tilde{t}_0^{i_0} \partial \tilde{x}_{10}^{i_1} \ldots \partial \tilde{x}_{n0}^{i_n}} - \frac{\partial^m \varphi(t, t_0, x_{10}, \ldots, x_{n0})}{\partial t^i \partial t_0^{i_0} \partial x_{10}^{i_1} \ldots \partial x_{n0}^{i_n}} \right| < \varepsilon,$$
$$i + i_0 + i_1 + \ldots + i_n = m \leqslant r+1, \quad i_1 + i_2 + \ldots + i_n \leqslant r. \tag{23}$$

Proof. The partial derivatives of the functions φ_k satisfy the following system of differential equations

$$\begin{gathered} \frac{d\varphi_k}{dt} = P_k(t, \varphi_1, \varphi_2, \ldots, \varphi_n), \\ \frac{d}{dt}\left(\frac{\partial \varphi_k}{\partial t_0}\right) = \sum_{j=1}^{n} \frac{\partial P_k}{\partial x_j} \frac{\partial \varphi_j}{\partial t_0}, \\ \frac{d}{dt} \frac{\partial \varphi_k}{\partial x_{i0}} = \sum_{j=1}^{n} \frac{\partial P_k}{\partial x_j} \frac{\partial \varphi_j}{\partial x_{i0}}. \\ \cdots\cdots\cdots \end{gathered} \tag{24}$$

with the initial conditions

$$\begin{gathered} \varphi_k(t_0, t_0, x_{10}, x_{20}, \ldots, x_{n0}) = x_{k0}, \\ \left.\frac{\partial \varphi_k}{\partial x_{i0}}\right|_{t=t_0} = \delta_k^i, \\ \cdots\cdots \end{gathered} \tag{25}$$

(see QT, Appendix, §8.3). The situation is analogous with respect to the partial derivatives of the functions $\tilde{\varphi}_k$. The validity of Theorem 3 directly follows from Theorem 2, when the latter is applied to system (24) with initial conditions (25).

Consider the case when the right-hand sides of the system are functions of some parameter μ, i.e., systems of the form

$$\frac{dx_k}{dt} = P_k(t, x_1, x_2, \ldots, x_n, \mu) \qquad (k = 1, 2, \ldots, n). \tag{A_μ}$$

The functions P_k are defined for all μ, $\mu_1 < \mu < \mu_2$, everywhere in some region G of the $(n+1)$-dimensional space $t, x_1, x_2, \ldots, x_n$. These functions are assumed to be continuous in all their arguments and to have continuous partial derivatives with respect to $x_1, x_2, \ldots, x_n$ and μ to order r inclusive.

Let

$$x_k = \varphi_k(t, t_0, x_{10}, \ldots, x_{n0}, \mu), \qquad k = 1, 2, \ldots, n, \tag{26}$$

be a solution of (A_μ).

By Corollary 2 of Theorem 2, φ_k are continuous functions in all their arguments.

Theorem 4. If the functions $P_k(t, x_1, \ldots, x_n, \mu)$ *have continuous partial derivatives to order* r *inclusive in the variables* $x_1, x_2, \ldots, x_n, \mu$, *the functions* $\varphi_k(t, t_0, x_{10}, \ldots, x_{n0}, \mu)$ *have continuous partial derivatives to order* r *inclusive for all those* t *for which they are defined, and these partial derivatives satisfy the differential equations*

$$
\begin{gathered}
\frac{d\varphi_k}{dt} = P_k(t, \varphi_1, \varphi_2, \ldots, \varphi_n, \mu),\\
\frac{d}{dt}\frac{\partial \varphi_k}{\partial \mu} = \sum_{l=1}^{n} \frac{\partial P_k(t, \varphi_1, \ldots, \varphi_n, \mu)}{\partial x_l}\frac{\partial \varphi_l}{\partial \mu} + \frac{\partial P_k}{\partial \mu},\\
\frac{d}{dt}\left(\frac{\partial \varphi_k}{\partial x_{i0}}\right) = \sum_{j=1}^{n} \frac{\partial P_k}{\partial x_j}\frac{\partial \varphi_j}{\partial x_{i0}},\\
\frac{d}{dt}\left(\frac{\partial^2 \varphi_k}{\partial x_{i0}\,\partial \mu}\right) = \sum_{j=1}^{n} \frac{\partial P_k}{\partial x_j}\frac{\partial^2 \varphi_j}{\partial x_{i0}\,\partial \mu} + \sum_{j,\,l=1}^{n} \frac{\partial^2 P_k}{\partial x_j\,\partial x_l}\frac{\partial \varphi_l}{\partial \mu}\frac{\partial \varphi_j}{\partial x_{i0}} + \sum_{j=1}^{n} \frac{\partial^2 P_k}{\partial x_j\,\partial \mu}\frac{\partial \varphi_j}{\partial x_{i0}},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered} \tag{27}
$$

with the initial conditions

$$
\begin{gathered}
\varphi_k(t_0, t_0, x_{10}, \ldots, x_{n0}, \mu) = x_{k0},\\
\left.\frac{\partial \varphi_k}{\partial \mu}\right|_{t=t_0} = 0, \qquad \left.\frac{\partial \varphi_k}{\partial x_{i0}}\right|_{t=t_0} = \delta_i^k,\\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered} \tag{28}
$$

The proof of Theorem 4 will be found in /17/.

2. A proposition regarding functions of many variables

Let $f(x, y)$ be a function of class $N \geqslant 1$ in a region containing the point $(0, 0)$.

Theorem 5. For every k, $1 \leqslant k \leqslant N$, *the function* $f(x, y)$ *may be represented in a sufficiently small neighborhood of the point* $(0, 0)$ *in the form*

$$f(x, y) = P_0(x, y) + P_1(x, y) + \ldots + P_k(x, y) + P^*(x, y), \tag{1}$$

where P_i, $i = 0, 1, \ldots, k$, *are homogeneous polynomials of degree* i,

$$P^*(x, y) = \sum_{\alpha=0}^{k} x^{k-\alpha} y^{\alpha} P_\alpha^*(x, y), \tag{2}$$

and $P_\alpha^*(x, y)$ *is a continuous function which vanishes at* $x = y = 0$.

Proof. We will use the Maclaurin formula for functions of a single variable with the residual term in integral form:

$$\varphi(t) = \varphi(0) + \frac{\varphi'(0)}{1!}t + \ldots + \frac{\varphi^{(k-1)}(0)}{(k-1)!}t^{k-1} + \frac{1}{(k-1)!}\int_0^t \varphi^k(z)(t-z)^{k-1}\,dz$$

(see /11/, Vol. 2, Sec. 306).

Applying this formula to the function $\varphi(t)=f(tx, ty)$ and taking $t=1$, we find

$$f(x, y)=f(0, 0)+\sum_{l=1}^{k-1}\frac{1}{l!}\left(\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y\right)^l f(0, 0)+$$
$$+\frac{1}{(k-1)!}\int_0^1\left(\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y\right)^k f(zx, zy)(1-z)^{k-1}\,dz. \qquad (3)$$

The last term in this equality is designated $Q^*(x, y)$. Clearly,

$$Q^*(x, y)=\frac{1}{(k-1)!}\sum_{\alpha=0}^{k}\left(C_k^\alpha\int_0^1\frac{\partial f^k(zx, zy)}{\partial x^{k-\alpha}\,\partial y^\alpha}(1-z)^{k-1}\,dz\right)x^{k-\alpha}y^\alpha. \qquad (4)$$

The coefficient before $x^{k-\alpha}y^\alpha$ in the last expression is a continuous function of x and y. We designate it $g_\alpha(x, y)$. Let

$$P_\alpha^*(x, y)=g_\alpha(x, y)-g_\alpha(0, 0).$$

Then

$$g_\alpha(x, y)=g_\alpha(0, 0)+P_\alpha^*(x, y) \qquad (5)$$

and

$$P_\alpha^*(0, 0)=0. \qquad (6)$$

Inserting (4) and (5) in (3), we obtain equation (1), where

$$P_0(x, y)=f(0, 0),$$
$$P_l(x, y)=\frac{1}{l!}\left(\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y\right)^l f(0, 0), \qquad l=1, 2, \ldots, k-1,$$
$$P_k(x, y)=\sum_{\alpha=0}^{k}g_\alpha(0, 0)x^{k-\alpha}y^\alpha$$

and

$$P^*(x, y)=\sum_{\alpha=0}^{k}[g_\alpha(x, y)-g_\alpha(0, 0)]x^{k-\alpha}y^\alpha.$$

The theorem is proved. A similar proposition clearly applies to functions of n variables.

3. The lemma about the normals of a simple smooth closed curve

Let L_0 be a simple closed curve defined by the equations

$$x=\varphi(s), \qquad y=\psi(s),$$

where φ and ψ are periodic functions of period τ. We assume that φ and ψ are functions of the second class, and that $\varphi'(s)$ and $\psi'(s)$ do not vanish simultaneously for any s.

Through each point $M(\varphi(s), \psi(s))$ of the path L_0 we draw a normal to the path at that point and lay off segments of the length $\delta\sqrt{\varphi'(s)^2+\psi'(s)^2}$ on either side of the point M.

Lemma 1. If $\delta > 0$ is sufficiently small, no two segments of normals drawn through different points of the curve L_0 intersect.

Proof. Since $\sqrt{\varphi'(s)^2+\psi'(s)^2}$ is bounded from above, it suffices to prove the lemma under the assumption that segments of length δ are laid off all the normals. Moreover, it suffices to consider the case when these segments are laid off on one side of the curve L_0 (either all inside the curve or all outside the curve). We assume that both these conditions are satisfied. Let the proposition of the lemma be false. Then there exists a sequence of numbers $\delta_n \to 0$ $(\delta_n > 0)$ and a sequence of pairs of points P_n, Q_n of the curve L_0 such that the segments of normals of length δ_n drawn through the points P_n and Q_n intersect. Since the curve L_n is compact, we may take $P_n \to P$ and $Q_n \to Q$, where P and Q are points of L_0.

Let us first consider the case when P and Q are two different points.

Let M_n be the intersection point of the normal segments through the points P_n and Q_n. Then $\rho(P_n, M_n) \leqslant \delta_n$, $\rho(Q_n, M_n) \leqslant \delta_n$, and by the triangle inequality

$$\rho(P, Q) < \rho(P, P_n) + \rho(P_n, M_n) + \rho(M_n, Q_n) + \rho(Q_n, Q) \leqslant \\ \leqslant \rho(P, P_n) + 2\delta_n + \rho(Q_n, Q)$$

(Figure 207). Since for a sufficiently large n, the right-hand side of the inequality is as small as desired, whereas the left-hand side is constant, this inequality cannot be satisfied. P and Q thus may not be two different points, and we have to consider only the case when they coincide.

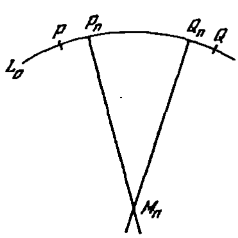

FIGURE 207

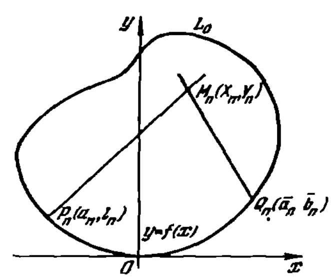

FIGURE 208

We change over to a new rectangular coordinate system, placing the origin O at the point P (which coincides with Q), and directing the abscissa axis along the tangent to L_0 at O (Figure 208). We retain the same notation as before, i.e., the new coordinates are designated x and y and the parametric equations of the curve L_0 are still $x = \varphi(s)$, $y = \psi(s)$. Clearly, φ and ψ are functions of the second class and $\varphi'(s)^2 + \psi'(s)^2 \neq 0$ for any s.

Let the point O of curve L_0 correspond to the value s_0 of the parameter s. Near the point O, the equation of the curve L_0 is $y = f(x)$. Then

$$f'(x) = \frac{\psi'(s)}{\varphi'(s)}, \quad f'(0) = 0, \quad \psi'(s_0) = 0, \quad \varphi'(s_0) \neq 0.$$

Therefore, near the point $x = 0$, the function $f(x)$ has a continuous second derivative $f''(x)$.

By assumption, the normal segments of length $\delta_n > 0$ drawn through the points P_n and Q_n intersect for every n (here $\delta_n \to 0$, $P_n \to 0$, $Q_n \to 0$). Let $M_n(X_n, Y_n)$ be an intersection point, and let $P_n(a_n, b_n)$ and $Q_n(\bar{a}_n, \bar{b}_n)$. Then $\bar{a}_n \neq a_n$, and for $n \to 0$, we have $a_n \to 0$, $\bar{a}_n \to 0$, $b_n \to 0$, $\bar{b}_n \to 0$.

The equations of the normals to the curve L_0 at the points P_n and Q_n are respectively written in the form

$$x - a_n = -f'(a_n)(y - b_n), \quad x - \bar{a}_n = -f'(\bar{a}_n)(y - \bar{b}_n).$$

Inserting for x and y in these equations the coordinates X_n and Y_n of the point M_n and subtracting term by term, we obtain after simple manipulations

$$Y_n[f'(\bar{a}_n) - f'(a_n)] = \bar{a}_n - a_n + (\bar{b}_n - b_n) f'(\bar{a}_n) + b_n [f'(\bar{a}_n) - f'(a_n)]. \qquad (1)$$

From the Lagrange formula

$$f'(\bar{a}_n) - f'(a_n) = f''(\xi_n)(\bar{a}_n - a_n) \qquad (2)$$

and

$$\bar{b}_n - b_n = f(\bar{a}_n) - f(a_n) = f'(\eta_n)(\bar{a}_n - a_n), \qquad (3)$$

where ξ_n and η_n lie between a_n and $\bar{a}_n$ and therefore go to zero for $n \to \infty$. Inserting (2) and (3) in (1) and dividing through by $(\bar{a}_n - a_n)$, we obtain

$$Y_n f''(\xi_n) = 1 + f'(\eta_n) f'(a_n) + b_n f''(\xi_n)$$

and taking the limit

$$\lim_{n\to\infty} Y_n f''(\xi_n) = 1. \qquad (4)$$

On the other hand,

$$\sqrt{(Y_n - b_n)^2} \leqslant \sqrt{(Y_n - b_n)^2 + (X_n - a_n)^2} \leqslant \delta_n.$$

Since $b_n \to 0$ and $\delta_n \to 0$, we have $Y_n \to 0$. Therefore

$$\lim_{n\to 0} Y_n f''(\xi_n) = 0,$$

which contradicts (4). This proves the lemma.

4. Proof of the differentiability of the function $R(\rho, \theta)$ with respect to ρ

We will first prove an auxiliary proposition. Let the function $f(x, y)$ be defined in the circle $x^2 + y^2 \leqslant R^2$ where it is continuously differentiable with respect to x and y to some order $N \geqslant 1$, and $f(0, 0) = 0$.

Consider the function $f^*(\rho, \theta)$ defined as follows:

$$\begin{aligned} f^*(\rho, \theta) &= \frac{1}{\rho} f(\rho\cos\theta, \rho\sin\theta) \quad \text{for } \rho \neq 0, \\ f^*(0, \theta) &= f'_x(0, 0)\cos\theta + f'_y(0, 0)\sin\theta. \end{aligned} \qquad (1)$$

Since by assumption $f(x, y)$ is a function of class N, then for $\rho \neq 0$ and for all θ, $f^*(\rho, \theta)$ has continuous partial derivatives with respect to ρ to order N. The next lemma establishes the existence and the continuity of the derivatives of the function $f^*(\rho, 0)$ with respect to ρ for $\rho = 0$.

Lemma 2. If $f(x, y)$ is a function of class $N \geqslant 1$ in the circle $x^2 + y^2 \leqslant R^2$, then

(a) for every θ, $f^*(\rho, \theta)$ is continuously differentiable in ρ to order $N - 1$;

(b) for $\rho \to 0$, the function $\rho \frac{\partial^N f^*(\rho, \theta)}{\partial \rho^N}$ goes to zero uniformly in θ.

Proof. If $f(x, y)$ is a homogeneous polynomial of degree $n \geqslant 1$, the lemma can be directly verified (it suffices to check the proposition for a single term of the form $x^k y^{n-k}$). It is moreover clear that if the lemma is true for two functions, it is also true for their sum. It is therefore sufficient to prove the lemma for the function

$$f_1(x, y) = f(x, y) - \sum_{k=1}^{N} \frac{1}{k!} \left(\sum_{q=0}^{k} C_k^q f^{(k)}_{x^{k-q} y^q}(0, 0) x^{k-q} y^q \right).$$

All the derivative of this function to order N inclusive vanish at the point $(0, 0)$. Thus, in our proof of the lemma, we may assume without loss of generality that all the derivatives of the function $f(x, y)$ to order N inclusive vanish at the point $(0, 0)$. If this is so, we have from Taylor's formula

$$f(x, y) = \frac{1}{N!} \sum_{p=0}^{N} C_N^p f^{(N)}_{x^{N-p} y^p}(\bar{x}, \bar{y}) x^{N-p} y^p, \tag{2}$$

where $\bar{x}$ and $\bar{y}$ are numbers lying between 0 and x and 0 and y, respectively, which in general depend on the point (x, y). Inserting for x and y the polar coordinates $\rho \cos \theta$ and $\rho \sin \theta$, we find $f(x, y) = \rho^N F(\bar{x}, \bar{y})$, where

$$F(\xi, \eta) = \frac{1}{N!} \sum_{p=0}^{N} C_N^p f^N_{x^{N-p} y^p}(\xi, \eta) \cos^{N-p} \theta \sin^p \theta. \tag{3}$$

Similarly, using the Taylor series of the function $f^{(q)}_{x^{q-r} y^r}$, $1 \leqslant q \leqslant N-1$, $0 \leqslant r \leqslant q$, under the same assumption and remembering that the partial derivatives of the function $f(x, y)$ to order N inclusive vanish at $(0, 0)$, we obtain

$$f^{(q)}_{x^{q-r} y^r}(x, y) = \frac{1}{(N-q)!} \sum_{p=0}^{N-q} C_{N-q}^p f^{(N)}_{x^{N-r-p} y^{r+p}}(\bar{x}_{qr}, \bar{y}_{qr}) x^{N-q-p} y^p$$

or

$$f^{(q)}_{x^{q-r} y^r}(x, y) = \rho^{N-q} F_{qr}(\bar{x}_{qr}, \bar{y}_{qr}), \tag{4}$$

where

$$F_{qr}(\xi, \eta) = \frac{1}{(N-q)!} \sum_{p=0}^{N-q} C_{N-q}^p f^{(N)}_{x^{N-r-p} y^{r+p}}(\xi, \eta) \cos^{N-q-p} \theta \sin^p \theta, \tag{5}$$

and $\bar{x}_{qr}$ and $\bar{y}_{qr}$ lie between 0 and x and 0 and y, respectively. $F(\xi, \eta)$ and $F_{qr}(\xi, \eta)$ are evidently continuous functions of ξ, η which vanish at $\xi = \eta = 0$.

Differentiating the function

$$f(x,\ y)=f(\rho\cos\theta,\ \rho\sin\theta)$$

q times with respect to ρ $(1\leqslant q\leqslant N-1)$, we obtain

$$\frac{\partial^q}{\partial\rho^q} f(x,\ y)=\sum_{r=0}^{q} C_q^r f^{(q)}_{x^{q-r}y^r}(x,\ y)\cos^{q-r}\theta\sin^r\theta. \tag{6}$$

Inserting for $f^{(q)}_{x^{q-r}y^r}(x,\ y)$ its expression from (4) and dividing through by ρ^{N-q}, we obtain

$$\frac{1}{\rho^{N-q}}\frac{\partial^q f(x,\ y)}{\partial\rho^q}=\sum_{r=0}^{q} C_q^r F_{qr}(\bar{x}_{qr},\ \bar{y}_{qr})\cos^{q-r}\theta\sin^r\theta.$$

If $\rho\to 0$, we have $x\to 0$, $y\to 0$, $\bar{x}_{qr}\to 0$, $\bar{y}_{qr}\to 0$, and therefore $F_{qr}(\bar{x}_{qr},\bar{y}_{qr})\to 0$. The expression

$$\frac{1}{\rho^{N-q}}\frac{\partial^q f(x,\ y)}{\partial\rho^q}=\rho^{q-N}\frac{\partial^q f(x,\ y)}{\partial\rho^q} \tag{7}$$

$(1\leqslant q\leqslant N-1)$ thus also goes to zero for $\rho\to 0$ (uniformly in θ). It further follows from (2) that the expression

$$\frac{1}{\rho^N} f(x,\ y)=\rho^{-N} f(x,\ y) \tag{8}$$

also goes to zero for $\rho\to 0$, uniformly in θ.

Let us now prove proposition (a) of the lemma. We first take $\rho\neq 0$ and apply the Leibnitz rule to obtain expressions for derivatives of order k of the function

$$f^*(\rho,\ \theta)=\frac{1}{\rho} f(\rho\cos\theta,\ \rho\sin\theta) \text{ (see (1)), where } 1\leqslant k\leqslant N-1.$$

The Leibnitz rule gives

$$\frac{\partial^k f^*(\rho,\ \theta)}{\partial\rho^k}=\frac{\partial^k\left(\dfrac{f(\rho\cos\theta,\ \rho\sin\theta)}{\rho}\right)}{\partial\rho^k}=\sum_{q=0}^{k} C_k^q\frac{\partial^q f}{\partial\rho^q}\frac{d^{k-q}(\rho^{-1})}{d\rho^{k-q}}=$$
$$=\left(\sum_{q=0}^{k} C_k^q(-1)^{k-q}(k-q)!\,\rho^{q-N}\frac{\partial^q f}{\partial\rho^q}\right)\rho^{N-k-1}. \tag{9}$$

Since (8) and (9) for $\rho\to 0$ go to zero uniformly in θ, we clearly have for $\rho\to 0$,

$$\frac{\partial^k f^*(\rho,\ \theta)}{\partial\rho^k}\to 0 \tag{10}$$

uniformly in θ. Moreover, by (1) and (2), for $\rho\to 0$,

$$f^*(\rho,\ \theta)\to 0 \tag{11}$$

uniformly in θ.

By (10) and (11), proposition (a) of the lemma will be proved if we show that $f^*(0,\theta)=0$ and that the function $f^*(\rho,\theta)$ for $\rho=0$ (i.e., at any point $(0,\theta)$)

is differentiable with respect to ρ to order $N-1$ and the corresponding derivatives vanish. Clearly,

$$f^*(0, \theta)=0$$

according to the original definition of the function $f^*(\rho, \theta)$ (since f'_x and f'_y by assumption vanish at the point $(0, \theta)$). Taking $N>1$, we will prove the existence of the first partial derivative with respect to ρ of the function $f^*(\rho, \theta)$ at the point $(0, 0)$. Expression (9) is clearly inapplicable, since it holds true for $\rho \neq 0$ only. We should thus compute the partial derivative directly, as the limit of the expression

$$\frac{f^*(\rho, \theta)-f^*(0, \theta)}{\rho}=\frac{f^*(\rho, \theta)}{\rho}=\frac{f(x, y)}{\rho^2}=\rho^{N-2}(\rho^{-N}f(x, y)). \tag{12}$$

The factor in parentheses is (8), which for $\rho \to 0$ goes to zero uniformly in θ. The limit of (12), i.e., the derivative of the function $f^*(\rho, \theta)$ at the point $(0, \theta)$, thus exists and is equal to zero. By (10), we thus conclude that the first derivative with respect to ρ of the function $f^*(\rho, \theta)$ is continuous at $\rho=0$.

Now suppose that we have established the existence and the continuity of the partial derivatives with respect to ρ to order $p \leqslant N-2$ for the function $f^*(\rho, \theta)$ at the point $(0, \theta)$. Then, by (10), all these derivatives (to order p inclusive) vanish.

Let us find the $(p+1)$-th derivative with respect to ρ of the function $f(\rho, \theta)$ at the point $(0, \theta)$ by the direct technique, i.e., as the limit of the expression

$$\frac{\frac{\partial^p f^*(\rho, \theta)}{\partial \rho^p}-\frac{\partial^p f^*(0, \theta)}{\partial \rho^p}}{\rho}=\rho^{-1}\frac{\partial^p\left(\frac{f}{\rho}\right)}{\partial \rho^p}. \tag{13}$$

Using equality (9) (for $k=p \leqslant N-2$) and the properties of (7) and (8), we can directly show that the last expression goes to zero for $\rho \to 0$. This indicates that the derivative $\frac{\partial^{p+1} f^*(\rho, \theta)}{\partial \rho^{p+1}}$ at the point $(0, \theta)$ exists and is equal to zero. Then by (10), this derivative is continuous at the point $(0, \theta)$. We have thus proved proposition (a) of the lemma.

Let us now prove proposition (b). For $\rho \neq 0$, $f(x, y)=f(\rho\cos\theta, \rho\sin\theta)= =f^*(\rho, \theta)\rho$. Using the Leibnitz rule,

$$\frac{\partial^N f}{\partial \rho^N}=\rho\frac{\partial^N f^*}{\partial \rho^N}+N\frac{\partial^{N-1} f^*}{\partial \rho^{N-1}}. \tag{14}$$

For $\rho \to 0$, the left-hand side $\frac{\partial^N f}{\partial \rho^N}$ goes to zero uniformly in θ (this follows from (6), which is also valid for $q=N$, and from our assumption concerning the function $f(x, y)$). The second term on the right in (14) differs from (10) by a constant factor only, and it therefore also goes to zero uniformly in θ for $\rho \to 0$. But then the first term on the right $\rho\frac{\partial^N f^*}{\partial \rho^N}$ necessarily goes to zero uniformly in θ. Q. E. D.

Let us now proceed with the main proposition of this subsection. Consider the system

$$\frac{dx}{dt}=\alpha x-\beta y+\varphi(x, y), \quad \frac{dy}{dt}=\beta x+\alpha y+\psi(x, y), \tag{15}$$

where $\beta \neq 0$, and the functions φ and ψ together with their first derivatives vanish at the point $(0, 0)$. Let

$$F(\rho, \theta) = \alpha\rho + \varphi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta, \tag{16}$$

$$\left.\begin{aligned} \Phi(\rho, \theta) &= \frac{\psi(\rho\cos\theta, \rho\sin\theta)}{\rho}\cos\theta - \frac{\varphi(\rho\cos\theta, \rho\sin\theta)}{\rho}\sin\theta \\ &\text{for } \rho \neq 0 \text{ and} \\ \Phi(0, \theta) &\equiv 0, \end{aligned}\right\} \tag{17}$$

$$R(\rho, \theta) = \frac{F(\rho, \theta)}{\beta + \Phi(\rho, \theta)}. \tag{18}$$

Lemma 3. If (15) *is a system of class* $N \geqslant 1$ *and* $\rho^* > 0$ *is sufficiently small, the function* $R(\rho, \theta)$ *has continuous partial derivatives with respect to* ρ *to order* N *inclusive at every point* (ρ, θ) *of the region* $-\infty < \theta < +\infty$, $0 \leqslant |\rho| < \rho^*$ *(Lemma 3, § 24).*

Proof. The condition $\Phi(0, \theta) \equiv 0$ shows that for $\rho = 0$ the functions $\frac{\varphi(\rho\cos\theta, \rho\sin\theta)}{\rho}$ and $\frac{\psi(\rho\cos\theta, \rho\sin\theta)}{\rho}$ in (17) are defined and are equal to zero. It is thus clear that these functions are generated from the functions $\varphi(x, y)$ and $\psi(x, y)$, respectively, like the functions $f^*(\rho, \theta)$ from $f(x, y)$ (see (1)). Therefore, Lemma 2 is applicable to the functions

$$\frac{\varphi(\rho\cos\theta, \rho\sin\theta)}{\rho} \quad \text{and} \quad \frac{\psi(\rho\cos\theta, \rho\sin\theta)}{\rho}$$

and, using equations (15)–(18) and the inequality $\beta + \Phi(\rho, \theta) \neq 0$ (which is satisfied for every point (ρ, θ) of the relevant region for a sufficiently small ρ^*), we conclude that the function $R(\rho, \theta)$ also has continuous partial derivatives with respect to ρ to order $N - 1$ inclusive at every point (ρ, θ) $(0 \leqslant |\rho| < \rho^*$ for any $\theta)$ and to order N inclusive at every point where $\rho \neq 0$. We thus only have to establish the existence of a continuous N-th derivative with respect to ρ at points where $\rho = 0$. We represent $R(\rho, \theta)$ as a product of two functions

$$R(\rho, \theta) = F(\rho, \theta)\frac{1}{\beta + \Phi(\rho, \theta)},$$

and find $\frac{\partial^{N-1}R}{\partial\rho^{N-1}}$ using the Leibnitz rule (this is permissible, since each of the factors has partial derivatives with respect to ρ to order $N-1$ inclusive at every point (ρ, θ)). Simple manipulations show that

$$\frac{\partial^{N-1}R}{\partial\rho^{N-1}} = \frac{H(\rho, \theta)}{(\beta + \Phi)^N} - \frac{F(\rho, \theta)}{(\beta + \Phi)^2}\frac{\partial^{N-1}\Phi}{\partial\rho^{N-1}}, \tag{19}$$

where $H(\rho, \theta)$ is a polynomial in the functions

$$F, \frac{\partial F}{\partial\rho}, \ldots, \frac{\partial^{N-1}F}{\partial\rho^{N-1}}, \quad \Phi, \frac{\partial\Phi}{\partial\rho}, \ldots, \frac{\partial^{N-2}\Phi}{\partial\rho^{N-2}}.$$

The expression $\frac{H(\rho, \theta)}{(\beta + \Phi)^N}$ is clearly continuously differentiable in ρ at every point (ρ, θ) $(0 \leqslant |\rho| < \rho^*$ for any $\theta)$. To prove the lemma, we thus have to

establish existence and continuity of the derivative with respect to ρ of the expression

$$\frac{F(\rho,\theta)}{(\beta+\Phi)^2}\frac{\partial^{N-1}\Phi(\rho,\theta)}{\partial\rho^{N-1}} \tag{20}$$

at the point $(0, \theta)$. We will first prove the existence of this derivative for $\rho=0$. For $\rho=0$, $F(\rho,\theta)=0$, and expression (20) thus vanishes. Therefore,

$$\left\{\frac{\partial}{\partial\rho}\left[\frac{F(\rho,\theta)}{(\beta+\Phi)^2}\frac{\partial^{N-1}\Phi(\rho,0)}{\partial\rho^{N-1}}\right]\right\}_{\rho=0}=\lim_{\rho\to0}\left[\frac{F(\rho,\theta)}{\rho}\frac{1}{(\beta+\Phi)^2}\frac{\partial^{N-1}\Phi(\rho,0)}{\partial\rho^{N-1}}\right]=$$
$$=\lim_{\rho\to0}\left\{\frac{F(\rho,\theta)}{\rho}\frac{1}{(\beta+\Phi)^2}\left[\cos\theta\frac{\partial^{N-1}}{\partial\rho^{N-1}}\left(\frac{\psi}{\rho}\right)-\sin\theta\frac{\partial^{N-1}}{\partial\rho^{N-1}}\left(\frac{\varphi}{\rho}\right)\right]\right\}. \tag{21}$$

According to the above, the partial derivatives $\frac{\partial^{N-1}}{\partial\rho^{N-1}}\left(\frac{\psi}{\rho}\right)$ and $\frac{\partial^{N-1}}{\partial\rho^{N-1}}\left(\frac{\varphi}{\rho}\right)$ go to definite limits for $\rho\to0$, which are respectively equal to the values of these derivatives at the point $(0,\theta)$. Furthermore, $\lim\limits_{\rho\to0}\Phi(\rho,\theta)=0$, and $\lim\limits_{\rho\to0}\frac{F(\rho,\theta)}{\rho}=\alpha$. Hence it follows that the limit on the right in (21) exists and is equal to

$$\frac{\alpha}{\beta^2}\lim_{\rho\to0}\frac{\partial^{N-1}\Phi(\rho,\theta)}{\partial\rho^{N-1}},$$

i.e.,

$$\left\{\frac{\partial}{\partial\rho}\left[\frac{F(\rho,\theta)}{(\beta+\Phi)^2}\frac{\partial^{N-1}\Phi(\rho,\theta)}{\partial\rho^{N-1}}\right]\right\}_{\rho=0}=\frac{\alpha}{\beta^2}\lim_{\rho\to0}\frac{\partial^{N-1}\Phi(\rho,\theta)}{\partial\rho^{N-1}}. \tag{22}$$

We have thus established the existence of the partial derivative of (20) with respect to ρ for $\rho=0$.

We will now prove the continuity of this derivative at the point $(0,\theta)$. To this end, let us find

$$\lim_{\rho\to0}\frac{\partial}{\partial\rho}\left[\frac{F(\rho,\theta)}{(\beta+\Phi)^2}\frac{\partial^{N-1}\Phi(\rho,\theta)}{\partial\rho^{N-1}}\right]. \tag{23}$$

For $\rho\neq0$, we have

$$\frac{\partial}{\partial\rho}\left[\frac{F(\rho,\theta)}{(\beta+\Phi)^2}\frac{\partial^{N-1}\Phi(\rho,\theta)}{\partial\rho^{N-1}}\right]=$$
$$=\frac{F'_\rho(\rho,\theta)}{(\beta+\Phi)^2}\frac{\partial^{N-1}\Phi}{\partial\rho^{N-1}}-\frac{2F(\rho,\theta)\frac{\partial\Phi}{\partial\rho}}{(\beta+\Phi)^4}\frac{\partial^{N-1}\Phi}{\partial\rho^{N-1}}+$$
$$+\frac{F(\rho,\theta)}{(\beta+\Phi)^2}\left[\cos\theta\frac{\partial^N\left(\frac{\psi}{\rho}\right)}{\partial\rho^N}-\sin\frac{\partial^N\left(\frac{\varphi}{\rho}\right)}{\partial\rho^N}\right]. \tag{24}$$

The last term can be written in the form

$$\frac{1}{(\beta+\Phi)^2}\frac{F(\rho,\theta)}{\rho}\left[\rho\frac{\partial^N\left(\frac{\psi}{\rho}\right)}{\partial\rho^N}\cos\theta-\rho\frac{\partial^N\left(\frac{\varphi}{\rho}\right)}{\partial\rho^N}\sin\theta\right].$$

For $\rho \to 0$, we have $\frac{F(\rho, \theta)}{\rho} \to \alpha$, $\frac{1}{(\beta+\Phi)^2} \to \frac{1}{\beta^2}$, and the expression in brackets goes to zero by proposition (b) of Lemma 2. Moreover, $\lim\limits_{\rho\to 0} F(\rho, \theta) = 0$, and

$$\lim_{\rho\to 0} F'_\rho(\rho, \theta) = F'_\rho(0, \theta) = \lim_{\rho\to 0} \frac{F(\rho, \theta)}{\rho} = \alpha.$$

By (24) we then have

$$\lim_{\rho\to 0} \frac{\partial}{\partial\rho}\left[\frac{F(\rho, \theta)}{(\beta+\Phi)^2}\,\frac{\partial^{N-1}\Phi(\rho, \theta)}{\partial\rho^{N-1}}\right] = \frac{\alpha}{\beta^2}\lim_{\rho\to 0}\frac{\partial^{N-1}\Phi}{\partial\rho^{N-1}}.$$

Relations (22) and (25) show that the partial derivative with respect to ρ of the expression (20) and therefore of the function $\frac{\partial^{N-1}R}{\partial\rho^{N-1}}$ exists and is continuous at the point $(0, \theta)$. This completes the proof of the lemma.

5. A remark concerning the definition of a structurally stable dynamic system

In our definition of a structurally stable dynamic system in W (Definition 10, §6.1), we had to introduce a larger region H enclosing the relevant region W. We will now explain why this was unavoidable. Consider a dynamic system (A) defined in some region $\bar{G}$, which has an unstable triple limit cycle L_0 in this region (Definition 28, §26.2). Let W be the closed region bounded by the cycle L_0, and suppose that W contains a structurally stable focus O and that all the paths of system (A) in W, except the focus O and the cycle L_0, are spirals which go to O for $t \to +\infty$ and to L_0 for $t \to -\infty$ (an example of such a system is provided by system (B_k) with $k = 3$ in Example 10, §27.2, see Figure 121). There exist arbitrarily small modifications of system (A) which cause the cycle L_0 to decompose into three cycles (Theorem 42, §27.1), and system (A) therefore should be regarded as structurally unstable in W. On the other hand, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $(\tilde{A})$ is δ-close to (A) to rank 3, we have

$$(\tilde{W}, \tilde{A}) \stackrel{\varepsilon}{\equiv} (W, A), \tag{1}$$

where $\tilde{W}$ is some region. This proposition follows from the fact that any $(\tilde{A})$ sufficiently close to (A) has one, two, or three closed paths near L_0 (Theorems 42 and 43, §27). The innermost of these paths is a limit cycle which is unstable from the inside, and the region bounded by this cycle may be used as $\tilde{W}$. Relation (1) shows that if we were to consider W itself, rather than the larger region H, in Definition 10, §6.1, system (A) with a triple limit cycle would come out as a structurally unstable system.

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$$\frac{dy}{dx}=\frac{\sum a_{ij}x^iy^j}{\sum b_{ij}x^iy^j}, \quad 0\leqslant i+j\leqslant 2$$

(Second-Order Algebraic Limit Cycles for the Differential Equation

$$\frac{dy}{dx}=\frac{\sum a_{ij}x^iy^j}{\sum b_{ij}x^iy^j}, \quad 0\leqslant i+j\leqslant 2).$$

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